ON THE RATIONALITY OF CYCLE INTEGRALS OF MEROMORPHIC MODULAR FORMS

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ABSTRACT. We derive finite rational formulas for the traces of cycle integrals of certain meromorphic modular forms. Moreover, we prove the modularity of a completion of the generating function of such traces. The theoretical framework for these results is an extension of the Shintani theta lift to meromorphic modular forms of positive even weight.

1. Introduction and statement of results

1.1. **Rationality of traces of cycle integrals of meromorphic cusp forms.** Let $k \in \mathbb{N}$ be even. While considering the Doi-Naganuma lift, Zagier [21] encountered the functions

$$f_{k,d}(z) := \frac{|d|^{\frac{k+1}{2}}}{\pi} \sum_{Q \in \mathcal{Q}_d} Q(z,1)^{-k},$$

where $\mathcal{Q}_d$ denotes the set of all integral binary quadratic forms $Q = [a,b,c]$ of discriminant $d = b^2 - 4ac$. These are holomorphic cusp forms of weight $2k$ for $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ if $d > 0$, and meromorphic cusp forms of weight $2k$ for Γ if $d < 0$, i.e., they are meromorphic modular forms which decay like cusp forms towards $i\infty$.

The cusp forms $f_{k,d}$ for $d > 0$ give important examples of modular forms whose cycle integrals

$$\int_{c_Q} f_{k,d}(z) Q(z,1)^{k-1} dz$$

are rational, see [15]. Here $c_Q := \Gamma_Q \backslash C_Q$ is the image in $\Gamma\backslash\mathbb{H}$ of the geodesic

$$C_Q := \left\{ z \in \mathbb{H} : a|z|^2 + bx + c = 0 \right\} \qquad (z = x + iy)$$

associated to $Q = [a,b,c] \in \mathcal{Q}_D$ with $D > 0$. Complementing these results, we present rational formulas for the traces

$$\mathrm{tr}_{f_{k,\mathcal{A}}}(D) := \sum_{Q \in \mathcal{Q}_D/\Gamma} \int_{c_Q} f_{k,\mathcal{A}}(z) Q(z,1)^{k-1} dz$$

of cycle integrals of the refined functions

$$f_{k,\mathcal{A}}(z) := \frac{|d|^{\frac{k+1}{2}}}{\pi} \sum_{Q \in \mathcal{A}} Q(z,1)^{-k},$$

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where $\mathcal{A} \in \mathcal{Q}_d/\Gamma$ is a fixed equivalence class of quadratic forms of discriminant $d < 0$. The poles of $f_{k,\mathcal{A}}$ lie at the CM points $z_Q \in \mathbb{H}$ for $Q \in \mathcal{A}$, which are characterized by $Q(z_Q, 1) = 0$. We assume that they do not lie on any of the geodesics C_Q for $Q \in \mathcal{Q}_D$. Let $z_\mathcal{A} := x_\mathcal{A} + iy_\mathcal{A} \in \Gamma\backslash\mathbb{H}$ denote the equivalence class of a CM point z_Q for $Q \in \mathcal{A}$. We obtain the following rationality result for the traces of $f_{k,\mathcal{A}}$.

Theorem 1.1. *Let F be a weakly holomorphic modular form of weight $\frac{3}{2} - k$ for $\Gamma_0(4)$ satisfying the Kohnen plus space condition. Suppose that the Fourier coefficients $a_F(-D)$ vanish for all $D > 0$ which are squares and that $a_F(-D)$ is rational for $D > 0$. Moreover, assume that $z_\mathcal{A}$ does not lie on any of the geodesics C_Q for $Q \in \mathcal{Q}_D$ for any $D > 0$ for which $a_F(-D) \neq 0$. Then the linear combinations*

$$\sum_{D>0} a_F(-D) \operatorname{tr}_{f_{k,\mathcal{A}}}(D)$$

are rational.

We compute some numerical values of the above traces in Example 1.4 below. Theorem 1.1 follows from the following explicit formulas for the traces.

Theorem 1.2. *Assume the hypotheses of Theorem 1.1. Then we have the formula*

$$\sum_{D>0} a_F(-D) \operatorname{tr}_{f_{k,\mathcal{A}}}(D) = \frac{\sqrt{|d|}}{|\overline{\Gamma}_{z_\mathcal{A}}|} \sum_{D>0} a_F(-D) \left(\frac{c_k(D)}{y_\mathcal{A}^{k-1}} + 4\left(i\sqrt{D}\right)^{k-1} \sum_{\substack{Q=[a,b,c]\in\mathcal{Q}_D \\ Q_{z_\mathcal{A}}>0>a}} P_{k-1}\left(\frac{iQ_{z_\mathcal{A}}}{\sqrt{D}}\right) \right),$$

where P_ℓ is the ℓ-th Legendre polynomial, $|\overline{\Gamma}_{z_\mathcal{A}}|$ is the order of the stabilizer of $z_\mathcal{A}$ in $\overline{\Gamma} := \Gamma/\{\pm 1\}$, and

$$Q_z := \frac{1}{y}\left(a|z|^2 + bx + c\right)$$

for $Q = [a, b, c]$. The constant $c_k(D)$ is given by

$$c_k(D) := \frac{D^{k-\frac{1}{2}}\zeta(k)}{2^{k-3}(2k-1)\zeta(2k)} L_{D_0}(k) \sum_{m|f} \mu(m) \left(\frac{D_0}{m}\right) m^{-k} \sigma_{1-2k}\left(\frac{f}{m}\right). \tag{1.1}$$

Here we write $D = D_0 f^2$ with a fundamental discriminant D_0, μ is the Möbius function, $\sigma_\kappa(n) := \sum_{d|n} d^\kappa$ is the κ-th divisor sum, ζ is the Riemann zeta function, $(\frac{D_0}{\cdot})$ is the Kronecker symbol, and $L_{D_0}(s)$ is the associated Dirichlet L-function.

Remark 1.3. For $D > 0$ a non-square discriminant and $Q = [a, b, c] \in \mathcal{Q}_D$ the geodesic C_Q is a semi-circle centered at the real line. The condition $Q_{z_\mathcal{A}} > 0 > a$ means that C_Q is oriented clockwise and that $z_\mathcal{A}$ lies in the interior of the bounded component of $\mathbb{H} \setminus C_Q$. Since, for fixed non-square $D > 0$, every point $z \in \mathbb{H}$ lies in the interior of the bounded component of $\mathbb{H} \setminus C_Q$ for only finitely many $Q \in \mathcal{Q}_D$, the sum over $Q \in \mathcal{Q}_D$ in Theorem 1.2 is finite.

To illustrate Theorem 1.1 and Theorem 1.2, we treat two examples in low weights.

Example 1.4. We consider the cases $k \in \{2, 4\}$, since then the space S_{2k} of cusp forms of weight $2k$ for Γ is trivial. By the Shimura correspondence, the space of cusp forms of weight $k + \frac{1}{2}$ for $\Gamma_0(4)$ in the Kohnen plus space is isomorphic to S_{2k}, and hence trivial as well. This implies that for every discriminant $D > 0$ there exists a weakly holomorphic modular form F

of weight $\frac{3}{2}-k$ for $\Gamma_0(4)$ satisfying the Kohnen plus space condition such that $a_F(-D)=1$ and $a_F(\ell)=0$ for $-D\neq\ell<0$. Suppose that $D>0$ is a non-square discriminant and that $z_{\mathcal{A}}$ does not lie on any of the geodesics C_Q for $Q\in\mathcal{Q}_D$. Using that $P_1(x)=x$ and $P_3(x)=\frac{5}{2}x^3-\frac{3}{2}x$ we obtain the special cases

$$\mathrm{tr}_{f_{2,\mathcal{A}}}(D)=\frac{\sqrt{|d|}}{\left|\overline{\Gamma}_{z_{\mathcal{A}}}\right|}\left(\frac{c_2(D)}{y_{\mathcal{A}}}-4\sum_{\substack{Q=[a,b,c]\in\mathcal{Q}_D\\ Q_{z_{\mathcal{A}}}>0>a}}Q_{z_{\mathcal{A}}}\right),$$

$$\mathrm{tr}_{f_{4,\mathcal{A}}}(D)=\frac{\sqrt{|d|}}{\left|\overline{\Gamma}_{z_{\mathcal{A}}}\right|}\left(\frac{c_4(D)}{y_{\mathcal{A}}^3}-2\sum_{\substack{Q=[a,b,c]\in\mathcal{Q}_D\\ Q_{z_{\mathcal{A}}}>0>a}}\left(5Q_{z_{\mathcal{A}}}^3+3DQ_{z_{\mathcal{A}}}\right)\right).$$

We give some numerical values for the D-th trace of $f_{2,[A]}$ and $f_{4,[A]}$ with $A=[1,1,1]$ in the following table. To clear the denominators, we display the values of $\mathrm{tr}_{f_{2,[A]}}(D)$ and $3\,\mathrm{tr}_{f_{4,[A]}}(D)$.

D	5	8	13	17	20	21	24	29	32	33	37	40	41
$\mathrm{tr}_{f_{2,[A]}}(D)$	4	8	12	28	24	20	32	20	40	64	44	64	76
$3\,\mathrm{tr}_{f_{4,[A]}}(D)$	20	48	92	452	320	340	576	260	880	1664	1596	1920	2612

We left out those discriminants D which are squares, and $D=12$ and $D=28$ since in these cases the CM point z_A lies on one of the geodesics of discriminant D. We explain the numerical evaluation in Section 6. We checked the above values using Sage, by computing the traces of cycle integrals using their definition. As the values in the table above suggest, for $k\in\{2,4\}$ the numbers $|\overline{\Gamma}_{z_{\mathcal{A}}}|\cdot|d|^{\frac{k}{2}-1}\cdot\mathrm{tr}_{f_{k,\mathcal{A}}}(D)$ are always even integers (for any $\mathcal{A}$), which is not hard to show using the formula (5.7) for $c_k(D)$.

1.2. **The regularized Shintani theta lift of a meromorphic cusp form.** We now describe the theoretical foundation of our work. The classical Shimura-Shintani correspondence establishes a Hecke equivariant isomorphism between the spaces of cusp forms of half-integral weight $k+\frac{1}{2}$ and even integral weight $2k$, with $k\in\mathbb{N}$. Soon after its discovery by Shimura [18], this correspondence was realized by Shintani [19] as a theta lift, that is, as an integral constructed from a theta kernel in two variables. The classical Shintani theta lift for cusp forms was recently generalized to weakly holomorphic modular forms by Guerzhoy, Kane, and the second author [4], to harmonic Maass forms by the first and the third author [2], and to differentials of the third kind by Bruinier, Funke, Imamoglu, and Li [10]. Extending the results of [10], we also include meromorphic cusp forms of arbitrary positive even weight with poles of arbitrary order in the upper half-plane in the Shintani theta lift.

For $k\in\mathbb{N}$ we let $\mathbb{S}_{2k}$ denote the space of meromorphic cusp forms of weight $2k$ for Γ. Every meromorphic modular form can be written as a sum of a meromorphic cusp form, a weakly holomorphic modular form, and, if $k=1$, a multiple of j'/j, with j the usual modular j-invariant. Since the Shintani theta lifts of weakly holomorphic modular forms and of the meromorphic modular form j'/j have already been determined in [2, 10], we restrict our attention to theta lifts of meromorphic cusp forms.

For $k\in\mathbb{N}$ and a fundamental discriminant $\Delta\in\mathbb{Z}$ satisfying $(-1)^k\Delta>0$ we let $\Theta_{k,\Delta}(z,\tau)$ denote the Shintani theta function defined in (2.1). The function $\Theta_{k,\Delta}(-\overline{z},\tau)$ is real-analytic in both variables and transforms like a modular form of weight $2k$ in z for Γ and of weight $k+\frac{1}{2}$ in τ for $\Gamma_0(4)$. We define the regularized Shintani theta lift of $f \in \mathbb{S}_{2k}$ by ($z = x+iy$)

$$\Phi_{k,\Delta}(f,\tau) := \left\langle f, \overline{\Theta_{k,\Delta}(\,\cdot\,,\tau)}\right\rangle^{\mathrm{reg}}, \tag{1.2}$$

where the regularized inner product is defined in (2.3). The regularized integral in (1.2) exists by the following theorem.

Theorem 1.5. *For $f \in \mathbb{S}_{2k}$ the Shintani theta lift $\Phi_{k,\Delta}(f,\tau)$ is a real-analytic function on $\mathbb{H}$ that transforms like a modular form of weight $k+\frac{1}{2}$ for $\Gamma_0(4)$ and satisfies the Kohnen plus space condition.*

More generally, Proposition 3.2 shows that the regularized inner product in (1.2) converges even if we replace $\overline{\Theta_{k,\Delta}(\,\cdot\,,\tau)}$ by any real-analytic function g which transforms like a modular form of weight $2k$ and is of moderate growth at $i\infty$.

1.3. The Fourier expansion of the Shintani theta lift.

One of the main results of this paper is the Fourier expansion of the Shintani theta lift $\Phi_{k,\Delta}(f,\tau)$ of $f \in \mathbb{S}_{2k}$. I turns out that $\Phi_{k,\Delta}(f,\tau)$ yields a completion of the generating series of twisted traces of cycle integrals

$$\mathrm{tr}_{f,\Delta}(D) := \sum_{Q \in \mathcal{Q}_{|\Delta|D}/\Gamma} \chi_\Delta(Q) \int_{c_Q}^{\mathrm{reg}} f(z) Q(z,1)^{k-1} dz,$$

where χ_Δ is the usual genus character as defined on page 238 of [14], and the cycle integrals have to be regularized as explained in Section 2.3 if poles of f lie on the geodesic c_Q.

To describe the non-holomorphic part of the Shintani theta lift, we define for $z \in \mathbb{H}, v > 0$, and any integral binary quadratic form $Q = [a,b,c]$ the function

$$\phi_Q(z,v) := \frac{\sqrt{|\Delta|}}{4} Q(z,1)^{k-1}\left(\mathrm{sgn}(Q_z) - \mathrm{erf}\left(2\sqrt{\pi v}\frac{Q_z}{\sqrt{|\Delta|}}\right)\right), \tag{1.3}$$

where $\mathrm{erf}(x) := \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$ is the error function and $\mathrm{sgn}(0) := 0$. Note that $z \mapsto \phi_Q(z,v)$ is real-analytic up to a jump singularity along C_Q if Q has positive discriminant. More generally, for any $n \in \mathbb{N}_0$ and $z \in \mathbb{H}, v > 0$, we consider the function

$$R^n_{2-2k,z}\left(\phi_Q(z,v)\right) := \frac{\sqrt{|\Delta|}}{4}\left(\mathrm{sgn}(Q_z) R^n_{2-2k,z}\left(Q(z,1)^{k-1}\right) - R^n_{2-2k,z}\left(Q(z,1)^{k-1}\,\mathrm{erf}\left(2\sqrt{\pi v}\frac{Q_z}{\sqrt{|\Delta|}}\right)\right)\right),$$

where $R^n_\kappa := R_{\kappa+2n-2} \circ \cdots \circ R_\kappa$ is an iterated version of the Maass raising operator $R_\kappa := 2i\frac{\partial}{\partial z} + \frac{\kappa}{y}$. Furthermore, we define for $n \in \mathbb{N}_0$ and $z, \tau = u+iv \in \mathbb{H}$ the "theta function"

$$R^n_{2-2k,z}\left(\theta^*_{1-k,\Delta}(z,\tau)\right) := \sum_{D\in\mathbb{Z}}\sum_{Q\in\mathcal{Q}_{|\Delta|D}} \chi_\Delta(Q) R^n_{2-2k,z}\left(\phi_Q(z,v)\right) e^{2\pi i D\tau}.$$

We are now ready to state the Fourier expansion of the Shintani theta lift.

Theorem 1.6. *Let $f \in \mathbb{S}_{2k}$. Then the Fourier expansion of the Shintani theta lift of f is given by*

$$\Phi_{k,\Delta}(f,\tau) = \frac{\sqrt{|\Delta|}}{2}\sum_{D>0}\operatorname{tr}_{f,\Delta}(D)e^{2\pi i D\tau}$$
$$+ (-4)^{1-k}\pi \sum_{\varrho\in\Gamma\backslash\mathbb{H}} \frac{1}{\left|\overline{\Gamma}_{\varrho}\right|}\sum_{n\geq 1} c_{f,\varrho}(-n)\frac{\operatorname{Im}(\varrho)^{n-2k}}{(n-1)!}\left[R_{2-2k,z}^{n-1}\left(\theta^{*}_{1-k,\Delta}(z,\tau)\right)\right]_{z=\varrho},$$

where $c_{f,\varrho}(\ell)$ denotes the ℓ-th coefficient in the elliptic expansion (3.1) *of f at $\varrho \in \mathbb{H}$ and $\overline{\Gamma}_{\varrho}$ is the stabilizer of ϱ in $\overline{\Gamma} = \Gamma/\{\pm 1\}$.*

Remark 1.7.
(1) The sum over $\varrho \in \Gamma\backslash\mathbb{H}$ only runs over the finitely many poles of f modulo Γ.
(2) Using a vector-valued setting as in [2, 10], the methods of the present paper can be applied to compute the Shintani theta lift of meromorphic cusp forms for $\Gamma_0(N)$.

Next, we show that the Shintani theta lift of a meromorphic cusp form is related to the Millson theta function $\Theta^{*}_{k,\Delta}(z,\tau)$, defined in (2.2). A straightforward calculation yields the following result.

Corollary 1.8. *Under the lowering operator $L := -2iv^2\frac{\partial}{\partial\overline{\tau}}$ ($\tau = u + iv$) the Shintani theta lift $\Phi_{k,\Delta}(f,\tau)$ of $f \in \mathbb{S}_{2k}$ maps to*

$$L\left(\Phi_{k,\Delta}(f,\tau)\right) = (-1)^k 2^{1-2k}\pi \sum_{\varrho\in\Gamma\backslash\mathbb{H}} \frac{1}{\left|\overline{\Gamma}_{\varrho}\right|}\sum_{n\geq 1} c_{f,\varrho}(-n)\frac{\operatorname{Im}(\varrho)^{n-2k}}{(n-1)!}\left[R_{2-2k,z}^{n-1}\left(\Theta^{*}_{k,\Delta}(z,\tau)\right)\right]_{z=\varrho}.$$

Remark 1.9. Let $\Delta = 1$ and $k \in \mathbb{N}$ even. The authors of [8] constructed a function $\widehat{\Psi}(z,\tau)$ on $\mathbb{H}\times\mathbb{H}$ that is real-analytic in both variables and transforms like a modular form of weight $2-2k$ in z for Γ and of weight $k+\frac{1}{2}$ in τ for $\Gamma_0(4)$. Furthermore, they showed that it maps to a multiple of the Millson theta function $\Theta^{*}_{k,1}(z,\tau)$ under the lowering operator L_τ. Comparing their construction with Theorem 1.6 and Corollary 1.8, it is not hard to see that the Shintani theta lift $\Phi_{k,1}(f,\tau)$ of $f \in \mathbb{S}_{2k}$ is, up to addition of a holomorphic cusp form, given by a linear combination of the functions $[R^{n-1}_{2-2k,z}(\widehat{\Psi}(z,\tau))]_{z=\varrho}$, with ϱ running over the poles of f in $\Gamma\backslash\mathbb{H}$ and $n \geq 1$ with $c_{f,\varrho}(-n) \neq 0$.

1.4. **Organization of the paper.** We start with a section on the necessary preliminaries about theta functions, regularized inner products, and cycle integrals of meromorphic cusp forms. In the remaining part of this work, we give the proofs of the above results. To prove Theorem 1.5 we derive elliptic expansions of real-analytic functions on $\mathbb{H}$ and study regularized inner products of meromorphic and real-analytic modular forms in Section 3. The computation of the Fourier expansion from Theorem 1.6 is explained in Section 4. In Section 5, we give the proof of Theorem 1.1 and Theorem 1.2. Finally, in Section 6 we provide some details on the numerical evaluation of the rational formulas for the traces from Theorem 1.2.

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2. Preliminaries

2.1. **Theta functions.** For $k \in \mathbb{N}$ and $\Delta \in \mathbb{Z}$ a fundamental discriminant satisfying $(-1)^k\Delta > 0$, the Shintani theta function is defined as

$$\Theta_{k,\Delta}(z,\tau) := y^{-2k} v^{\frac{1}{2}} \sum_{D \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_{|\Delta| D}} \chi_\Delta(Q) Q(z,1)^k e^{-4\pi v \frac{Q_z^2}{|\Delta|}} e^{2\pi i D\tau}. \tag{2.1}$$

Note that $\Theta_{k,\Delta}(z,\tau)$ would vanish identically if $(-1)^k\Delta < 0$. The function $\Theta_{k,\Delta}(-\overline{z},\tau)$ is real-analytic in both variables and transforms like a modular form of weight $2k$ in z for Γ and weight $k+\frac{1}{2}$ in τ for $\Gamma_0(4)$ (see Proposition 3.2 of [7]). Moreover, as a function of z it has moderate growth at $i\infty$ (see Proposition 4.2 of [2]).

We also consider the Millson theta function

$$\Theta^*_{k,\Delta}(z,\tau) := v^{\frac{3}{2}} \sum_{D \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_{|\Delta| D}} \chi_\Delta(Q) Q_z Q(z,1)^{k-1} e^{-4\pi v \frac{Q_z^2}{|\Delta|}} e^{2\pi i D\tau}, \tag{2.2}$$

which transforms like a modular form of weight $2-2k$ in z for Γ and weight $k-\frac{3}{2}$ in τ for $\Gamma_0(4)$ (see Proposition 3.2 of [7]).

2.2. **Regularized inner products.** Next, we describe the regularized inner product in (1.2), which was first introduced by Petersson [17] and later rediscovered and extended by Harvey and Moore [13], Borcherds [3], Bruinier [9], and others. We denote by $[\varrho_1], \dots, [\varrho_r] \in \Gamma\backslash\mathbb{H}$ the equivalence classes of all of the poles of f on $\mathbb{H}$ and we choose a fundamental domain $\mathcal{F}^*$ for $\Gamma\backslash\mathbb{H}$ such that $\varrho_\ell \in \overline{\Gamma}_{\varrho_\ell}\mathcal{F}^*$ for all $1 \leq \ell \leq r$. For any $\varrho \in \mathbb{H}$ and $\varepsilon > 0$ we consider the ε-ball around ϱ,

$$B_\varepsilon(\varrho) := \{z \in \mathbb{H} : |X_\varrho(z)| < \varepsilon\}, \qquad X_\varrho(z) := \frac{z-\varrho}{z-\overline{\varrho}}.$$

Let $g : \mathbb{H} \to \mathbb{C}$ be real-analytic and assume that g transforms like a modular form of weight $2k$ for Γ and is of moderate (i.e., polynomial) growth at $i\infty$. We define the regularized Petersson inner product of f and g by

$$\langle f, g\rangle^{\mathrm{reg}} := \lim_{\varepsilon_1,\dots,\varepsilon_r \to 0} \int_{\mathcal{F}^* \setminus \bigcup_{\ell=1}^r B_{\varepsilon_\ell}(\varrho_\ell)} f(z)\overline{g(z)} y^{2k} \frac{dxdy}{y^2}. \tag{2.3}$$

We see in Proposition 3.2 that the regularized inner product exists.

Remark 2.1. Similar regularized inner products in the case that both f and g are meromorphic cusp forms or weakly holomorphic modular forms have recently been studied, for example, in [6, 22].

2.3. **Regularized cycle integrals of meromorphic cusp forms.** Let $D > 0$ and let $Q = [a,b,c] \in \mathcal{Q}_D$. The associated geodesic C_Q is a semi-circle centered at the real line if $a \neq 0$, and a vertical line if $a = 0$. We orient it counterclockwise if $a > 0$ and from $-\frac{c}{b}$ to $i\infty$ if $a = 0$ and $b > 0$. If poles of f lie on C_Q, then we modify it as follows. For every pole ϱ of f lying on C_Q choose $\varepsilon > 0$ sufficiently small such that no other poles of f lie on $B_\varepsilon(\varrho)$. We denote by $C^{\pm}_{Q,\varepsilon}$ the path that agrees with C_Q outside of every such ball but circumvents every pole ϱ of f along the boundary arc of $B_\varepsilon(\varrho)$ that lies in the connected component of $\mathbb{H} \setminus C_Q$ with $\pm Q_z > 0$. Moreover $c_Q := \Gamma_Q\backslash C_Q$ is the image of C_Q in the modular curve $\Gamma\backslash\mathbb{H}$ and $c^{\pm}_{Q,\varepsilon} := \Gamma_Q\backslash C^{\pm}_{Q,\varepsilon}$.

We define the regularized geodesic cycle integral of $f \in \mathbb{S}_{2k}$ along c_Q by

$$\int_{c_Q}^{\mathrm{reg}} f(z)Q(z,1)^{k-1}dz := \frac{1}{2}\lim_{\varepsilon\to 0}\left(\int_{c_{Q,\varepsilon}^{+}} f(z)Q(z,1)^{k-1}dz + \int_{c_{Q,\varepsilon}^{-}} f(z)Q(z,1)^{k-1}dz\right). \quad (2.4)$$

This is sometimes also called the Cauchy principal value of the geodesic cycle integral, see Section 2.4 of [10]. Furthermore, if no pole of f lies on c_Q, then the above definition agrees with the usual definition of geodesic cycle integrals.

3. Proof of Theorem 1.5

A meromorphic function $f : \mathbb{H} \to \mathbb{C}$ has an elliptic expansion near $\varrho \in \mathbb{H}$ of the shape

$$f(z) = (z - \overline{\varrho})^{-2k} \sum_{n \gg -\infty} c_{f,\varrho}(n) X_{\varrho}^{n}(z), \quad (3.1)$$

with coefficients $c_{f,\varrho}(n) \in \mathbb{C}$. For a proof see Proposition 17 in Zagier's part of [11], note that the required modularity of f in the cited proposition is actually not necessary.

In order to prove Theorem 1.5, we would like to plug in elliptic expansions of f and $\Theta_{k,\Delta}(z,\tau)$ near the poles of f. Note that $z \mapsto \Theta_{k,\Delta}(z,\tau)$ is real-analytic. The shape of elliptic expansions of real-analytic functions are described in the following lemma.

Lemma 3.1. *Let $\kappa \in \mathbb{Z}$, $\varrho \in \mathbb{H}$, and $g : \mathbb{H} \to \mathbb{C}$ be real-analytic near ϱ. Then g has an elliptic expansion of the shape (near ϱ)*

$$g(z) = (z - \overline{\varrho})^{-\kappa} \sum_{n \in \mathbb{Z}} c_{g,\varrho}(|X_{\varrho}(z)|, n) X_{\varrho}^{n}(z), \quad (3.2)$$

with coefficients $c_{g,\varrho}(r,n) \in \mathbb{C}$, which are analytic as functions of the real variable r. Near $r = 0$ we have the Taylor expansion

$$c_{g,\varrho}(r,n) = \sum_{m \geq \max\{0,-n\}} a_{g,\varrho,n}(m) r^{2m}, \quad (3.3)$$

with coefficients $a_{g,\varrho,n}(m) \in \mathbb{C}$ given in (3.6). The constant term $c_{g,\varrho}(0,n)$ is given by

$$c_{g,\varrho}(0,n) = \begin{cases} \frac{1}{n!}(2i)^{\kappa} \operatorname{Im}(\varrho)^{n+\kappa} R_{\kappa}^{n}(g)(\varrho) & \text{if } n \geq 0, \\ 0 & \text{if } n < 0. \end{cases} \quad (3.4)$$

Proof. We generalize the proof of Proposition 17 in Zagier's part in [11]. Expanding as a Taylor series we obtain

$$g(\varrho + W) = \sum_{a,b \geq 0} \left[\frac{\partial^b}{\partial \overline{z}^b} \frac{\partial^a}{\partial z^a} g(z)\right]_{z=\varrho} \frac{W^a}{a!} \frac{\overline{W}^b}{b!}.$$

Writing $\frac{\varrho - \overline{\varrho} w}{1-w} = \varrho + \frac{2i\varrho_2 w}{1-w}$ ($\varrho_2 := \operatorname{Im}(\varrho)$) we obtain, for $|w|$ sufficiently small, the formula

$$(1-w)^{-\kappa} g\left(\frac{\varrho - \overline{\varrho} w}{1-w}\right) = (1-w)^{-\kappa} \sum_{a,b \geq 0} \left[\frac{\partial^b}{\partial \overline{z}^b} \frac{\partial^a}{\partial z^a} g(z)\right]_{z=\varrho} \frac{\left(\frac{2i\varrho_2 w}{1-w}\right)^a}{a!} \frac{\left(-\frac{2i\varrho_2 \overline{w}}{1-\overline{w}}\right)^b}{b!}. \quad (3.5)$$

Expanding $(1-w)^{-\kappa-a}$ and $(1-\overline{w})^{-b}$ using the Binomial Theorem, the right-hand side of (3.5) becomes

$$\sum_{a,b,j,\ell\geq 0}\binom{\kappa+a+j-1}{j}\binom{b+\ell-1}{\ell}\left[\frac{\partial^b}{\partial\overline{z}^b}\frac{\partial^a}{\partial z^a}g(z)\right]_{z=\varrho}\frac{(2i\varrho_2)^a}{a!}\frac{(-2i\varrho_2)^b}{b!}w^{a+j-b-\ell}|w|^{2b+2\ell}.$$

We reorder the summation by setting $n=a+j-b-\ell$ and $m=b+\ell$. Plugging in $w=X_\varrho(z)$ and using the formulas

$$1-w=\frac{2i\varrho_2}{z-\overline{\varrho}},\qquad \frac{\varrho-\overline{\varrho}w}{1-w}=z,$$

we obtain the elliptic expansion

$$g(z)=(z-\overline{\varrho})^{-\kappa}\sum_{n\in\mathbb{Z}}\left(\sum_{m\geq\max\{0,-n\}}a_{g,\varrho,n}(m)|X_\varrho(z)|^{2m}\right)X_\varrho^n(z)$$

with coefficients

$$a_{g,\varrho,n}(m):=(2i\varrho_2)^\kappa\sum_{\substack{0\leq a\leq m+n\\ 0\leq b\leq m}}\binom{\kappa+m+n-1}{m+n-a}\binom{m-1}{m-b}\left[\frac{\partial^a}{\partial z^a}\frac{\partial^b}{\partial\overline{z}^b}g(z)\right]_{z=\varrho}\frac{(2i\varrho_2)^a}{a!}\frac{(-2i\varrho_2)^b}{b!}. \tag{3.6}$$

The formula for $c_{g,\varrho}(0,n)=a_{g,\varrho,n}(0)$ follows from (3.6) and (56) in Zagier's part in [11]. □

Using the elliptic expansion of a real-analytic function given in Lemma 3.1, we can now prove that the regularized Petersson inner product of a meromorphic cusp form and a real-analytic modular form exists.

Proposition 3.2. *The regularized Petersson inner product, defined in* (2.3)*, exists.*

Proof. We divide $\mathcal{F}^*$ into a compact domain containing all of the poles $\varrho_1,\ldots,\varrho_r$ of f, and a remaining set on which f is holomorphic. The integral over the second set converges since f decays like a cusp form towards $i\infty$ and g is of moderate growth at $i\infty$. Hence it suffices to show that for every pole $\varrho:=\varrho_\ell$ of f and $\delta>0$ sufficiently small the limit

$$\lim_{\varepsilon\to 0}\int_{A_\varepsilon^\delta(\varrho)\cap\mathcal{F}^*}f(z)\overline{g(z)}y^{2k}\frac{dxdy}{y^2},\qquad A_\varepsilon^\delta(\varrho):=B_\delta(\varrho)\setminus\overline{B_\varepsilon(\varrho)},$$

exists. In order to obtain an integral over the whole annulus $A_\varepsilon^\delta(\varrho)$, we recall that, by (2a.15) of [16], we have the disjoint union

$$B_\varepsilon(\varrho)=\dot{\bigcup}_{M\in\overline{\Gamma}_\varrho}M\left(B_\varepsilon(\varrho)\cap\mathcal{F}^*\right). \tag{3.7}$$

Therefore we can write, using the modularity of f and g,

$$\int_{A_\varepsilon^\delta(\varrho)\cap\mathcal{F}^*}f(z)\overline{g(z)}y^{2k}\frac{dxdy}{y^2}=\frac{1}{\left|\overline{\Gamma}_\varrho\right|}\int_{A_\varepsilon^\delta(\varrho)}f(z)\overline{g(z)}y^{2k}\frac{dxdy}{y^2}. \tag{3.8}$$

We plug in the elliptic expansions of f from (3.1) and g from (3.2) with $\kappa=2k$ to rewrite the right-hand side of (3.8) as

$$\frac{1}{\left|\overline{\Gamma}_\varrho\right|}\int_{A_\varepsilon^\delta(\varrho)}\sum_{\substack{n\gg-\infty\\ m\in\mathbb{Z}}}c_{f,\varrho}(n)\overline{c_{g,\varrho}(|X_\varrho(z)|,m)}X_\varrho^n(z)\overline{X_\varrho^m(z)}\frac{y^{2k}}{|z-\overline{\varrho}|^{4k}}\frac{dxdy}{y^2}.$$

We next make the change of variables $X_\varrho(z) = e^{i\vartheta} r$ with $0 < \vartheta < 2\pi$ and $\varepsilon < r < \delta$. Then a short calculation shows that

$$\frac{y^{2k}}{|z-\overline{\varrho}|^{4k}} = \left(\frac{1-r^2}{4\varrho_2}\right)^{2k}, \qquad \frac{dxdy}{y^2} = \frac{4r}{(1-r^2)^2} d\vartheta dr,$$

where again $\varrho_2 = \mathrm{Im}(\varrho)$. Thus

$$\frac{4}{(4\varrho_2)^{2k}\left|\overline{\Gamma}_\varrho\right|}\int_\varepsilon^\delta \int_0^{2\pi} \sum_{\substack{n\gg-\infty\\ m\in\mathbb{Z}}} c_{f,\varrho}(n)\overline{c_{g,\varrho}(r,m)} e^{i\vartheta(n-m)} r^{m+n+1}\left(1-r^2\right)^{2k-2} d\vartheta dr. \tag{3.9}$$

The integral over ϑ vanishes unless $m = n$, in which case it equals 2π, thus (3.9) becomes

$$\frac{8\pi}{(4\varrho_2)^{k}\left|\overline{\Gamma}_\varrho\right|}\int_\varepsilon^\delta \sum_{n\gg-\infty} c_{f,\varrho}(n)\overline{c_{g,\varrho}(r,n)} r^{2n+1}\left(1-r^2\right)^{2k-2} dr.$$

The limit as $\varepsilon \to 0$ of the contribution from $n \geq 0$ clearly exists. Hence we need to show that the integral

$$\int_0^\delta \overline{c_{g,\varrho}(r,n)} r^{2n+1}\left(1-r^2\right)^{2k-2} dr$$

exists for the finitely many $n < 0$ with $c_{f,\varrho}(n) \neq 0$. Indeed, the integral exists since for $n < 0$ we have $c_{g,\varrho}(r,n) = O(r^{|2n|})$ by (3.3). This finishes the proof. □

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. Noting that $g(z) = \overline{\Theta_{k,\Delta}(z,\tau)}$ is real-analytic in z and of moderate growth at $i\infty$, Proposition 3.2 shows that the regularized Shintani theta lift (1.2) exists and hence transforms like a modular form of weight $k + \frac{1}{2}$ for $\Gamma_0(4)$. The fact that the Shintani theta lift is real-analytic follows from its Fourier expansion given in Theorem 1.6. This finishes the proof of Theorem 1.5. □

4. Proof of Theorem 1.6

In this section we compute the Fourier expansion of the Shintani theta lift and thereby prove Theorem 1.6.

4.1. **Preliminary computations.**

We start with a useful formula which can be viewed as a generalization of the Residue Theorem.

Lemma 4.1. *Let $\varrho \in \mathbb{H}$, let $f : \mathbb{H} \to \mathbb{C}$ be meromorphic near ϱ, and let $g : \mathbb{H} \to \mathbb{C}$ be real-analytic near ϱ. Then we have the formula*

$$\lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(\varrho)} f(z)g(z)dz = \frac{\pi}{\mathrm{Im}(\varrho)}\sum_{n<0} c_{f,\varrho}(n) c_{g,\varrho}(0,-n-1),$$

where $c_{f,\varrho}(n)$ and $c_{g,\varrho}(r,n)$ are the coefficients of the elliptic expansions (3.1) *of f and* (3.2) *of g at ϱ (with $\kappa = 2 - 2k$).*

Proof. For $\varepsilon > 0$ sufficiently small we can plug in the elliptic expansions of f and of g to obtain that

$$\int_{\partial B_\varepsilon(\varrho)} f(z)g(z)dz = \int_{\partial B_\varepsilon(\varrho)} (z-\overline{\varrho})^{-2} \sum_{\substack{n\gg -\infty \\ m\in\mathbb{Z}}} c_{f,\varrho}(n)c_{g,\varrho}(|X_\varrho(z)|, m)X_\varrho^{m+n}(z)dz$$
$$= \sum_{\substack{n\gg -\infty \\ m\in\mathbb{Z}}} c_{f,\varrho}(n)c_{g,\varrho}(\varepsilon, m)\int_{\partial B_\varepsilon(\varrho)} \frac{(z-\varrho)^{m+n}}{(z-\overline{\varrho})^{m+n+2}}dz.$$

In the last step we use that $|X_\varrho(z)| = \varepsilon$ on $\partial B_\varepsilon(\varrho)$. By the Residue Theorem the last integral vanishes unless $m = -n-1$, in which case it equals $\frac{2\pi i}{\varrho-\overline{\varrho}}$. Taking the limit $\varepsilon \to 0$ and using that $c_{g,\varrho}(0,-n-1) = 0$ for $n \geq 0$ by (3.4), we obtain the stated formula. □

To ease notation, define

$$\varphi_Q(z,v) := y^{-2k}v^{\frac{1}{2}}Q(z,1)^k e^{-4\pi v\frac{Q_z^2}{|\Delta|}},$$

so that the Shintani theta function can be written as

$$\Theta_{k,\Delta}(z,\tau) = \sum_{D\in\mathbb{Z}}\sum_{Q\in\mathcal{Q}_{|\Delta|D}} \chi_\Delta(Q)\varphi_Q(z,v)e^{2\pi iD\tau}.$$

A short calculation, using that $\frac{\partial}{\partial\overline{z}}Q_z = -y^2\frac{i}{2}Q(z,1)$, yields the following result.

Lemma 4.2. *For $z\in\mathbb{H}$ with $Q_z \neq 0$ and all $v > 0$ the function ϕ_Q, defined in* (1.3)*, satisfies*

$$L_z\left(\phi_Q(z,v)\right) = \varphi_Q(z,v)y^{2k}.$$

Remark 4.3. The function $\phi_Q(z,v)$ was used in [2] to compute the Shintani theta lift of harmonic Maass forms and in [10] for $k = 1$ to compute the Shintani theta lift of meromorphic modular forms of weight two that are holomorphic at the cusps and have at most simple poles in $\mathbb{H}$.

Using Lemma 4.2, we see that the D-th Fourier coefficient of the regularized Shintani theta lift (1.2) of $f \in \mathbb{S}_{2k}$ with poles at $[\varrho_1],\dots,[\varrho_r] \in \Gamma\backslash\mathbb{H}$ is given by

$$\lim_{\varepsilon_1,\dots,\varepsilon_r\to 0}\int_{\mathcal{F}^*\setminus\bigcup_{\ell=1}^r B_{\varepsilon_\ell}(\varrho_\ell)} f(z)L_z\left(\sum_{Q\in\mathcal{Q}_{|\Delta|D}}\chi_\Delta(Q)\phi_Q(z,v)\right)\frac{dxdy}{y^2}. \tag{4.1}$$

We compute the Fourier coefficients for $D \leq 0$ and $D > 0$ separately.

4.2. The coefficients of index $D \leq 0$.

For $D \leq 0$ and $Q \in \mathcal{Q}_{|\Delta|D}$ the function $z \mapsto \phi_Q(z,v)$ is real-analytic. Using Stokes' Theorem, we find that (4.1) equals

$$-\lim_{\varepsilon_1,\dots,\varepsilon_r\to 0}\int_{\partial\left(\mathcal{F}^*\setminus\bigcup_{\ell=1}^r B_{\varepsilon_\ell}(\varrho_\ell)\right)} f(z)\sum_{Q\in\mathcal{Q}_{|\Delta|D}}\chi_\Delta(Q)\phi_Q(z,v)dz. \tag{4.2}$$

Here we also use the fact that f decays like a cusp form towards $i\infty$. If $\varepsilon_1,\dots,\varepsilon_r$ are sufficiently small, then the boundary of $\mathcal{F}^*\setminus\bigcup_{\ell=1}^r B_{\varepsilon_\ell}(\varrho_\ell)$ consists of a disjoint union of the boundary arcs $-\partial B_{\varepsilon_\ell}(\varrho_\ell)\cap\mathcal{F}^*$ for $1 \leq \ell \leq r$ and further remaining boundary pieces, which come in Γ-equivalent pairs with opposite orientation and hence cancel out in the integral due to the modularity of the integrand. Therefore, (4.2) equals

$$\sum_{1\leq \ell\leq r}\sum_{Q\in\mathcal{Q}_{|\Delta|D}}\chi_\Delta(Q)\lim_{\varepsilon_\ell\to 0}\int_{\partial B_{\varepsilon_\ell}(\varrho_\ell)\cap\mathcal{F}^*} f(z)\phi_Q(z,v)dz. \tag{4.3}$$

Using the disjoint union (3.7) again, we can rewrite (4.3) as

$$\sum_{1\leq \ell\leq r}\frac{1}{\left|\overline{\Gamma}_{\varrho_\ell}\right|}\sum_{Q\in\mathcal{Q}_{|\Delta|D}}\chi_\Delta(Q)\lim_{\varepsilon_\ell\to 0}\int_{\partial B_{\varepsilon_\ell}(\varrho_\ell)} f(z)\phi_Q(z,v)dz.$$

Since $\phi_Q(z,v)$ is real-analytic at $z=\varrho_\ell$, Lemma 4.1 combined with (3.4) yields that

$$\lim_{\varepsilon_\ell\to 0}\int_{\partial B_{\varepsilon_\ell}(\varrho_\ell)} f(z)\phi_Q(z,v)dz = (-4)^{1-k}\pi\sum_{n\geq 1}c_{f,\varrho_\ell}(-n)\frac{\operatorname{Im}(\varrho_\ell)^{n-2k}}{(n-1)!}\left[R_{2-2k,z}^{n-1}\left(\phi_Q(z,v)\right)\right]_{z=\varrho_\ell}. \tag{4.4}$$

We obtain the coefficients of index $D\leq 0$ as stated in Theorem 1.6.

4.3. **The coefficients of index $D>0$.** We first assume that the poles of f do not lie on any of the geodesics C_Q for $Q\in\mathcal{Q}_{|\Delta|D}$. Since $\varphi_Q(z,v)$ has a jump singularity along the geodesic C_Q, one cannot directly apply Stokes' Theorem. Hence, for $\delta>0$ and every $Q\in\mathcal{Q}_{|\Delta|D}$ we first cut out a δ-tube

$$U_\delta(C_Q) := \{z\in\mathbb{H} : |Q_z|<\delta\}$$

around C_Q from $\mathcal{F}^*$. Applying Stokes' Theorem, we can rewrite (4.1) as

$$\begin{aligned}&\sum_{Q\in\mathcal{Q}_{|\Delta|D}}\chi_\Delta(Q)\lim_{\delta\to 0}\int_{\partial U_\delta(C_Q)\cap\mathcal{F}^*} f(z)\phi_Q(z,v)dz\\ &\quad+\sum_{1\leq \ell\leq r}\frac{1}{\left|\overline{\Gamma}_{\varrho_\ell}\right|}\sum_{Q\in\mathcal{Q}_{|\Delta|D}}\chi_\Delta(Q)\lim_{\varepsilon_\ell\to 0}\int_{\partial B_{\varepsilon_\ell}(\varrho_\ell)} f(z)\phi_Q(z,v)dz.\end{aligned} \tag{4.5}$$

Since $z\mapsto\phi_Q(z,v)$ is real-analytic at every pole of f by assumption, the integral in the second term can be evaluated using Lemma 4.1 as in (4.4) above. The first term in (4.5) is computed in the following proposition.

Proposition 4.4. *Assume that no pole of f lies on any geodesic C_Q for $Q\in\mathcal{Q}_{|\Delta|D}$. Then we have*

$$\sum_{Q\in\mathcal{Q}_{|\Delta|D}}\chi_\Delta(Q)\lim_{\delta\to 0}\int_{\partial U_\delta(C_Q)\cap\mathcal{F}^*} f(z)\phi_Q(z,v)dz = \frac{\sqrt{|\Delta|}}{2}\operatorname{tr}_{f,\Delta}(D). \tag{4.6}$$

Proof. By definition (1.3) the left-hand side of (4.6) equals

$$\frac{\sqrt{|\Delta|}}{4}\sum_{Q\in\mathcal{Q}_{|\Delta|D}}\chi_\Delta(Q)\lim_{\delta\to 0}\int_{\partial U_\delta(C_Q)\cap\mathcal{F}^*} f(z)Q(z,1)^{k-1}\left(\operatorname{sgn}(Q_z)-\operatorname{erf}\left(2\sqrt{\pi v}\frac{Q_z}{\sqrt{|\Delta|}}\right)\right)dz. \tag{4.7}$$

The path $\partial U_\delta(C_Q)\cap\mathcal{F}^*$ can be divided into two paths which are distinguished by the sign of Q_z. The orientation of C_Q is given such that the connected component of $\mathbb{H}\setminus C_Q$ with $Q_z>0$ lies on the right in the direction of travel. The part of $\partial U_\delta(C_Q)\cap\mathcal{F}^*$ which lies in the component of $\mathbb{H}\setminus C_Q$ with $Q_z > 0$ has the same orientation. Using $\operatorname{erf}(0) = 0$, the expression in (4.7) is therefore given by

$$\frac{\sqrt{|\Delta|}}{2} \sum_{Q\in\mathcal{Q}_{|\Delta|D}} \chi_\Delta(Q) \int_{C_Q\cap\mathcal{F}^*} f(z)Q(z,1)^{k-1}dz. \tag{4.8}$$

We split $Q \in \mathcal{Q}_{|\Delta|D}$ as $Q \circ M$ with $Q \in \mathcal{Q}_{|\Delta|D}/\Gamma$ and $M \in \Gamma_Q\backslash\Gamma$, and rewrite (4.8) as

$$\begin{aligned}
&\frac{\sqrt{|\Delta|}}{2} \sum_{Q\in\mathcal{Q}_{|\Delta|D}/\Gamma} \sum_{M\in\Gamma_Q\backslash\Gamma} \chi_\Delta(Q\circ M) \int_{C_{Q\circ M}\cap\mathcal{F}^*} f(z)(Q\circ M)(z,1)^{k-1}dz \\
&= \frac{\sqrt{|\Delta|}}{2} \sum_{Q\in\mathcal{Q}_{|\Delta|D}/\Gamma} \chi_\Delta(Q) \sum_{M\in\Gamma_Q\backslash\Gamma} \int_{C_Q\cap M\mathcal{F}^*} f(z)Q(z,1)^{k-1}dz \\
&= \frac{\sqrt{|\Delta|}}{2} \sum_{Q\in\mathcal{Q}_{|\Delta|D}/\Gamma} \chi_\Delta(Q) \int_{\Gamma_Q\backslash C_Q} f(z)Q(z,1)^{k-1}dz.
\end{aligned}$$

For the first equality we use that $\chi_\Delta(Q\circ M) = \chi_\Delta(Q)$, $(Q\circ M)(z,1) = Q(z,1)|_{-2}M$, and $C_{Q\circ M} = M^{-1}C_Q$. We obtain the formula as stated in the proposition. $\square$

We next consider the case that the poles of f do lie on a geodesic C_Q for some $Q \in \mathcal{Q}_{|\Delta|D}$. By similar arguments as above we find that (4.1) for $D > 0$ equals

$$\begin{aligned}
&\frac{\sqrt{|\Delta|}}{2} \sum_{\substack{Q\in\mathcal{Q}_{|\Delta|D}\\ \varrho_1,\ldots,\varrho_r\notin C_Q}} \chi_\Delta(Q) \int_{C_Q\cap\mathcal{F}^*} f(z)Q(z,1)^{k-1}dz \\
&+ \sum_{\substack{Q\in\mathcal{Q}_{|\Delta|D}\\ \varrho_1,\ldots,\varrho_r\notin C_Q}} \chi_\Delta(Q) \sum_{1\le\ell\le r} \frac{1}{\left|\overline{\Gamma}_{\varrho_\ell}\right|} \lim_{\varepsilon_\ell\to 0} \int_{\partial B_{\varepsilon_\ell}(\varrho_\ell)} f(z)\phi_Q(z,v)dz \\
&+ \sum_{\substack{Q\in\mathcal{Q}_{|\Delta|D}\\ \varrho_\ell\in C_Q\\ \text{for some } \ell}} \chi_\Delta(Q) \lim_{\varepsilon_1,\ldots,\varepsilon_r\to 0} \lim_{\delta\to 0} \int_{\partial\left(U_\delta(C_Q)\cup\bigcup_{\ell=1}^r B_{\varepsilon_\ell}(\varrho_\ell)\right)\cap\mathcal{F}^*} f(z)\phi_Q(z,v)dz.
\end{aligned} \tag{4.9}$$

As in (4.4), the second term of (4.9) can again be computed using Lemma 4.1. By plugging in the definition (1.3) of $\phi_Q(z,v)$, we can rewrite the third term of (4.9) as

$$\begin{aligned}
&\frac{\sqrt{|\Delta|}}{4}\sum_{\substack{Q\in\mathcal{Q}_{|\Delta|D}\\ \varrho_\ell\in C_Q\\ \text{for some }\ell}}\chi_\Delta(Q)\lim_{\varepsilon\to 0}\left(\int_{C^+_{Q,\varepsilon}\cap\mathcal{F}^*}f(z)Q(z,1)^{k-1}dz+\int_{C^-_{Q,\varepsilon}\cap\mathcal{F}^*}f(z)Q(z,1)^{k-1}dz\right)\\
&-\frac{\sqrt{|\Delta|}}{4}\sum_{\substack{Q\in\mathcal{Q}_{|\Delta|D}\\ \varrho_\ell\in C_Q\\ \text{for some }\ell}}\chi_\Delta(Q)\sum_{\substack{1\leq\ell\leq r\\ \varrho_\ell\in C_Q}}\frac{1}{\left|\overline{\Gamma}_{\varrho_\ell}\right|}\lim_{\varepsilon_\ell\to 0}\int_{\partial B_{\varepsilon_\ell}(\varrho_\ell)}f(z)Q(z,1)^{k-1}\operatorname{erf}\left(2\sqrt{\pi v}\frac{Q_z}{\sqrt{|\Delta|}}\right)dz\\
&+\frac{\sqrt{|\Delta|}}{4}\sum_{\substack{Q\in\mathcal{Q}_{|\Delta|D}\\ \varrho_\ell\in C_Q\\ \text{for some }\ell}}\chi_\Delta(Q)\sum_{\substack{1\leq\ell\leq r\\ \varrho_\ell\notin C_Q}}\frac{1}{\left|\overline{\Gamma}_{\varrho_\ell}\right|}\lim_{\varepsilon_\ell\to 0}\int_{\partial B_{\varepsilon_\ell}(\varrho_\ell)}f(z)\phi_Q(z,v)dz,
\end{aligned}\tag{4.10}$$

where $C^{\pm}_{Q,\varepsilon}$ is the modified geodesic defined in Section 2.3. The first term of (4.10) combines with the first term of (4.9) to the D-th Δ-twisted trace of the regularized cycle integrals of f. Since the integrands in the second and third line of (4.10) are real-analytic near ϱ_ℓ, the integrals can again be evaluated using Lemma 4.1. We obtain that the second and third term of (4.10) combine to

$$(-4)^{1-k}\pi\sum_{\substack{Q\in\mathcal{Q}_{|\Delta|D}\\ \varrho_\ell\in C_Q\\ \text{for some }\ell}}\chi_\Delta(Q)\sum_{1\leq\ell\leq r}\frac{1}{\left|\overline{\Gamma}_{\varrho_1}\right|}\sum_{n<0}c_{f,\varrho_\ell}(n)\frac{\operatorname{Im}(\varrho_\ell)^{-n-2k}}{(-n-1)!}\left[R^{-n-1}_{2-2k,z}\left(\phi_Q(z,v)\right)\right]_{z=\varrho_\ell}.$$

This finishes the proof of Theorem 1.6.

5. Proofs of Theorem 1.1 and Theorem 1.2

Throughout this section we let $\Delta=1$. Furthermore, let $d<0$ be a discriminant and let $\mathcal{A}\in\mathcal{Q}_d/\Gamma$ be a fixed class of quadratic forms.

We consider the Shintani theta lift of the meromorphic cusp form $f_{k,\mathcal{A}}$. A short calculation verifies the formula

$$Q(z,1)=\frac{\sqrt{|d|}}{2\operatorname{Im}(z_Q)}(z-\overline{z_Q})^2X_{z_Q}(z)$$

for every $Q\in\mathcal{Q}_d$. In particular, the only poles of $f_{k,\mathcal{A}}$ in $\mathbb{H}$ lie at the CM points z_Q for $Q\in\mathcal{A}$, and the elliptic expansion of $f_{k,\mathcal{A}}$ at z_Q has the shape

$$f_{k,\mathcal{A}}(z)=(z-\overline{z_Q})^{-2k}\left(\frac{\sqrt{|d|}}{\pi}\left(2\operatorname{Im}\left(z_Q\right)\right)^kX^{-k}_{z_Q}(z)+O(1)\right).$$

Hence, by Theorem 1.6 the Shintani theta lift of $f_{k,\mathcal{A}}$ is given by

$$\Phi_k(f_{k,\mathcal{A}},\tau)=\frac{1}{2}\sum_{D>0}\operatorname{tr}_{f_{k,\mathcal{A}}}(D)e^{2\pi iD\tau}-\frac{2^{2-k}\sqrt{|d|}}{(k-1)!\left|\overline{\Gamma}_{z_\mathcal{A}}\right|}\left[R^{k-1}_{2-2k,z}\left(\theta^*_{1-k}(z,\tau)\right)\right]_{z=z_\mathcal{A}}.\tag{5.1}$$

Furthermore, Corollary 1.8 yields

$$L(\Phi_k(f_{k,\mathcal{A}},\tau)) = \frac{2^{1-k}\sqrt{|d|}}{(k-1)!\,\left|\overline{\Gamma}_{z_A}\right|}\left[R_{2-2k,z}^{k-1}\left(\Theta_k^*(z,\tau)\right)\right]_{z=z_{\mathcal{A}}}. \tag{5.2}$$

Following [5], we define the function

$$\mathcal{F}_{1-k,D}(z) := \frac{D^{\frac{1}{2}-k}}{\binom{2k-2}{k-1}\pi}\sum_{Q\in\mathcal{Q}_D}\operatorname{sgn}(Q_z)Q(z,1)^{k-1}\psi\left(\frac{Dy^2}{|Q(z,1)|^2}\right),$$

where $\psi(v):=\frac{1}{2}\beta(v;k-\frac{1}{2},\frac{1}{2})$ is a special value of the incomplete β-function. The function $\mathcal{F}_{1-k,D}$ is a locally harmonic Maass form (in the sense of [5]) of weight $2-2k$ for Γ with jump singularities along the geodesics C_Q for $Q\in\mathcal{Q}_D$. By Theorem 1.2 of [5] the function $\mathcal{F}_{1-k,D}$ maps to a constant multiple of $f_{k,D}$ under ξ_{2-2k}.

Proposition 5.1. *Under the assumptions of Theorem 1.1, we have*

$$\sum_{D>0} a_F(-D)\operatorname{tr}_{f_{k,\mathcal{A}}}(D) = \frac{2^k\sqrt{|d|}}{(k-1)!\,\left|\overline{\Gamma}_{z_\mathcal{A}}\right|}\sum_{D>0} a_F(-D)D^{k-\frac{1}{2}}R_{2-2k}^{k-1}\left(\mathcal{F}_{1-k,D}\right)(z_\mathcal{A}).$$

Proof. Let $P_{\frac{3}{2}-k,D}$ be the unique harmonic Maass form of weight $\frac{3}{2}-k$ for $\Gamma_0(4)$ satisfying the Kohnen plus space condition which maps to a cusp form of weight $k+\frac{1}{2}$ under the ξ-operator and which has a Fourier expansion of the form $\frac{2}{3}e^{-2\pi iD\tau}+O(1)$ at $i\infty$. Then we obtain that $F=\frac{3}{2}\sum_{D>0}a_F(-D)P_{\frac{3}{2}-k,D}$. By Theorem 1.3 (2) [7], we have

$$2^{2k-2}D^{k-\frac{1}{2}}\mathcal{F}_{1-k,D}(z) = \lim_{T\to\infty}\int_{\mathcal{F}_T(4)} P_{\frac{3}{2}-k,D}(\tau)\Theta_k^*(z,\tau)\frac{dudv}{v^2}, \tag{5.3}$$

where

$$\mathcal{F}_T(4) := \bigcup_{M\in\Gamma_0(4)\backslash\Gamma} M\mathcal{F}_T, \qquad \mathcal{F}_T := \left\{\tau=u+iv\in\mathbb{H} : |u|\le\frac{1}{2}, |\tau|\ge 1, v\le T\right\},$$

is a truncated fundamental domain for $\Gamma_0(4)\backslash\mathbb{H}$. Note that the normalization of the functions $P_{\frac{3}{2}-k,D}$ and $\mathcal{F}_{1-k,D}$ in [7] differs from the normalization used in this paper, which explains the different constants when comparing equation (5.3) to Theorem 1.3 (2) in [7].

Now we apply the iterated raising operator and plug $z=z_\mathcal{A}$ into (5.3). A standard argument involving the dominated convergence theorem shows that, for $z_\mathcal{A}$ not lying on any of the geodesics C_Q for $Q\in\mathcal{Q}_D$, we have

$$2^{2k-2}D^{k-\frac{1}{2}}R_{2-2k}^{k-1}\left(\mathcal{F}_{1-k,D}\right)(z_\mathcal{A}) = \lim_{T\to\infty}\int_{\mathcal{F}_T(4)} P_{\frac{3}{2}-k,D}(\tau)\left[R_{2-2k,z}^{k-1}\left(\Theta_k^*(z,\tau)\right)\right]_{z=z_\mathcal{A}}\frac{dudv}{v^2}.$$

Using (5.2), Stokes' Theorem in the form given in Lemma 2 in [12], and the fact that $F = \frac{3}{2}\sum_{D>0} a_F(-D)P_{\frac{3}{2}-k,D}$ is holomorphic on $\mathbb{H}$, we obtain

$$\begin{aligned}
&\frac{2^{k-1}\sqrt{|d|}}{(k-1)!\left|\overline{\Gamma}_{z_\mathcal{A}}\right|}\sum_{D>0} a_F(-D)D^{k-\frac{1}{2}}R^{k-1}_{2-2k}\left(\mathcal{F}_{1-k,D}\right)(z_\mathcal{A})\\
&\quad=\frac{2}{3}\lim_{T\to\infty}\int_{\mathcal{F}_T(4)} F(\tau)\overline{\xi_{k+\frac{1}{2}}\left(\Phi_k(f_{k,\mathcal{A}},\tau)\right)}v^{\frac{3}{2}-k}\frac{dudv}{v^2}\\
&\quad=\frac{2}{3}\lim_{T\to\infty}\int_{-\frac{1}{2}+iT}^{\frac{1}{2}+iT}\left(F(\tau)\Phi_k(f_{k,\mathcal{A}},\tau)+\frac{1}{2}F^{\mathrm{e}}(\tau)\Phi_k^{\mathrm{e}}(f_{k,\mathcal{A}},\tau)+\frac{1}{2}F^{\mathrm{o}}(\tau)\Phi_k^{\mathrm{o}}(f_{k,\mathcal{A}},\tau)\right)d\tau. \quad (5.4)
\end{aligned}$$

Here we set, for $f(\tau) = \sum_{D\in\mathbb{Z}} a(v,D)e^{2\pi iD\tau}$,

$$f^{\mathrm{e}}(\tau) := \sum_{\substack{D\in\mathbb{Z}\\ D\equiv 0 \pmod 2}} a\left(\frac{v}{4},D\right)e^{2\pi iD\frac{\tau}{4}}, \qquad f^{\mathrm{o}}(\tau) := \sum_{\substack{D\in\mathbb{Z}\\ D\equiv 1 \pmod 2}} a\left(\frac{v}{4},D\right)e^{2\pi i\frac{D}{8}}e^{2\pi iD\frac{\tau}{4}}.$$

The integral in (5.4) picks out the constant term in the Fourier expansion of the integrand. Using (5.1) we get that (5.4) equals

$$\begin{aligned}
&\frac{1}{2}\sum_{D>0} a_F(-D)\,\mathrm{tr}_{f_{k,\mathcal{A}}}(D) - \lim_{T\to\infty}\frac{2^{3-k}\sqrt{|d|}}{3(k-1)!\left|\overline{\Gamma}_{z_\mathcal{A}}\right|}\sum_{D\in\mathbb{Z}} a_F(-D)\sum_{Q\in\mathcal{Q}_D}\left[R^{k-1}_{2-2k,z}\left(\phi_Q(z,T)\right)\right]_{z=z_\mathcal{A}}\\
&\qquad - \lim_{T\to\infty}\frac{2^{2-k}\sqrt{|d|}}{3(k-1)!\left|\overline{\Gamma}_{z_\mathcal{A}}\right|}\sum_{D\in\mathbb{Z}} a_F(-D)\sum_{Q\in\mathcal{Q}_D}\left[R^{k-1}_{2-2k,z}\left(\phi_Q\left(z,\tfrac{T}{4}\right)\right)\right]_{z=z_\mathcal{A}}. \quad (5.5)
\end{aligned}$$

One can show that

$$\left[R^{k-1}_{2-2k,z}\left(\phi_Q(z,T)\right)\right]_{z=z_\mathcal{A}} = P_1\left(\sqrt{T}\right)\mathrm{erfc}\left(2\sqrt{\pi T}|Q_{z_\mathcal{A}}|\right) + P_2\left(\sqrt{T}\right)e^{-4\pi TQ^2_{z_\mathcal{A}}},$$

where P_1 and P_2 are polynomials. For $D \leq 0$ and $Q \in \mathcal{Q}_D\backslash\{[0,0,0]\}$ we have $Q_z^2 > 0$ for all $z\in\mathbb{H}$, for $Q = [0,0,0]$ we have $\phi_Q(z,v) = 0$ by definition, and for those $Q\in\mathcal{Q}_D$ with $D > 0$ appearing above (i.e., $a_F(-D)\neq 0$) we have $Q^2_{z_\mathcal{A}} > 0$ since $z_\mathcal{A}$ does not lie on any geodesic C_Q for $Q\in\mathcal{Q}_D$ with $a_F(-D)\neq 0$, by assumption. Since $\mathrm{erfc}(C\sqrt{T})$ and e^{-CT} for $C > 0$ are rapidly decreasing as $T\to\infty$, the limits on the right-hand side of (5.5) vanish, and we obtain the formula stated in the proposition. ☐

By Theorem 7.1 in [5], the function $\mathcal{F}_{1-k,D}$ is locally a polynomial if $S_{2k} = \{0\}$. More generally, by taking suitable linear combinations, we get the following result.

Lemma 5.2. *Assuming the conditions as in Theorem 1.1, we have*

$$\sum_{D>0} a_F(-D)D^{k-\frac{1}{2}}\mathcal{F}_{1-k,D}(z) = \sum_{D>0} a_F(-D)\left(-\frac{c_k(D)}{2^k\binom{2k-2}{k-1}} + 2^{3-2k}\sum_{\substack{Q=[a,b,c]\in\mathcal{Q}_D\\ Q_z>0>a}} Q(z,1)^{k-1}\right), \quad (5.6)$$

where $c_k(D)$ is the constant defined in (1.1).

Proof. By Theorem 1.1 (2) of [7], we have

$$\frac{2^{2k-3}}{6\binom{2k-2}{k-1}} D^{\frac{k}{2}-1} f_{k,D}(z) = \Phi_k^*\left(\xi_{\frac{3}{2}-k}\left(P_{\frac{3}{2}-k,D}\right), z\right),$$

where

$$\Phi_k^*(f,z) := \frac{1}{6}\lim_{T\to\infty}\int_{\mathcal{F}_T(4)} f(\tau)\overline{\Theta_k(z,\tau)} v^{k+\frac{1}{2}}\frac{dudv}{v^2}$$

is the Shimura theta lift of a cusp form f of weight $k+\frac{1}{2}$ for $\Gamma_0(4)$. Using that $F = \frac{3}{2}\sum_{D>0} a_F(-D) P_{\frac{3}{2}-k,D}$ is holomorphic on $\mathbb{H}$, we find that

$$\frac{2^{2k-3}}{6\binom{2k-2}{k-1}} \sum_{D>0} a_F(-D) D^{\frac{k}{2}-1} f_{k,D}(z) = \Phi_k^*\left(\xi_{\frac{3}{2}-k}\left(\sum_{D>0} a_F(-D) P_{\frac{3}{2}-k,D}\right), z\right) = 0,$$

which implies that the analogous linear combinations of the holomorphic and non-holomorphic Eichler integrals $\mathcal{E}_{f_{k,D}}$ and $f_{k,D}^*$ (as defined in the introduction of [5]) vanish. The formula (5.6) then follows from Theorem 7.1 in [5]. Note that there is a typo in the constant c_∞ defined in (7.3) in [5]. It should be equal to $\sum_{\mathcal{A}} c_\infty(\mathcal{A})$, where the sum runs over all classes of binary quadratic forms of discriminant D. Since $\sum_{\mathcal{A}} r_{a,b}(\mathcal{A}) = 2$, the constant c_∞ is missing a factor 2. □

In view of Proposition 5.1, we need to apply the iterated raising operator R_{2-2k}^{k-1} to the right-hand side of (5.6). The following lemma is well-known and not hard to prove.

Lemma 5.3. *For $k \in \mathbb{N}$ we have*

$$R_{2-2k}^{k-1}(1) = (-1)^{k+1}(k-1)!\binom{2k-2}{k-1} y^{1-k}.$$

Next, we compute the iterated raising operator applied to $Q(z,1)^{k-1}$.

Lemma 5.4. *For $k \in \mathbb{N}$ and $Q \in \mathcal{Q}_D$ with $D > 0$ we have*

$$R_{2-2k}^{k-1}\left(Q(z,1)^{k-1}\right) = \left(2i\sqrt{D}\right)^{k-1}(k-1)! P_{k-1}\left(\frac{iQ_z}{\sqrt{D}}\right).$$

Proof. Since $D > 0$ there exists a matrix $M \in \mathrm{SL}_2(\mathbb{R})$ such that $Q \circ M = [0, \sqrt{D}, 0]$. Using that $Q(z,1)|_{-2}M = (Q\circ M)(z,1), Q_{Mz} = (Q \circ M)_z$, and the fact that the slash operator commutes with the raising operator, we can assume without loss of generality that $Q = [0, \sqrt{D}, 0]$. Then $Q(z,1) = \sqrt{D}z$ and $Q_z = \sqrt{D}\frac{x}{y}$. We rewrite the iterated raising operator using formula (56) in Zagier's part in [11] to get

$$\begin{aligned} R_{2-2k}^{k-1}\left(z^{k-1}\right) &= (-4\pi)^{k-1}\sum_{j=0}^{k-1}(-1)^{k+1+j}\binom{k-1}{j}\frac{(2-2k+j)_{k-1-j}}{(4\pi y)^{k-1-j}(2\pi i)^j}\frac{\partial^j}{\partial z^j} z^{k-1} \\ &= \sum_{j=0}^{k-1}(2i)^j\binom{k-1}{j}\frac{(2-2k+j)_{k-1-j}}{y^{k-1-j}}\frac{(k-1)!}{(k-1-j)!} z^{k-1-j}, \end{aligned}$$

where $(a)_m := a(a+1)\cdots(a+m-1)$ is the Pochhammer symbol. We replace $j \mapsto k-1-j$ and obtain, after some simplification, that this equals

$$(2i)^{k-1}(k-1)!\sum_{j=0}^{k-1}\binom{k-1}{j}\binom{k-1+j}{j}\left(\frac{\frac{ix}{y}-1}{2}\right)^{j} = (2i)^{k-1}(k-1)!P_{k-1}\left(\frac{ix}{y}\right),$$

where we use a formula for the Legendre polynomials which can be obtained by combining (22.3.2) and (22.5.24) in [1]. Recalling that $\frac{ix}{y} = \frac{iQ_z}{\sqrt{D}}$ finishes the proof. □

We are now ready to prove Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.2. By combining Proposition 5.1, Lemma 5.2, Lemma 5.3, and Lemma 5.4, we obtain the formulas given in Theorem 1.2. □

Proof of Theorem 1.1. From the functional equation of the Dirichlet L-function and its evalutation at negative integers in terms of Bernoulli polynomials $B_k(x) \in \mathbb{Q}[x]$ (see §7 in [20]), and the well-known evaluation of the Riemann zeta function at even natural numbers in terms of Bernoulli numbers $B_k \in \mathbb{Q}$, we obtain

$$c_k(D) = -\frac{f^{2k-1}D_0^{k-1}\binom{2k}{k}B_k}{2^{k-2}(2k-1)B_{2k}}\sum_{\ell=1}^{D_0}\left(\frac{D_0}{\ell}\right)B_k\left(\frac{\ell}{D_0}\right)\sum_{m|f}\mu(m)\left(\frac{D_0}{m}\right)m^{-k}\sigma_{1-2k}\left(\frac{f}{m}\right), \quad (5.7)$$

which is rational. Furthermore, we have that $|z_{\mathcal{A}}|^2, x_{\mathcal{A}} \in \mathbb{Q}$ and $y_{\mathcal{A}} \in \sqrt{|d|}\,\mathbb{Q}$, hence $Q_{z_{\mathcal{A}}} \in \sqrt{|d|}\,\mathbb{Q}$. This implies that $\sum_{D>0} a_F(-D)\operatorname{tr}_{f_{k,\mathcal{A}}}(D) \in \mathbb{Q}$. The proof of Theorem 1.1 is finished. □

6. Numerical evaluation of $\operatorname{tr}_{f_{k,\mathcal{A}}}(D)$

In order to emphasize the explicit nature of the rational formulas given in Theorem 1.2, we give some details on their numerical evaluation. The constant $c_k(D)$ can be computed using (5.7). To evaluate the sum over $Q = [a,b,c] \in \mathcal{Q}_D$ appearing in the formulas, recall that $Q_{z_A} > 0 > a$ implies that the CM point $z_A = x_A + iy_A$ lies in the interior of the bounded component of $\mathbb{H} \setminus C_Q$. This can only happen if x_A lies between the two real endpoints of the semi-circle C_Q and if y_A is smaller than the radius of C_Q. It is not hard to see that for $a < 0$ this implies the conditions

$$2|a|x_A - \sqrt{D} \le b \le 2|a|x_A + \sqrt{D}, \qquad |a| \le \frac{\sqrt{D}}{2y_A}, \qquad c := \frac{D-b^2}{4|a|} \in \mathbb{Z}. \quad (6.1)$$

There are only finitely many integral binary quadratic forms $[a,b,c]$ with $a < 0$ satisfying these conditions. Hence it remains to check whether $Q_{z_A} > 0$ for these finitely many quadratic forms.

For example, let $k = 2, \mathcal{A} = [A]$ with $A = [1,1,1]$, and $D = 5$. Then we have $d = -3, z_A = \frac{-1+\sqrt{3}}{2} = e^{\frac{2\pi i}{3}}$, and $|\overline{\Gamma}_{z_{\mathcal{A}}}| = 3$. A short calculation gives that $c_2(5) = 8$. The conditions (6.1) are only satisfied by the four quadratic forms $[-1,\pm 1,1], [-1,\pm 3,-1]$, and only $Q = [-1,-1,1]$ satisfies $Q_{z_A} > 0$. Hence the sum in $\operatorname{tr}_{f_{2,[A]}}(5)$ has only one summand, whose value is $[-1,-1,1]_{z_A} = \frac{1}{\sqrt{3}}$. Altogether, we obtain $\operatorname{tr}_{f_{2,[A]}}(5) = 4$.

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