

REGULARIZED INNER PRODUCTS OF MEROMORPHIC MODULAR FORMS AND HIGHER GREEN'S FUNCTIONS

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ABSTRACT. In this paper we study generalizations of quadratic form Poincaré series, which naturally occur as outputs of theta lifts. Integrating against them yields evaluations of higher Green's functions. For this we require a new regularized inner product, which is of independent interest.

1. INTRODUCTION AND STATEMENT OF RESULTS

While investigating the Doi-Naganuma lift, Zagier [42] encountered interesting weight $2k$ cusp forms for $\mathrm{SL}_2(\mathbb{Z})$ ($k \in \mathbb{N}_{\geq 2}$, $\ell \in \mathbb{Z}$) for $\ell = \delta > 0$ (throughout we write $\delta > 0$ for positive discriminants and let $-D < 0$ denote negative discriminants)

$$f_{k,\ell} := \sum_{\mathcal{A} \in \mathcal{Q}_\ell / \mathrm{SL}_2(\mathbb{Z})} f_{k,\ell,\mathcal{A}}. \quad (1.1)$$

Here \mathcal{Q}_ℓ is the set of integral binary quadratic forms of discriminant $\ell \in \mathbb{Z}$ and for \mathcal{A} an $\mathrm{SL}_2(\mathbb{Z})$ -equivalence class of quadratic forms of discriminant ℓ , we set ($z \in \mathbb{H}$)

$$f_{\mathcal{A}}(z) = f_{k,\ell,\mathcal{A}}(z) := |\ell|^{\frac{k}{2}} \sum_{Q \in \mathcal{A}} Q(z, 1)^{-k}.$$

Kohnen and Zagier [32], using a different normalization, showed that the even periods of $f_{k,\delta}$ are rational and Kramer [30] proved that the $f_{k,\delta,\mathcal{A}}$ span the space of weight $2k$ cusp forms. Furthermore, Kohnen and Zagier [31] used the functions $f_{k,\delta}$ to construct a kernel function for the Shimura and Shintani lifts. These may also be realized as so-called theta lifts.

Roughly speaking, a theta lift maps between modular objects in different spaces. One begins with a theta kernel $\Theta(z, \tau)$, which is modular in both variables. In our setting, both variables lie in \mathbb{H} and $\Theta(z, \tau)$ has integral weight in z and a half-integral weight in τ . Given a function $\tau \mapsto P(\tau)$ transforming of the same weight as Θ in the τ -variable, one may then define the *theta lift* of P by taking the Petersson inner product $\langle \cdot, \cdot \rangle$ between Θ and P :

$$\Phi(\Theta; P)(z) := \langle P, \Theta(z, \cdot) \rangle.$$

Niwa [34] wrote the Shimura and Shintani lifts as theta lifts by using a theta kernel corresponding to an integral quadratic form of signature $(2, 1)$, which was later extended by Oda [35] to signature $(2, n)$ for $n \in \mathbb{N}$. These lifts fit into the general framework of the theta correspondence between automorphic forms associated to two groups of a dual reductive pair [26]. Theta lifts have appeared

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in a variety of applications, including a relation of Katok and Sarnak [28] between central values of L -functions and Fourier coefficients. Paralleling the results in [28], the realization of $f_{k,\delta}$ as theta lifts gave non-negativity of twisted central L -values [31].

Natural inputs for theta lifts are Poincaré series. These are, in the simplest case, defined for a translation-invariant function φ (in the case of absolute convergence) as

$$\sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \varphi|_\kappa \gamma(\tau),$$

where $\Gamma_\infty := \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z}\}$, $\kappa \in \frac{1}{2}\mathbb{Z}$ (throughout the paper we use κ for arbitrary weight in $\frac{1}{2}\mathbb{Z}$ and reserve k for restricted weights), and $|_\kappa$ denotes the usual slash operator. A natural choice for φ is a term from the Fourier expansions of forms in the space of automorphic forms one is interested in. In this paper, we consider in particular half-integral weight modular forms and *harmonic Maass forms*, which transform and grow like modular forms but instead of being meromorphic they are annihilated by the weight κ *hyperbolic Laplace operator* ($z = x + iy \in \mathbb{H}$)

$$\Delta_\kappa := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + i\kappa y \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right). \quad (1.2)$$

We denote the Poincaré series constructed by choosing a particular φ from the Fourier expansions of these forms by $P_{k+1/2,m}$ and $\mathcal{P}_{3/2-k,m}$ (see (2.20) and (2.21) for the explicit definitions). This gives in particular four relevant cases: for positive weight one can average a cusp form coefficient or a coefficient which grows towards $i\infty$, while in negative weight one can define two kinds of Poincaré series, one of which grows in the holomorphic part and one which grows in the non-holomorphic part (see (2.13) for the decomposition).

We start with positive weight and define the theta kernel ($\tau = u + iv \in \mathbb{H}$)

$$\Theta_k(z, \tau) := y^{-2k} v^{\frac{1}{2}} \sum_{D \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_D} Q(z, 1)^k e^{-4\pi Q_z^2 v} e^{2\pi i D \tau}. \quad (1.3)$$

Here, for $Q = [a, b, c]$, we set

$$Q_z := y^{-1} (a|z|^2 + bx + c). \quad (1.4)$$

It is well-known that the function $z \mapsto \Theta_k(-\bar{z}, \tau)$ is modular of weight $2k$ and $\tau \mapsto \Theta_k(z, \tau)$ is modular of weight $k + 1/2$. Hence taking the inner product in either variable yields a lift between integral and half-integral weights. For this, we define the following theta lift

$$\Phi_k(f)(z) := \Phi(\Theta_k; f)(z).$$

Using as input positive weight cuspidal Poincaré series, one recovers the functions $f_{k,\delta}$:

$$f_{k,\delta} = C_{k,\delta} \cdot \Phi_k \left(P_{k+\frac{1}{2},\delta} \right)$$

with $C_{k,\delta}$ an explicit constant. By the Petersson coefficient formula, holomorphic projection (re-called below) of the theta kernel Θ_k yields the generating function

$$\Omega_k(z, \tau) := \sum_{\delta > 0} \delta^{\frac{k-1}{2}} f_{k,\delta}(z) e^{2\pi i \delta \tau}.$$

Kohnen and Zagier [31] proved that $z \mapsto \Omega_k(z, \tau)$ is a weight $2k$ cusp form and $\tau \mapsto \Omega_k(z, \tau)$ a weight $k + 1/2$ cusp form. By integrating in either variable, Ω_k yields theta lifts from weight $2k$ to $k + 1/2$ and from weight $k + 1/2$ to $2k$; these are the well-known (first) Shintani [40] and Shimura [39] lifts. Hereby, the idea underlying holomorphic projection is simple. Suppose that f is a weight κ real-analytic modular form with moderate growth at cusps. Then a linear functional on the space of weight κ cusp forms is given by $g \mapsto \langle g, f \rangle$. Since the Petersson inner product is non-degenerate,

this functional must be given by $\langle \cdot, F \rangle$ for some weight κ cusp form F . This F is essentially the holomorphic projection of f .

If one takes as inputs of the theta lifts weakly holomorphic Poincaré series, one obtains, instead of the $f_{k,\delta}$'s, the analogous meromorphic modular forms $f_{k,-D}$, defined in (1.1). We note that some care is needed if the inputs are no longer cusp forms. Although the naive definition of the inner product usually diverges when taking weakly holomorphic modular forms (meromorphic modular forms with poles only at cusps), one can extend it, and hence also theta lifts, by doing a process called regularization which extends the inner product to include more general inputs; we describe this in Section 3. To obtain the functions $f_{k,-D}$ as theta lifts, we use a regularization of Borcherds. Note that the functions $f_{k,-D}$ were first constructed by Bengoechea [3] in her thesis.

Theorem 1.1. *We have*

$$\Phi_k \left(P_{k+\frac{1}{2},-D} \right) = f_{k,-D}.$$

Remarks.

- (1) The theta lift in Theorem 1.1 is a special case of a more general theta lift introduced by Borcherds in Theorem 14.3 of [4]. We choose a Poincaré series as a distinguished input, while Borcherds had more general inputs. Moreover, Borcherds unfolded against the theta function, while we apply the unfolding method to the Poincaré series. As a result, the two approaches yield different representations of the functions $f_{k,-D}$.
- (2) The theta lift Φ_k maps (parabolic) Poincaré series $P_{k+1/2,\ell}$ to other types of Poincaré series; the functions $f_{k,\delta}$ are sums of hyperbolic Poincaré series which appeared in previous work of Petersson [36] (see also [27]), while we see in (2.24) that the $f_{k,-D}$ are sums of elliptic Poincaré series defined by Petersson in [37]. This implies that they are elements of \mathbb{S}_{2k} , the space of meromorphic cusp forms of weight $2k$ for $\mathrm{SL}_2(\mathbb{Z})$, which are meromorphic modular forms that decay like cusp forms towards $i\infty$.

We next turn to negative weight. For this, we use the theta kernel ($k \in \mathbb{N}_{\geq 2}$)

$$\Theta_{1-k}^*(z, \tau) := v^k \sum_{D \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_D} Q_z Q(z, 1)^{k-1} e^{-\frac{4\pi|Q(z,1)|^2 v}{y^2}} e^{-2\pi i D \tau}.$$

The function $z \mapsto \Theta_{1-k}^*(z, \tau)$ is modular of weight $2 - 2k$ and $\tau \mapsto \Theta_{1-k}^*(z, \tau)$ is modular of weight $3/2 - k$. We set

$$\Phi_{1-k}^*(f)(z) := \Phi(\Theta_{1-k}^*; f)(-\bar{z}).$$

We then define negative weight analogues of the functions $f_{k,\ell}$, namely ($\ell \in \mathbb{Z}$)

$$\mathcal{F}_{1-k,\ell} := \sum_{\mathcal{A} \in \mathcal{Q}_\ell / \mathrm{SL}_2(\mathbb{Z})} \mathcal{F}_{1-k,\ell,\mathcal{A}},$$

where

$$\mathcal{F}_{\mathcal{A}}(z) = \mathcal{F}_{1-k,\ell,\mathcal{A}}(z) := \sum_{Q \in \mathcal{A}} \mathbb{P}_{1-k,\ell,Q}(z) \tag{1.5}$$

with

$$\mathbb{P}_{1-k,\ell,Q}(z) := \frac{i(-1)^k}{2} |\ell|^{\frac{1-k}{2}} \mathrm{sgn}(Q_z) Q(z, 1)^{k-1} \beta \left(\frac{\ell y^2}{|Q(z, 1)|^2}; k - \frac{1}{2}, \frac{1}{2} \right). \tag{1.6}$$

Here, $\beta(Z; a, b)$ denotes the *incomplete beta-function*, which is defined for $a, b \in \mathbb{C}$ satisfying $\mathrm{Re}(a), \mathrm{Re}(b) > 0$ by $\beta(Z; a, b) := \int_0^Z t^{a-1} (1-t)^{b-1} dt$. Note that we can also write the incomplete beta-function in terms of hypergeometric functions ${}_2F_1$ (see (2.11)).

We recall some of the properties of these functions for $\ell = \delta > 0$. The $\mathcal{F}_{1-k,\delta,\mathcal{A}}$, with a different normalization, were investigated by Kohnen and the first two authors in [7] and a variant of these

was studied by Hövel [25] for $k = 1$. It turns out that they are locally harmonic Maass forms. Locally harmonic Maass forms allow jump singularities in the upper half-plane. They also appeared as theta lifts in both physics (or rather their higher-dimensional analogues occurred) via work of Angelantonj, Florakis, and Pioline [1] and in mathematics from work of Viazovska and the first two authors [8]. Namely, analogous to the positive weight case we have [8, 25]

$$\mathcal{F}_{1-k,\delta} = C_{1-k,\delta} \cdot \Phi_{1-k} \left(\mathcal{P}_{\frac{3}{2}-k,\delta} \right)$$

with $C_{1-k,\delta}$ an explicit constant and $\mathcal{P}_{\frac{3}{2}-k,\delta}$ defined in (2.21). In addition to their relationship via theta lifts, the functions $\mathcal{F}_{\mathcal{A}}$ are connected to the functions $f_{\mathcal{A}}$ via the differential operators ξ_{2-2k} and \mathcal{D}^{2k-1} , where

$$\xi_{\kappa} := 2iy^{\kappa} \frac{\bar{\partial}}{\partial \bar{z}} \quad \text{and} \quad \mathcal{D} := \frac{1}{2\pi i} \frac{\partial}{\partial z}. \quad (1.7)$$

Specifically, we have

$$\xi_{2-2k}(\mathcal{F}_{\mathcal{A}}) = \mathcal{C}_{1,k,\delta} \cdot f_{\mathcal{A}}, \quad \mathcal{D}^{2k-1}(\mathcal{F}_{\mathcal{A}}) = \mathcal{C}_{2,k,\delta} \cdot f_{\mathcal{A}},$$

where the $\mathcal{C}_{j,k,\delta}$ are explicit constants.

It is unusual for a harmonic Maass form to map to a constant multiple of the same function under ξ_{2-2k} and \mathcal{D}^{2k-1} . However, given their uniform definition in (1.5), it is not a surprise to find out that for discriminants $-D < 0$, the functions $\mathcal{F}_{1-k,-D,\mathcal{A}}$ have many properties similar to those of $\mathcal{F}_{1-k,\delta,\mathcal{A}}$. As a difference between negative and positive discriminants, note that for negative discriminants the functions have poles instead of jump singularities. We call functions which behave like harmonic Maass forms away from singularities of this type *polar harmonic Maass forms*.

Theorem 1.2.

(1) *We have*

$$\Phi_{1-k}^* \left(\mathcal{P}_{\frac{3}{2}-k,-D} \right) = \mathcal{F}_{1-k,-D}.$$

(2) *For $\mathcal{A} \in \mathcal{Q}_{-D}/\mathrm{SL}_2(\mathbb{Z})$, the functions $\mathcal{F}_{\mathcal{A}}$ are weight $2 - 2k$ polar harmonic Maass forms whose only singularities occur at τ_Q for $Q \in \mathcal{A}$. Furthermore, we have*

$$\xi_{2-2k}(\mathcal{F}_{\mathcal{A}}) = f_{\mathcal{A}}, \quad \mathcal{D}^{2k-1}(\mathcal{F}_{\mathcal{A}}) = -\frac{(2k-2)!}{(4\pi)^{2k-1}} f_{\mathcal{A}}. \quad (1.8)$$

Remarks.

- (1) The difference in the singularities of $\mathcal{F}_{1-k,\ell,\mathcal{A}}$ for discriminants $\ell > 0$ and $\ell < 0$ comes from the sign factor in (1.6). For $\ell > 0$, $Q_z = 0$ along a geodesic S_Q , and the function “jumps” as one crosses from one side of S_Q to the other. For $\ell < 0$, $\mathrm{sgn}(Q_z) \neq 0$ and $\mathrm{sgn}(Q_z)$ is independent of z ; namely, $\mathrm{sgn}(Q_z) = 1$ for all $z \in \mathbb{H}$ if Q is positive-definite and $\mathrm{sgn}(Q_z) = -1$ for all $z \in \mathbb{H}$ if Q is negative-definite.
- (2) A key step in proving Theorem 1.2 (2) is to relate $\mathcal{F}_{\mathcal{A}}$ to higher Green’s functions G_k defined in Subsection 2.6 (see Corollary 5.2). These have appeared in a number of interesting applications and their evaluations at pairs of CM-points has been heavily studied. Values of higher Green’s functions at CM-points are conjectured to be basically logarithms of algebraic numbers, and a number of cases are known by work of Mellit [33] and Viazovska [41].

Let us now return to the positive weight cusp forms $f_{k,\delta,\mathcal{A}}$. Integrating against them gives cycle integrals. To be more precise we have ($\mathcal{C}_{k,\delta} \in \mathbb{R}$ an explicit constant)

$$\langle f, f_{k,\delta,\mathcal{A}} \rangle = \mathcal{C}_{k,\delta} \int_{\Gamma_{Q_0} \setminus S_{Q_0}} f(z) Q_0(z, 1)^{k-1} dz, \quad (1.9)$$

where $Q_0 \in \mathcal{A}$ is arbitrary, S_{Q_0} is an orientated geodesic joining the 2 real roots of Q_0 and $\Gamma_{Q_0} \subset \mathrm{SL}_2(\mathbb{R})$ is the stabilizer group of Q_0 . These cycle integrals then occur as coefficients of the (first) Shintani lift. In this paper we take the Petersson inner product against $f_{\mathcal{A}}$.

Since the classical inner product diverges, one needs to regularize. In addition to their inherent interest, extensions of Petersson's inner product yield applications to other areas including generalized Kac–Moody algebras [21] and the arithmetic of Shimura varieties [14]. Those applications used a regularization of Petersson [38], later rediscovered and generalized by Borchers [4] and Harvey–Moore [23]. However, Petersson's inner product $\langle f, f \rangle$ for any (non-cuspidal) $f \in \mathbb{S}_{2k}$ always diverges (see Satz 1 of [38]), so one cannot use it to extend the classical inner product to an inner product on any larger subspace. For the application in this paper, we introduce a new regularized inner product, again denoted by $\langle \cdot, \cdot \rangle$ and formally defined in (3.5) below in all cases.

Theorem 1.3. *The regularized inner product $\langle f, g \rangle$ exists for all $f, g \in \mathbb{S}_{2k}$, it is Hermitian, and it equals Petersson's regularized inner product whenever his exists.*

We next consider an application of the inner product to higher Green's functions. To state the formula, for $\varrho \in \mathbb{H}$ we let $\omega_{\varrho} := \#\Gamma_{\varrho}$, where Γ_{ϱ} is the stabilizer group of ϱ of the action of $\mathrm{PSL}_2(\mathbb{Z})$ on \mathbb{H} . We require the *elliptic expansion* of a meromorphic modular form f around $\varrho \in \mathbb{H}$, namely

$$f(z) = (z - \bar{\varrho})^{-2k} \sum_{n \gg -\infty} c_{f, \varrho}(n) X_{\varrho}(z)^n, \quad \text{with} \quad X_{\varrho}(z) := \frac{z - \varrho}{z - \bar{\varrho}}. \quad (1.10)$$

Furthermore, set

$$b_{k, n} := \frac{(-1)^k \Gamma(k - \frac{1}{2})}{2^k \sqrt{\pi}} \begin{cases} \frac{1}{n!} & \text{if } n \geq k - 1, \\ \frac{1}{(2k-2-n)!} & \text{if } n < k - 1. \end{cases} \quad (1.11)$$

Throughout, we let

$$R_{\kappa} := 2i \frac{\partial}{\partial z} + \frac{\kappa}{y} \quad (1.12)$$

be the *Maass raising operator* and denote repeated raising by $R_{\kappa}^n := R_{\kappa+2n-2} \circ \cdots \circ R_{\kappa}$.

Theorem 1.4. *If $Q_0 \in \mathcal{A} \in \mathcal{Q}_{-D}/\mathrm{SL}_2(\mathbb{Z})$ and $f \in \mathbb{S}_{2k}$ with poles in $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ at $\mathfrak{z}_1, \dots, \mathfrak{z}_r$ with $\mathfrak{z}_{\ell} \neq \tau_Q$ for all $Q \in \mathcal{A}$ and $\ell \in \{1, \dots, r\}$, then $(\mathfrak{z}_{\ell} = \mathfrak{x}_{\ell} + i\mathfrak{y}_{\ell})$*

$$\langle f, f_{\mathcal{A}} \rangle = \frac{\pi}{\omega_{\tau_{Q_0}}} \sum_{\ell=1}^r \frac{1}{\omega_{\mathfrak{z}_{\ell}}} \left(\sum_{n \geq k} b_{k, n-1} \mathfrak{y}_{\ell}^{-2k+n} c_{f, \mathfrak{z}_{\ell}}(-n) R_0^{n-k} (G_k(z, \tau_{Q_0})) \right. \\ \left. + \sum_{n=1}^{k-1} b_{k, n-1} \mathfrak{y}_{\ell}^{-n} c_{f, \mathfrak{z}_{\ell}}(-n) \overline{R_0^{n-k} (G_k(z, \tau_{Q_0}))} \right).$$

Particularly interesting is the following special case.

Corollary 1.5. *For every $Q_j \in \mathcal{A}_j \in \mathcal{Q}_{-D_j}/\mathrm{SL}_2(\mathbb{Z})$ ($j = 1, 2$) with $\mathcal{A}_1 \neq \mathcal{A}_2$,*

$$\langle f_{\mathcal{A}_1}, f_{\mathcal{A}_2} \rangle = \pi b_{k, k-1} \frac{G_k(\tau_{Q_1}, \tau_{Q_2})}{\omega_{\tau_{Q_1}} \omega_{\tau_{Q_2}}}.$$

Remark. Replacing $f_{\mathcal{A}_j}$ by more general functions $\Psi_{2k, -k}(\cdot, z_j)$, defined in (2.22) below, which have poles at arbitrary $z_1, z_2 \in \mathbb{H}$, one obtains a relation to $G_k(z_1, z_2)$.

Remark. Given the interest in G_k evaluated at CM-points, one may wonder what further implications Corollary 1.5 has. Possible future directions of study along these lines are discussed in Section 7 below. The relation between higher Green's functions and inner products in Corollary 1.5 leads one to search for connections with geometry. For $k = 1$, which is excluded here, Gross and Zagier

related the Green's function evaluated at CM-points to the height pairing of certain Heegner points on modular curves (see Proposition 2.22 in Section II of [22]). This has been generalized to higher k by Zhang, who defined a global height pairing between CM-cycles in certain Kuga–Sato varieties using arithmetic intersection theory, as developed by Gillet and Soulé [19]. The archimedean part of this height pairing is then given by the values of higher Green's functions evaluated at CM-points (see Propositions 3.4.1 and 4.1.2 of [45]).

The paper is organized as follows. In Section 2, we recall basic geometric facts and certain special functions and introduce the relevant modular objects. In Section 3, we study regularized inner products and prove Theorem 1.3. In Section 4, we investigate theta lifts, proving Theorem 1.1 and Theorem 1.2 (1). Theorem 1.2 (2) is established while studying the functions $\mathcal{F}_{\mathcal{A}}$ in Section 5. In Section 6, we compute regularized inner products in order to prove Theorem 1.4 and Corollary 1.5. We conclude the paper with a discussion of natural questions in Section 7.

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2. PRELIMINARIES

2.1. CM-points and hyperbolic distance. For a positive-definite $Q = [a, b, c] \in \mathcal{Q}_{-D}$, we write the associated CM-point (note $a > 0$)

$$\tau_Q = u_Q + iv_Q, \quad \text{with} \quad u_Q = -\frac{b}{2a} \quad \text{and} \quad v_Q = \frac{\sqrt{D}}{2a}. \quad (2.1)$$

We note that for $z = x + iy \in \mathbb{H}$, with X_{τ_Q} defined in (1.10),

$$Q(z, 1) = \frac{\sqrt{D}}{2v_Q} (z - \overline{\tau_Q})^2 X_{\tau_Q}(z). \quad (2.2)$$

Moreover, for $Q \in \mathcal{Q}_{\ell}$, we often make use of the identity

$$y^{-2} |Q(z, 1)|^2 = Q_z^2 + \ell, \quad (2.3)$$

with Q_z given in (1.4). The quantity Q_z naturally occurs when computing the hyperbolic distance $d(z, \mathfrak{z})$ between z and $\mathfrak{z} = \mathfrak{x} + i\mathfrak{y} \in \mathbb{H}$, which is expressed through (see p. 131 of [2])

$$\cosh(d(z, \mathfrak{z})) = 1 + \frac{|z - \mathfrak{z}|^2}{2y\mathfrak{y}}.$$

Choosing $\mathfrak{z} = \tau_Q$ to be a CM-point with $Q \in \mathcal{Q}_{-D}$ in particular, we have the equality

$$\cosh(d(z, \tau_Q)) = \frac{Q_z}{\sqrt{D}}. \quad (2.4)$$

Using (2.3) and (2.4) gives

$$(1 - \cosh(d(z, \tau_Q))^2)^{-1} = -\frac{Dy^2}{|Q(z, 1)|^2}. \quad (2.5)$$

Finally, for $z \in \mathbb{H}$ (and $\varrho \in \mathbb{H}$ fixed), we set

$$r_{\varrho}(z) := \tanh\left(\frac{d(z, \varrho)}{2}\right) = |X_{\varrho}(z)|.$$

We note that (2.4) implies that

$$1 - r_{\tau_Q}(z)^2 = \frac{2y\sqrt{D}}{Q_z + \sqrt{D}}. \quad (2.6)$$

2.2. Properties of hypergeometric functions. In this subsection we recall relations of the hypergeometric function to other functions and transformations required for this paper.

Firstly, one directly sees, by inspecting the series representation of the ${}_2F_1$, that

$${}_2F_1(a, b; c; Z) = {}_2F_1(b, a; c; Z). \quad (2.7)$$

Furthermore, by 15.4.6 of [15], we have

$${}_2F_1(a, b; b; Z) = (1 - Z)^{-a}. \quad (2.8)$$

We also require the following transformation law from 15.8.1 of [15] (valid for $1 - Z \notin \mathbb{R}^-$)

$${}_2F_1(a, b; c; Z) = (1 - Z)^{-b} {}_2F_1\left(c - a, b; c; \frac{Z}{Z - 1}\right). \quad (2.9)$$

By 15.5.3 (with $n = 1$) of [15], we have

$$\frac{\partial}{\partial Z} (Z^a {}_2F_1(a, b; c; Z)) = aZ^{a-1} {}_2F_1(a + 1, b; c; Z). \quad (2.10)$$

By 8.17.7 of [15], the hypergeometric function is related to the incomplete β -function via

$$\beta(Z; a, b) = \frac{Z^a}{a} {}_2F_1(a, 1 - b; a + 1; Z). \quad (2.11)$$

Using 15.8.14 of [15] then yields that for $1 - Z \notin \mathbb{R}^-$, we have

$$\beta(Z; 2k - 1, 1 - k) = 2^{2k-2} i (-1)^k \beta\left(\frac{Z^2}{4(Z-1)}; k - \frac{1}{2}, \frac{1}{2}\right). \quad (2.12)$$

2.3. Polar harmonic Maass forms and their elliptic expansions. To define polar harmonic Maass forms, for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $f: \mathbb{H} \rightarrow \mathbb{C}$, the weight $\kappa \in \frac{1}{2}\mathbb{Z}$ slash-action is

$$f|_{\kappa}\gamma(\tau) := (c\tau + d)^{-\kappa} f(\gamma\tau) \begin{cases} 1 & \text{if } \kappa \in \mathbb{Z}, \\ \left(\frac{c}{d}\right) \varepsilon_d^{2\kappa} & \text{if } \kappa \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z} \text{ and } \gamma \in \Gamma_0(4), \end{cases}$$

with the extended Legendre symbol (\cdot) and $\varepsilon_d = 1$ or $\varepsilon_d = i$ for $d \equiv 1 \pmod{4}$ or $d \equiv 3 \pmod{4}$.

Definition. For $N \in \mathbb{N}$, a weight $\kappa \in \frac{1}{2}\mathbb{Z}$ polar harmonic Maass form (on $\Gamma_0(N)$) is a real-analytic function $\mathcal{M}: \mathbb{H} \rightarrow \mathbb{C}$ which, outside of finitely many singularities in $\Gamma_0(N) \backslash (\mathbb{H} \cup \mathbb{Q} \cup \{i\infty\})$, satisfies:

- (1) For all $\gamma \in \Gamma_0(N)$, we have $\mathcal{M}|_{\kappa}\gamma = \mathcal{M}$.
- (2) We have $\Delta_{\kappa}(\mathcal{M}) = 0$ with the hyperbolic Laplacian defined in (1.2).
- (3) For every $\varrho \in \mathbb{H}$, there exists $n \in \mathbb{N}$ such that $(\tau - \varrho)^n \mathcal{M}(\tau)$ is bounded for $r_{\varrho}(\tau) \ll_{\mathcal{M}} 1$.
- (4) The function \mathcal{M} grows at most linear exponentially at the cusps.

If the only singularities of \mathcal{M} occur at the cusps, then \mathcal{M} is a *harmonic Maass form*.

For $\kappa < 1$, the Fourier expansion of a polar harmonic Maass form at $i\infty$ has a natural splitting of the shape (convergent for v sufficiently large, with $c_{\mathcal{M}}^{\pm}(n) \in \mathbb{C}$)

$$\mathcal{M}(\tau) = \mathcal{M}^+(\tau) + \mathcal{M}^-(\tau), \quad (2.13)$$

where the *holomorphic* and *non-holomorphic parts* (at $i\infty$) are defined, respectively, by

$$\begin{aligned}\mathcal{M}^+(\tau) &:= \sum_{n \gg -\infty} c_{\mathcal{M}}^+(n) e^{2\pi i n \tau}, \\ \mathcal{M}^-(\tau) &:= c_{\mathcal{M}}^-(0) v^{1-\kappa} + \sum_{0 \neq n \ll \infty} c_{\mathcal{M}}^-(n) \Gamma(1-\kappa, -4\pi n v) e^{2\pi i n \tau}.\end{aligned}$$

Here $\Gamma(r, Z) := \int_Z^\infty t^{r-1} e^{-t} dt$ denotes the *incomplete gamma function*. There are similar expansions at the other cusps. One reason that the splitting into holomorphic and non-holomorphic parts is natural is that the holomorphic part is annihilated by ξ_κ defined in (1.7). Both parts can have singularities; the singularities in the holomorphic part are poles, while one can determine the kind of singularities in the non-holomorphic part by noting that its image under ξ_κ is meromorphic. The terms in the expansion which grow as $v \rightarrow \infty$ are called the *principal part of \mathcal{M}* (at $i\infty$); namely, for $\kappa < 1$ these are the terms in \mathcal{M}^+ with $n < 0$ and those terms in \mathcal{M}^- with $n \geq 0$. The coefficients $c_{\mathcal{M}}^-$ are closely related to coefficients of meromorphic modular forms of weight $2 - \kappa$, following from the fact that if \mathcal{M} is modular of weight κ , then $\xi_\kappa(\mathcal{M})$ is modular of weight $2 - \kappa$. Thus ξ_κ maps weight κ polar harmonic Maass forms to weight $2 - \kappa$ meromorphic modular forms.

Solving the second-order differential equation coming from (1.2), one obtains an elliptic expansion of polar harmonic Maass forms which parallels the expansion (1.10) for meromorphic cusp forms. The resulting expansion is given in Proposition 2.2 of [6], which Pioline later pointed out appears as a special case of Theorem 1.1 of [18]. To describe it, for $0 \leq Z < 1$, $a \in \mathbb{N}$, and $b \in \mathbb{Z}$, we let

$$\beta_0(Z; a, b) := \beta(Z; a, b) - \mathcal{C}_{a,b} \quad \text{with} \quad \mathcal{C}_{a,b} := \sum_{\substack{0 \leq j \leq a-1 \\ j \neq -b}} \binom{a-1}{j} \frac{(-1)^j}{j+b}. \quad (2.14)$$

Making the change of variables $t \mapsto 1 - t$ in the integral representation and then applying the Binomial Theorem, we obtain

$$\beta_0(Z; a, b) = \sum_{\substack{0 \leq j \leq a-1 \\ j \neq -b}} \binom{a-1}{j} \frac{(-1)^{j+1}}{j+b} (1-Z)^{j+b} + \delta_{1-a \leq b \leq 0} \binom{a-1}{-b} (-1)^{b+1} \log(1-Z). \quad (2.15)$$

Here for a property S , $\delta_S = 1$ if and only if S is satisfied and $\delta_S = 0$ otherwise.

For a polar harmonic Maass form \mathcal{M} of weight $2 - 2k$ (or more generally any function \mathcal{M} which is annihilated by Δ_{2-2k}) and $\varrho \in \mathbb{H}$, there exist $c_{\mathcal{M}, \varrho}^\pm(n) \in \mathbb{C}$ such that for $r_\varrho(z) \ll_\varrho 1$

$$\mathcal{M}(z) = (z - \bar{\varrho})^{2k-2} \left(\sum_{n \gg -\infty} c_{\mathcal{M}, \varrho}^+(n) X_\varrho(z)^n + \sum_{n \ll \infty} c_{\mathcal{M}, \varrho}^-(n) \beta_0(1 - r_\varrho(z)^2; 2k-1, -n) X_\varrho(z)^n \right). \quad (2.16)$$

The *meromorphic* and the *non-meromorphic parts* of the elliptic expansion around ϱ are

$$\begin{aligned}\mathcal{M}_\varrho^+(z) &:= (z - \bar{\varrho})^{2k-2} \sum_{n \gg -\infty} c_{\mathcal{M}, \varrho}^+(n) X_\varrho(z)^n, \\ \mathcal{M}_\varrho^-(z) &:= (z - \bar{\varrho})^{2k-2} \sum_{n \ll \infty} c_{\mathcal{M}, \varrho}^-(n) \beta_0(1 - r_\varrho(z)^2; 2k-1, -n) X_\varrho(z)^n.\end{aligned}$$

We refer to the terms in (2.16) which grow as $z \rightarrow \varrho$ as the *principal part around ϱ* and denote them by $\mathbb{P}_{\mathcal{M},\varrho}$; the corresponding meromorphic and non-meromorphic parts of $\mathbb{P}_{\mathcal{M},\varrho}$ are

$$\begin{aligned}\mathbb{P}_{\mathcal{M},\varrho}^+(z) &:= (z - \bar{\varrho})^{2k-2} \sum_{n < 0} c_{\mathcal{M},\varrho}^+(n) X_{\varrho}(z)^n, \\ \mathbb{P}_{\mathcal{M},\varrho}^-(z) &:= (z - \bar{\varrho})^{2k-2} \sum_{n \geq 0} c_{\mathcal{M},\varrho}^-(n) \beta_0(1 - r_{\varrho}(z)^2; 2k-1, -n) X_{\varrho}(z)^n.\end{aligned}$$

Remark. Note that the principal parts of the Fourier expansions around all cusps and the principal parts of the elliptic expansions uniquely determine the form. Indeed, Proposition 3.5 of [11] implies that harmonic Maass forms \mathcal{M} without any singularities must satisfy $\xi_{2-2k}(\mathcal{M}) = 0$ and there are no non-trivial negative-weight holomorphic modular forms.

2.4. Differential operators. Recall the raising operator, defined in (1.12). If g has eigenvalue λ and weight κ , then $R_{\kappa}^{\ell}(g)$ ($\ell \in \mathbb{N}_0$) has weight $\kappa + 2\ell$ and eigenvalue $\lambda + \kappa\ell + \ell(\ell - 1)$. The following lemma may easily be verified by induction on ℓ .

Lemma 2.1. *For $\ell \in \mathbb{N}_0$ and $g : \mathbb{H} \rightarrow \mathbb{C}$, satisfying $\Delta_{\kappa}(g) = \lambda g$, we have*

$$R_{-\kappa-2\ell}^{\ell} \left(y^{2\ell+\kappa} \overline{R_{\kappa}^{\ell}(g(z))} \right) = y^{\kappa} \prod_{j=1}^{\ell} (-\bar{\lambda} - j(j + \kappa - 1)) \overline{g(z)}.$$

The next lemma rewrites the elliptic coefficients of a meromorphic function f in terms of the raising operator and $\eta := \text{Im}(\varrho)$. The idea of proof may be found on pages 87–89 of [13].

Lemma 2.2. *If $f : \mathbb{H} \rightarrow \mathbb{C}$ is a meromorphic function which is holomorphic in some neighborhood around $\varrho \in \mathbb{H}$ and $\kappa \in \mathbb{Z}$, then for z in this neighborhood of ϱ , we have*

$$f(z) = (2i\eta)^{\kappa} (z - \bar{\varrho})^{-\kappa} \sum_{n \geq 0} \frac{\eta^n}{n!} R_{\kappa}^n(f(\varrho)) X_{\varrho}(z)^n.$$

We also recall that raising and differentiation are related through Bol's identity ($k \in \mathbb{N}$)

$$\mathcal{D}^{2k-1} = -(4\pi)^{1-2k} R_{2-2k}^{2k-1}. \quad (2.17)$$

We note that the constant $C_{a,b}$ in (2.14) is chosen so that the operator \mathcal{D}^{2k-1} acts nicely on \mathcal{M}_{ϱ}^- . Namely, most of the terms in \mathcal{M}_{ϱ}^- are annihilated by \mathcal{D}^{2k-1} . One can use this to conclude that \mathcal{D}^{2k-1} maps polar harmonic Maass forms to meromorphic modular forms. One can also easily show that ξ_{2-2k} maps polar harmonic Maass forms to meromorphic modular forms. This operator may also be written in terms of the raising operator. To be precise, we have, for $g : \mathbb{H} \rightarrow \mathbb{C}$,

$$\xi_{\kappa} \left(y^{-\kappa} \overline{g(z)} \right) = R_{-\kappa}(g(z)). \quad (2.18)$$

2.5. Poincaré series. In this section, we recall Maass–Poincaré series [18] with singularities at the cusps, which are used as inputs of the theta lifts of Theorems 1.1 and 1.2 (1), and Petersson's meromorphic Poincaré series [36, 37], which are closely connected to $f_{\mathcal{A}}$.

To construct the Maass–Poincaré series, for $Z \in \mathbb{R} \setminus \{0\}$, let

$$\mathcal{M}_{\kappa,s}(Z) := |Z|^{-\frac{\kappa}{2}} M_{\frac{\kappa}{2}, \text{sgn}(Z), s-\frac{1}{2}}(|Z|),$$

with $M_{\mu,\nu}$ the usual M -Whittaker function. For $\mu, s \in \mathbb{C}$ with $\text{Re}(s \pm \mu) > 0$ and $Z \in \mathbb{R}^+$, we have

$$M_{\mu, s-\frac{1}{2}}(Z) = Z^s e^{\frac{Z}{2}} \frac{\Gamma(2s)}{\Gamma(s+\mu)\Gamma(s-\mu)} \int_0^1 t^{s+\mu-1} (1-t)^{s-\mu-1} e^{-Zt} dt;$$

and if $s = \mu$, then we have $M_{\mu, \mu-1/2}(Z) = e^{-\frac{Z}{2}} Z^\mu$. For $m \in \mathbb{Z} \setminus \{0\}$ and $\kappa \in \frac{1}{2}\mathbb{Z}$, the function

$$\psi_{\kappa, m, s}(\tau) := (4\pi|m|)^{\frac{\kappa}{2}} \mathcal{M}_{\kappa, s}(4\pi m v) e^{2\pi i m v}$$

is then an eigenfunction of Δ_κ with eigenvalue $(s - \kappa/2)(1 - s - \kappa/2)$. Denoting by $|_\kappa \text{pr}$ the identity for $\kappa \in \mathbb{Z}$ and Kohnen's projection operator (see p. 250 of [29]) for $\kappa \notin \mathbb{Z}$, one concludes that for $\sigma := \text{Re}(s) > 1$, the following Poincaré series are also eigenfunctions of Δ_κ and have weight κ :

$$P_{\kappa, m, s} := \sum_{\gamma \in \Gamma_\infty \setminus \Gamma_0(4)} \psi_{\kappa, \text{sgn}(\kappa)m, s} \Big|_\kappa \gamma \Big|_\kappa \text{pr}. \quad (2.19)$$

For $s = 1 - \kappa/2$ or $s = \kappa/2$, the functions $P_{\kappa, m, s}$ are harmonic. The Poincaré series satisfy the growth condition

$$P_{\kappa, m, s}(\tau) - \psi_{\kappa, m, s}(\tau) \Big|_\kappa \text{pr} = O\left(v^{1 - \text{Re}(s) - \frac{\kappa}{2}}\right).$$

In the special case that $\kappa = k + 1/2 > 1$ and $s = \kappa/2$, we normalize the resulting weakly holomorphic Poincaré series as

$$P_{k+\frac{1}{2}, m} := \frac{6(4\pi)^{\frac{k}{2} - \frac{1}{4}}}{(k-1)!|m|^{\frac{1}{4}}} P_{k+\frac{1}{2}, m, \frac{k}{2} + \frac{1}{4}}. \quad (2.20)$$

For $\kappa = 3/2 - k < 1$, we set

$$\mathcal{P}_{\frac{3}{2}-k, m} := -\frac{6(4\pi)^{\frac{k}{2} - \frac{1}{4}}}{(k-1)!|m|^{\frac{1}{4}}(2k-1)} P_{\frac{3}{2}-k, m, \frac{k}{2} + \frac{1}{4}}. \quad (2.21)$$

We next turn to Poincaré series with singularities in the upper half plane, defining, for $n \in \mathbb{Z}$,

$$\Psi_{\kappa, n}(z, \mathfrak{z}) := \sum_{\gamma \in \text{SL}_2(\mathbb{Z})} \left((z - \bar{\mathfrak{z}})^{-\kappa} X_{\mathfrak{z}}(z)^n \right) \Big|_{\kappa, z} \gamma. \quad (2.22)$$

For $\kappa > 2$ even, $z \mapsto \Psi_{\kappa, n}(z, \mathfrak{z}) \in \mathbb{S}_\kappa$ and $\mathfrak{z} \mapsto \mathfrak{y}^{-\kappa-n} \Psi_{\kappa, n}(z, \mathfrak{z})$ are modular of weight $-2n - \kappa$ (see page 72 of [37]). Moreover, the functions $z \mapsto \Psi_{\kappa, n}(z, \mathfrak{z}) \in \mathbb{S}_\kappa$ vanish identically if $n \not\equiv -\kappa/2 \pmod{\omega_{\mathfrak{z}}}$ and are cusp forms in z if $n \in \mathbb{N}_0$. Furthermore, the set $\{\Psi_{\kappa, n}(z, \mathfrak{z}) : \mathfrak{z} \in \mathbb{H}, n \in \mathbb{Z}\}$ spans \mathbb{S}_κ (see Sätze 7 and 9 of [38]). The principal part of $z \mapsto \Psi_{\kappa, m}(z, \mathfrak{z})$ has a simple shape. To be more precise, writing $c(n) := c_{f, \varrho}(n) - \delta_{n=m} \delta_{\mathfrak{z}=\varrho}$ (throughout we consider \mathfrak{z} and ϱ as elements in $\text{SL}_2(\mathbb{Z}) \setminus \mathbb{H}$) for $f(z) = (2\omega_{\mathfrak{z}})^{-1} \Psi_{\kappa, m}(z, \mathfrak{z})$, Satz 7 of [38] implies that

$$(2\omega_{\mathfrak{z}})^{-1} \Psi_{\kappa, m}(z, \mathfrak{z}) = (z - \bar{\varrho})^{-\kappa} \left(\delta_{\mathfrak{z}=\varrho} X_{\varrho}(z)^m + \sum_{n \geq 0} c(n) X_{\varrho}(z)^n \right). \quad (2.23)$$

Moreover, $f_{\mathcal{A}}$ is a specialization of $\Psi_{2k, -k}$, as given in the following straightforward lemma.

Lemma 2.3. *With $Q_0 \in \mathcal{A} \in \mathcal{Q}_{-D}/\text{SL}_2(\mathbb{Z})$, we have*

$$f_{\mathcal{A}}(z) = \frac{(2v_{Q_0})^k}{2\omega_{\tau_{Q_0}}} \Psi_{2k, -k}(z, \tau_{Q_0}). \quad (2.24)$$

2.6. Higher Green's functions. For $z, \mathfrak{z} \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\sigma > 1$, the *automorphic Green's function* G_s on $\text{SL}_2(\mathbb{Z}) \setminus \mathbb{H}$ is given by

$$G_s(z, \mathfrak{z}) := \sum_{\gamma \in \text{SL}_2(\mathbb{Z})} g_s^{\mathbb{H}}(z, \gamma \mathfrak{z}),$$

where

$$g_s^{\mathbb{H}}(z, \mathfrak{z}) := -\frac{2^{s-1} \Gamma(s)^2}{\Gamma(2s)} \cosh(d(z, \mathfrak{z}))^{-s} {}_2F_1\left(\frac{s}{2}, \frac{s+1}{2}; s + \frac{1}{2}; \frac{1}{\cosh(d(z, \mathfrak{z}))^2}\right).$$

Note that we have the equality $g_s^{\mathbb{H}}(z, \mathfrak{z}) = -Q_{s-1}(\cosh(d(z, \mathfrak{z})))$ with Q_ν the associated Legendre function of the second kind. Further note that there are different normalizations of G_s in the literature; our normalization agrees with that in [33]. Automorphic Green's functions can be defined for arbitrary Fuchsian groups of the first kind, and hence in particular for any congruence groups, and arise as the resolvent kernel for the hyperbolic Laplacian (see, e.g. [18, 24]).

In the case $s = k \in \mathbb{N}_{>1}$, the function $G_k : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{C}$ is called a *higher Green's function*. It is uniquely characterized by the following properties:

- (1) The function G_k is smooth and real-valued on $\mathbb{H} \times \mathbb{H} \setminus \{(z, \gamma z) : \gamma \in \mathrm{SL}_2(\mathbb{Z}), z \in \mathbb{H}\}$.
- (2) For $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{Z})$, we have $G_k(\gamma_1 z, \gamma_2 \mathfrak{z}) = G_k(z, \mathfrak{z})$.
- (3) We have

$$\Delta_{0,z}(G_k(z, \mathfrak{z})) = \Delta_{0,\mathfrak{z}}(G_k(z, \mathfrak{z})) = k(1-k)G_k(z, \mathfrak{z}).$$

- (4) As $z \rightarrow \mathfrak{z}$, we have

$$G_k(z, \mathfrak{z}) = 2\omega_{\mathfrak{z}} \log(r_{\mathfrak{z}}(z)) + O(1).$$

- (5) As z approaches a cusp, $G_k(z, \mathfrak{z}) \rightarrow 0$.

These higher Green's functions have a long history (cf. [18, 22, 24, 33]). For example, Gross and Zagier [22] conjectured that their evaluations at CM-points are essentially logarithms of algebraic numbers. If $S_{2k}(\Gamma) = \{0\}$, with $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ of finite index, then the conjecture reads as

$$G_k(z, \mathfrak{z}) = (D_1 D_2)^{\frac{1-k}{2}} \log(\alpha)$$

for CM-points z, \mathfrak{z} of discriminants D_1 and D_2 respectively and some algebraic number α . Various cases of this conjecture have been solved. For example, Mellit [33] proved the case $k = 2$, $\mathfrak{z} = i$, for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, and also interpreted α as an intersection number of certain algebraic cycles. Further cases were then investigated by Viazovska [41].

3. REGULARIZED PETERSSON INNER PRODUCTS AND THE PROOF OF THEOREM 1.3

3.1. Known regularized inner products. The classical Petersson inner product of two weight $\kappa \in \frac{1}{2}\mathbb{Z}$ (holomorphic) modular forms f and g on $\Gamma_0(N)$ (f, g a cusp form) is given by

$$\langle f, g \rangle := \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]} \int_{\Gamma_0(N) \backslash \mathbb{H}} f(z) \overline{g(z)} y^\kappa \frac{dx dy}{y^2}. \quad (3.1)$$

Although (3.1) generally diverges for meromorphic modular forms, one may still define a regularized inner product in some cases. The first to do so appears to be Petersson [38]. If all of the poles of f and g are at the cusps, the regularization of Petersson was later rediscovered and extended by Harvey-Moore [23] and Borchers [4] and subsequently used by Bruinier [10] and others to obtain a regularized integral which exists in many cases. Indeed, Petersson gave explicit necessary and sufficient conditions for existence of his regularized inner product in Satz 1a of [38]. In particular, for $f, g \in \mathbb{S}_\kappa$, his regularization exists if and only if, for every $n < 0$ and $\varrho \in \mathbb{H}$,

$$c_{f,\varrho}(n) c_{g,\varrho}(n) = 0; \quad (3.2)$$

the conditions are similar if f and g have singularities at the cusps. In this paper, we use this regularization to take theta lifts of functions with singularities.

To give a full definition, we require some notation. We let F_T be the restriction of the standard fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ to those z with $y \leq T$ and let $F_T(N) := \bigcup_{\gamma \in \Gamma_0(N) \backslash \mathrm{SL}_2(\mathbb{Z})} \gamma F_T$. Then, for functions f and g of weight $\kappa \in \frac{1}{2}\mathbb{Z}$ for $\Gamma_0(N)$, we define (if it exists)

$$\langle f, g \rangle := \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]} \lim_{T \rightarrow \infty} \int_{F_T(N)} f(z) \overline{g(z)} y^\kappa \frac{dx dy}{y^2}. \quad (3.3)$$

The definition above may be interpreted as cutting out neighborhoods around cusps and letting the hyperbolic volume of the neighborhood shrink to zero. If poles occur in \mathbb{H} , the construction in [38] is similar. For $f, g \in \mathbb{S}_{2k}$ with poles at $\mathfrak{z}_1, \dots, \mathfrak{z}_r \in \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$, we choose a fundamental domain F^* such that the representatives of $\mathfrak{z}_1, \dots, \mathfrak{z}_r$ in F^* (also denoted by \mathfrak{z}_ℓ) all lie in the interior of $\Gamma_{\mathfrak{z}_\ell} F^*$. We then define Petersson's regularized inner product as $(\mathcal{B}_\varepsilon(\mathfrak{z}) := \{z \in \mathbb{H} : r_{\mathfrak{z}}(z) < \varepsilon\})$

$$\langle f, g \rangle := \lim_{\varepsilon_1, \dots, \varepsilon_r \rightarrow 0^+} \int_{F^* \setminus \bigcup_{\ell=1}^r \mathcal{B}_{\varepsilon_\ell}(\mathfrak{z}_\ell)} f(z) \overline{g(z)} y^{2k} \frac{dx dy}{y^2}. \quad (3.4)$$

Like (3.3), the regularized inner product (3.4) does not always exist. The inner product (3.3) has recently been further extended to an inner product on all weakly holomorphic modular forms by Diamantis, Ehlen, and the first author in [5], and we address the extension of (3.4) to all meromorphic cusp forms in the next section.

3.2. A new regularization. In this section we restrict to $\kappa = 2k \in 2\mathbb{Z}$ and $N = 1$, but the construction can easily be generalized to subgroups. We also assume that f, g decay like cusp forms towards the cusps, which is however no restriction by combining with the regularization in Subsection 3.1. We choose a fundamental domain F^* as in Subection 3.1 and denote the poles of f and g in F^* by $\mathfrak{z}_1, \dots, \mathfrak{z}_r$.

For an analytic function $A(s)$ in $s = (s_1, \dots, s_r)$, denote by $\mathrm{CT}_{s=0} A(s)$ the constant term of the meromorphic continuation of $A(s)$ around $s_1 = \dots = s_r = 0$ and define

$$\langle f, g \rangle := \mathrm{CT}_{s=0} \left(\int_{\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}} f(z) H_s(z) \overline{g(z)} y^{2k} \frac{dx dy}{y^2} \right) \quad (3.5)$$

where

$$H_s(z) = H_{s_1, \dots, s_r, \mathfrak{z}_1, \dots, \mathfrak{z}_r}(z) := \prod_{\ell=1}^r h_{s_\ell, \mathfrak{z}_\ell}(z).$$

Here, for $\mathfrak{z}_\ell \in F^*$ and $z \in \mathbb{H}$, $h_{s_\ell, \mathfrak{z}_\ell}(z) := r_{\mathfrak{z}_\ell}(\gamma z)^{2s_\ell}$, with $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ chosen such that $\gamma z \in F^*$. Note that, as $z \rightarrow \gamma^{-1} \mathfrak{z}_\ell$, we have $r_{\mathfrak{z}_\ell}(\gamma z) \rightarrow 0$, so the integral in (3.5) converges for $\sigma \gg 0$, where this notation means that for every $1 \leq \ell \leq r$, $\sigma_\ell := \mathrm{Re}(s_\ell) \gg 0$. One can show that the regularization is independent of the choice of fundamental domain.

Proof of Theorem 1.3. For $\delta > 0$ sufficiently small, we may assume that the $\mathcal{B}_\delta(\mathfrak{z}_\ell)$ are disjoint and split off the integral over those z that lie in one of these balls.

If $z \notin \mathcal{B}_\delta(\mathfrak{z}_\ell)$ for all ℓ , then one can bound the integrand locally uniformly for $\sigma > -\varepsilon$ ($\varepsilon > 0$). Hence we conclude that

$$\mathrm{CT}_{s=0} \left(\int_{F^* \setminus \bigcup_{\ell=1}^r \mathcal{B}_\delta(\mathfrak{z}_\ell)} f(z) H_s(z) \overline{g(z)} y^{2k} \frac{dx dy}{y^2} \right) = \int_{F^* \setminus \bigcup_{\ell=1}^r \mathcal{B}_\delta(\mathfrak{z}_\ell)} f(z) \overline{g(z)} y^{2k} \frac{dx dy}{y^2}. \quad (3.6)$$

Thus we are left to show existence of the meromorphic continuation to those $s \in \mathbb{C}$ with $\sigma \leq 0$ of

$$\int_{\mathcal{B}_\delta(\mathfrak{z}_\ell) \cap F^*} f(z) H_s(z) \overline{g(z)} y^{2k} \frac{dx dy}{y^2}. \quad (3.7)$$

By construction, \mathfrak{z}_ℓ lies in the interior of $\Gamma_{\mathfrak{z}_\ell} F^*$, so we may assume that $\mathcal{B}_\delta(\mathfrak{z}_\ell) \subseteq \Gamma_{\mathfrak{z}_\ell} F^*$. To rewrite (3.7) as an integral over the entire ball $\mathcal{B}_\delta(\mathfrak{z}_\ell)$, we decompose

$$\mathcal{B}_\delta(\mathfrak{z}_\ell) = \dot{\bigcup}_{\gamma \in \Gamma_{\mathfrak{z}_\ell}} \gamma(\mathcal{B}_\delta(\mathfrak{z}_\ell) \cap F^*). \quad (3.8)$$

Moreover, bounding $h_{s_m, \delta m}(z)$ locally uniformly for $\sigma_m > -\varepsilon$ for $m \neq \ell$, we may plug in $s_m = 0$, and hence invariance of the integrand under $\mathrm{SL}_2(\mathbb{Z})$ implies that the constant term at $s = 0$ of (3.7) is the constant term at $s_\ell = 0$ of

$$\mathcal{I}_{s_\ell, \delta \ell, \delta}(f, g) := \frac{1}{\omega_{\delta \ell}} \int_{\mathcal{B}_\delta(\delta \ell)} f(z) h_{s_\ell, \delta \ell}(z) \overline{g(z)} y^{2k} \frac{dx dy}{y^2}.$$

Setting $R := r_{\delta \ell}(z)$ for $z \in \mathcal{B}_\delta(\delta \ell)$, we have for $\gamma \in \Gamma_{\delta \ell}$ the equality

$$h_{s_\ell, \delta \ell}(z) = r_{\delta \ell}(\gamma z)^{2s_\ell} = R^{2s_\ell}. \quad (3.9)$$

We now closely follow the proof of Satz 1a in [38]. Changing variables $X_{\delta \ell}(z) = R e^{i\vartheta}$ and inserting the elliptic expansions (1.10) of f, g around $\varrho = \delta \ell$ with $\delta \ell = \mathfrak{x}_\ell + i y_\ell$ yields

$$\begin{aligned} \mathcal{I}_{s_\ell, \delta \ell, \delta}(f, g) &= \frac{1}{\omega_{\delta \ell}} \int_{\mathcal{B}_\delta(\delta \ell)} f(z) \overline{g(z)} r_{\delta \ell}(z)^{2s_\ell} y^{2k} \frac{dx dy}{y^2} \\ &= \frac{4}{\omega_{\delta \ell} (4y_\ell)^{2k}} \sum_{m, n \gg -\infty} c_{f, \delta \ell}(n) \overline{c_{g, \delta \ell}(m)} \int_0^\delta \int_0^{2\pi} e^{i(n-m)\vartheta} R^{n+m+2s_\ell} (1-R^2)^{2k-2} R d\vartheta dR \\ &= \frac{8\pi}{\omega_{\delta \ell} (4y_\ell)^{2k}} \sum_{n \gg -\infty} c_{f, \delta \ell}(n) \overline{c_{g, \delta \ell}(n)} \int_0^\delta R^{1+2n+2s_\ell} (1-R^2)^{2k-2} dR. \end{aligned} \quad (3.10)$$

Plugging in the binomial expansion of $(1-R^2)^{2k-2}$, the remaining integral in (3.10) becomes

$$\sum_{j=0}^{2k-2} (-1)^j \binom{2k-2}{j} \int_0^\delta R^{1+2(n+j)+2s_\ell} dR = \sum_{j=0}^{2k-2} (-1)^j \binom{2k-2}{j} \frac{\delta^{2(n+j+1+s_\ell)}}{2(n+j+1+s_\ell)}.$$

Since this is meromorphic, its constant term exists, yielding existence of the inner product.

We next prove that the inner product is Hermitian. For $f, g \in \mathbb{S}_{2k}$, let $F_{f,g}$ denote the meromorphic continuation of the function defined for $s \in \mathbb{C}^r$ with $\sigma_\ell \gg 0$ by

$$F_{f,g}(s) := \int_{F^*} f(z) \overline{g(z)} H_s(z) y^{2k} \frac{dx dy}{y^2}.$$

Since $\langle f, g \rangle$ always exists, $F_{f,g}$ has an expansion around $s = 0$ of the shape

$$F_{f,g}(s) = \sum_{n=(n_1, \dots, n_r) \in \mathbb{Z}^r} a_{f,g}(n) s_1^{n_1} \cdots s_r^{n_r}$$

with $\langle f, g \rangle = a_{f,g}(0)$. Since $r_{\delta \ell}(z) \in \mathbb{R}$, we have $\overline{H_{\overline{s}}(z)} = H_s(z)$, and thus

$$\overline{\langle g, f \rangle} = \overline{a_{g,f}(0)} = \mathrm{CT}_{s=0}(\overline{F_{g,f}(\overline{s})}) = \mathrm{CT}_{s=0} \left(\int_{F^*} f(z) \overline{g(z)} \overline{H_{\overline{s}}(z)} y^{2k} \frac{dx dy}{y^2} \right) = \langle f, g \rangle.$$

We finally show that the new regularization agrees with Petersson's. Setting $\mathcal{B}(\delta \ell, \varepsilon, \delta) := \{z \in \mathbb{H} : \varepsilon < r_{\delta \ell}(z) < \delta\}$, Petersson's regularization equals

$$\begin{aligned} &\lim_{\varepsilon_1, \dots, \varepsilon_r \rightarrow 0^+} \int_{F^* \setminus \bigcup_{\ell=1}^r \mathcal{B}_{\varepsilon_\ell}(\delta \ell)} f(z) \overline{g(z)} y^{2k} \frac{dx dy}{y^2} \\ &= \int_{F^* \setminus \bigcup_{\ell=1}^r \mathcal{B}_\delta(\delta \ell)} f(z) \overline{g(z)} y^{2k} \frac{dx dy}{y^2} + \lim_{\varepsilon_1, \dots, \varepsilon_r \rightarrow 0^+} \int_{F^* \cap \bigcup_{\ell=1}^r \mathcal{B}(\delta \ell, \varepsilon_\ell, \delta)} f(z) \overline{g(z)} y^{2k} \frac{dx dy}{y^2}. \end{aligned} \quad (3.11)$$

The first term on the right-hand side of (3.11) is precisely the right-hand side of (3.6). It thus remains to prove that

$$\lim_{\varepsilon_\ell \rightarrow 0^+} \int_{F^* \cap \mathcal{B}(\delta \ell, \varepsilon_\ell, \delta)} f(z) \overline{g(z)} y^{2k} \frac{dx dy}{y^2} = \mathrm{CT}_{s_\ell=0} \mathcal{I}_{s_\ell, \delta \ell, \delta}(f, g). \quad (3.12)$$

By the existence condition (3.2), we may plug in $s_\ell = 0$ in (3.10), to obtain that

$$\text{CT}_{s_\ell=0} \mathcal{I}_{s_\ell, \mathfrak{z}_\ell, \delta}(f, g) = \frac{8\pi}{\omega_{\mathfrak{z}_\ell} (4y_\ell)^{2k}} \sum_{n \geq 0} c_{f, \mathfrak{z}_\ell}(n) \overline{c_{g, \mathfrak{z}_\ell}(n)} \int_0^\delta R^{1+2n} (1 - R^2)^{2k-2} dR.$$

Using (3.8) and then following the calculation in (3.10), plugging the elliptic expansion into the left-hand side of (3.12) yields that the two regularizations match since

$$\lim_{\varepsilon_\ell \rightarrow 0^+} \int_{\mathcal{B}(\mathfrak{z}_\ell, \varepsilon_\ell, \delta)} f(z) \overline{g(z)} y^{2k} \frac{dx dy}{y^2} = \frac{8\pi}{(4y_\ell)^{2k}} \sum_{n \geq 0} c_{f, \mathfrak{z}_\ell}(n) \overline{c_{g, \mathfrak{z}_\ell}(n)} \int_0^\delta R^{1+2n} (1 - R^2)^{2k-2} dR.$$

□

4. THETA LIFTS AND THE PROOFS OF THEOREMS 1.1 AND 1.2 (1)

4.1. Proof of Theorem 1.1. Before proving Theorem 1.1, we note a (well-known) fact about the theta kernel Θ_k .

Lemma 4.1. *The function $\tau \mapsto \Theta_k(z, \tau)$ grows at most polynomially towards the cusps and decays exponentially for $\tau \rightarrow i\infty$.*

Proof. To show that it exponentially decays towards $i\infty$, we use (2.3) to rewrite the absolute value of the exponential in definition (1.3) as $e^{-2\pi v(Q_z^2 + |Q(z, 1)|^2/y^2)}$ and then note that $Q_z^2 + |Q(z, 1)|^2/y^2 > 0$ for $Q \neq [0, 0, 0]$. Modularity then implies the claim at the other cusps. □

We specifically apply the theta lift Φ_k to the weight $k + 1/2$ weakly holomorphic Poincaré series $P_{k+1/2, -D}$ defined in (2.20) in order to obtain Theorem 1.1.

Proof of Theorem 1.1. A standard unfolding argument [43] gives that $\Phi_k(P_{k+1/2, -D, k/2+1/4})$ equals

$$\frac{(4\pi D)^{\frac{k}{2} + \frac{1}{4}}}{6} \lim_{T \rightarrow \infty} \left(\int_0^T \int_0^1 e^{-2\pi i D \tau} \overline{\Theta_k(z, \tau)} v^{k+\frac{1}{2}} \frac{du dv}{v^2} - \sum_{c \geq 1} \sum_{a \pmod{c}^*} \int_{S_{a/c}} e^{-2\pi i D \tau} \overline{\Theta_k(z, \tau)} v^{k+\frac{1}{2}} \frac{du dv}{v^2} \right), \quad (4.1)$$

where a runs over $a \pmod{c}$ which are coprime to c and $S_{a/c}$ are the discs of radius $(2c^2 T)^{-1}$ tangent to the real axis at a/c . The polynomial growth of $\tau \mapsto \Theta_k(z, \tau)$ towards the cusps, yields that the second term of (4.1) does not contribute. To evaluate the integral in the first term of (4.1), we plug in definition (1.3) and integrate over u , to obtain, as $T \rightarrow \infty$,

$$\frac{(4\pi D)^{\frac{k}{2} + \frac{1}{4}}}{6} y^{-2k} \sum_{Q \in \mathcal{Q}_{-D}} Q(\bar{z}, 1)^k \int_0^\infty e^{4\pi D v - 4\pi Q_z^2 v} v^{k-1} dv.$$

The claim now easily follows using (2.3) to show that the integral on v equals

$$\int_0^\infty e^{-\frac{4\pi |Q(z, 1)|^2 v}{y^2}} v^{k-1} dv = \frac{(k-1)!}{(4\pi)^k} y^{2k} |Q(z, 1)|^{-2k}.$$

□

4.2. Proof of Theorem 1.2 (1). The goal of this section is to compute the image of the Maass–Poincaré series $P_{3/2-k, -D, s}$, defined in (2.19), under Φ_{1-k}^* and to connect these to the functions $\mathcal{F}_{\mathcal{A}}$. We do so in the following theorem, which extends Theorem 1.2 (1).

Theorem 4.2. For $s \in \mathbb{C}$ with $\sigma > 1$, we have

$$\begin{aligned} & \Phi_{1-k}^* \left(P_{\frac{3}{2}-k, -D, s} \right) (z) \\ &= \frac{D^s \Gamma \left(s + \frac{k}{2} - \frac{1}{4} \right)}{6(4\pi)^{\frac{k}{2} - \frac{1}{4}}} \sum_{Q \in \mathcal{Q}_{-D}} Q_z^{-2s-k+\frac{3}{2}} Q(z, 1)^{k-1} {}_2F_1 \left(s + \frac{k}{2} - \frac{1}{4}, s + \frac{k}{2} - \frac{3}{4}; 2s; \frac{D}{Q_z^2} \right). \end{aligned} \quad (4.2)$$

In particular

$$\Phi_{1-k}^* \left(\mathcal{P}_{\frac{3}{2}-k, -D} \right) = \mathcal{F}_{1-k, -D}.$$

Remark. After seeing a preliminary version of this paper, Zemel [44] has obtained further theta lifts related to vector-valued versions of $\mathcal{F}_{\mathcal{A}}$.

Proof of Theorem 4.2. One can show that both sides of (4.2) converge absolutely and locally uniformly for $\sigma > 1$ and $z \neq \tau_Q$ (for every $Q \in \mathcal{A}$) and hence are analytic in s . It thus suffices to show (4.2) for $\sigma > k - 3/2$. Following the proof of Theorem 1.1, we obtain

$$\Phi_{1-k}^* \left(P_{\frac{3}{2}-k, -D, s} \right) (z) = \frac{(4\pi D)^{\frac{1}{4} - \frac{k}{2}}}{6} \sum_{Q \in \mathcal{Q}_{-D}} Q_z Q(z, 1)^{k-1} \mathcal{I}_s \left(\frac{Dy^2}{|Q(z, 1)|^2} \right), \quad (4.3)$$

where

$$\mathcal{I}_s(Z) := \int_0^\infty \mathcal{M}_{\frac{3}{2}-k, s}(v) v^{-\frac{1}{2}} e^{-\frac{v}{2}} e^{-\frac{v}{Z}} dv \quad (Z \in \mathbb{R}^+).$$

Applying formula 7.621.1. of [20] yields

$$\mathcal{I}_s(Z) = \Gamma \left(s + \frac{k}{2} - \frac{1}{4} \right) \left(\frac{Z}{Z+1} \right)^{s + \frac{k}{2} - \frac{1}{4}} {}_2F_1 \left(s + \frac{k}{2} - \frac{1}{4}, s + \frac{k}{2} - \frac{3}{4}; 2s; \frac{Z}{Z+1} \right).$$

Equation (4.2) then follows directly by plugging back into (4.3) and using (2.3) to prove that

$$\frac{Z}{Z+1} = \frac{D}{Q_z^2} \quad (4.4)$$

for $Z := Dy^2/|Q(z, 1)|^2$.

To prove the second claim, we use (2.9), (2.7), and (2.11) to obtain

$$\begin{aligned} {}_2F_1 \left(k, k - \frac{1}{2}; k + \frac{1}{2}; W \right) &= (1 - W)^{\frac{1}{2}-k} {}_2F_1 \left(\frac{1}{2}, k - \frac{1}{2}; k + \frac{1}{2}; \frac{W}{W-1} \right) \\ &= (-1)^{k-\frac{1}{2}} \left(k - \frac{1}{2} \right) W^{-k+\frac{1}{2}} \beta \left(\frac{W}{W-1}; k - \frac{1}{2}, \frac{1}{2} \right). \end{aligned} \quad (4.5)$$

Employing (4.5) with $W := D/Q_z^2$ and using (4.4) to evaluate $W/(W-1) = -Z = -Dy^2/|Q(z, 1)|^2$, we obtain the second claim, using the fact that $\text{sgn}(Q_z) = 1$. \square

5. PROPERTIES OF THE FUNCTION $\mathcal{F}_{\mathcal{A}}$ AND PROOF OF THEOREM 1.2 (2)

5.1. Relation to Green's functions. The goal of this section is to write the functions $\mathcal{F}_{\mathcal{A}}$ defined in (1.5) in terms of higher Green's functions. For this, set

$$a_{k,n} := -\frac{2^{k-1} \Gamma \left(k - \frac{1}{2} \right)}{\sqrt{\pi}} \begin{cases} 1 & \text{if } n \geq k - 1, \\ \frac{n!}{(2k-2-n)!} & \text{if } n < k - 1. \end{cases} \quad (5.1)$$

Lemma 5.1. For $Q \in \mathcal{Q}_{-D}$ and $n \in \mathbb{N}_0$, we have

$$R_{2-2k}^n(\mathbb{P}_{1-k,-D,Q}(z)) = a_{k,n} \begin{cases} R_0^{n+1-k}(g_k^{\mathbb{H}}(z, \tau_Q)) & \text{if } n \geq k-1, \\ y^{2k-2-2n} \overline{R_0^{k-1-n}(g_k^{\mathbb{H}}(z, \tau_Q))} & \text{if } n < k-1. \end{cases}$$

Proof. We first prove that raising on the higher Green's function evaluates as $(z, \mathfrak{z} \in \mathbb{H}, j \in \mathbb{N}_0)$

$$\begin{aligned} & R_{0,z}^j(g_k^{\mathbb{H}}(z, \mathfrak{z})) \\ &= \frac{-2^{k-j-1}(k-1)!(k+j-1)!(\bar{z}-\mathfrak{z})^j(\bar{z}-\bar{\mathfrak{z}})^j}{(2k-1)!y^{2j}\bar{y}^j \cosh(d(z, \mathfrak{z}))^{k+j}} {}_2F_1\left(\frac{k+j+1}{2}, \frac{k+j}{2}; k+\frac{1}{2}; \frac{1}{\cosh(d(z, \mathfrak{z}))^2}\right). \end{aligned} \quad (5.2)$$

We note that the images under repeated raising of the Green's function are known in the literature (see, e.g., [33]), but some rewriting is still required to derive the form (5.2). So, for the convenience of the reader we present a direct proof. To show (5.2), we first compute

$$R_{0,z}(\cosh(d(z, \mathfrak{z}))) = -\frac{(\bar{z}-\mathfrak{z})(\bar{z}-\bar{\mathfrak{z}})}{2y^2\bar{y}}, \quad (5.3)$$

from which we conclude that $R_{0,z}^2(\cosh(d(z, \mathfrak{z}))) = 0$. Employing these identities, we prove, by induction on $j \in \mathbb{N}_0$, that

$$\begin{aligned} & R_{0,z}^j(g_k^{\mathbb{H}}(z, \mathfrak{z})) \\ &= -\frac{2^{k-1}(k-1)!^2}{(2k-1)!} (R_{0,z}(\cosh(d(z, \mathfrak{z}))))^j \frac{\partial^j}{\partial Z^j} \left[Z^{-k} {}_2F_1\left(\frac{k}{2}, \frac{k+1}{2}; k+\frac{1}{2}; \frac{1}{Z^2}\right) \right]_{Z=\cosh(d(z, \mathfrak{z}))}. \end{aligned} \quad (5.4)$$

Next, again by induction on $j \in \mathbb{N}_0$, and employing (2.10), we obtain

$$\frac{\partial^j}{\partial Z^j} \left(Z^{-k} {}_2F_1\left(\frac{k}{2}, \frac{k+1}{2}; k+\frac{1}{2}; \frac{1}{Z^2}\right) \right) = \frac{(-1)^j(k+j-1)!}{(k-1)!Z^{k+j}} {}_2F_1\left(\frac{k+j+1}{2}, \frac{k+j}{2}; k+\frac{1}{2}; \frac{1}{Z^2}\right).$$

Plugging this and (5.3) into (5.4) gives (5.2).

Using (5.2), we next show the $n=0$ case of the assertion of this lemma, namely

$$\mathbb{P}_{1-k,-D,Q}(z) = -\frac{1}{2^{k-1}(k-1)!} y^{2k-2} \overline{R_0^{k-1}(g_k^{\mathbb{H}}(z, \tau_Q))}. \quad (5.5)$$

For this we let $j = k-1$ in (5.2), which yields that

$$R_{0,z}^{k-1}(g_k^{\mathbb{H}}(z, \mathfrak{z})) = -\frac{(k-1)!(\bar{z}-\mathfrak{z})^{k-1}(\bar{z}-\bar{\mathfrak{z}})^{k-1}}{(2k-1)y^{2k-2}\bar{y}^{k-1} \cosh(d(z, \mathfrak{z}))^{2k-1}} {}_2F_1\left(k, k-\frac{1}{2}; k+\frac{1}{2}; \frac{1}{\cosh(d(z, \mathfrak{z}))^2}\right). \quad (5.6)$$

From now on we choose $\mathfrak{z} = \tau_Q$. Hence, employing (4.5), (2.5), and (2.2), (5.6) becomes

$$R_0^{k-1}(g_k^{\mathbb{H}}(z, \tau_Q)) = i(-1)^k 2^{k-2}(k-1)! D^{\frac{1-k}{2}} y^{2-2k} \overline{Q(z, 1)}^{k-1} \beta\left(-\frac{Dy^2}{|Q(z, 1)|^2}; k-\frac{1}{2}, \frac{1}{2}\right).$$

Since Q is positive-definite, so that $\text{sgn}(Q_z) = 1$, this gives (5.5).

To finish the proof, we apply raising n times to (5.5), yielding

$$R_{2-2k}^n(\mathbb{P}_{1-k,-D,Q}(z)) = -\frac{1}{2^{k-1}(k-1)!} R_{2-2k}^n\left(y^{2k-2} \overline{R_0^{k-1}(g_k^{\mathbb{H}}(z, \tau_Q))}\right). \quad (5.7)$$

We now distinguish two cases depending on whether $n \geq k-1$ or $n < k-1$.

If $n \geq k-1$, we apply Lemma 2.1 with $\ell = k-1$ to (5.7) and note that $g_k^{\mathbb{H}}$ is real-valued. Using the duplication formula for the gamma function then yields the claim for $n \geq k-1$.

For $n < k - 1$ the eigenvalue of $R_0^{k-1-n}(g_k^{\mathbb{H}})$ is $(n+1)(n+2-2k)$. Thus, by Lemma 2.1 with $\ell = n$,

$$R_{2-2k}^n \left(y^{2k-2} \overline{R_{2k-2-2n}^n \left(R_0^{k-1-n}(g_k^{\mathbb{H}}(z, \tau_Q)) \right)} \right) = y^{2k-2-2n} \frac{n!(2k-2)!}{(2k-2-n)!} \overline{R_0^{k-1-n}(g_k^{\mathbb{H}}(z, \tau_Q))}.$$

Plugging back into (5.7) yields the claim, again using the duplication formula. \square

We next require regularized versions of G_k and $\mathcal{F}_{\mathcal{A}}$. For this, define

$$\sum_{w \in S}^{\text{reg}} h(w) := \sum_{\substack{w \in S \\ h(w) \neq \infty}} h(w),$$

where h is an arbitrary function taking inputs from some set S . For any operator \mathcal{O} we then let

$$\mathcal{O} \left(\sum_{w \in S}^{\text{reg}} h(w) \right) := \sum_{w \in S}^{\text{reg}} \mathcal{O}(h(w)).$$

Moreover, for $\mathcal{H}(z) := \sum_{w \in S} h_z(w)$, we set

$$\mathcal{H}^{\text{reg}}(z) := \sum_{w \in S}^{\text{reg}} h_z(w).$$

Note that the regularization $\mathcal{H}^{\text{reg}}(z)$ depends on the choice of h_z , so we emphasize that the regularization uses the choice of h_z given in the definition of \mathcal{H} .

We obtain the following corollary by applying Lemma 5.1 termwise.

Corollary 5.2. *For $Q_0 \in \mathcal{A} \in \mathcal{Q}_{-D}/\text{SL}_2(\mathbb{Z})$ and $n \in \mathbb{N}_0$, we have*

$$R_{2-2k}^n(\mathcal{F}_{\mathcal{A}}^{\text{reg}}(z)) = \frac{a_{k,n}}{2\omega_{\tau_{Q_0}}} \begin{cases} R_0^{n+1-k}(G_k^{\text{reg}}(z, \tau_{Q_0})) & \text{if } n \geq k-1, \\ y^{2k-2-2n} \overline{R_0^{k-1-n}(G_k^{\text{reg}}(z, \tau_{Q_0}))} & \text{if } n < k-1, \end{cases}$$

where $a_{k,n}$ is defined in (5.1). In particular

$$\mathcal{F}_{\mathcal{A}}^{\text{reg}}(z) = -\frac{1}{2^k(k-1)!\omega_{\tau_{Q_0}}} y^{2k-2} \overline{R_0^{k-1}(G_k^{\text{reg}}(z, \tau_Q))}.$$

Proof. Writing $Q = Q_0 \circ M$ with $M \in \Gamma_{\tau_{Q_0}} \setminus \text{SL}_2(\mathbb{Z})$, we have

$$R_{2-2k}^n(\mathcal{F}_{\mathcal{A}}^{\text{reg}}) = \frac{1}{2\omega_{Q_0}} \sum_{M \in \text{SL}_2(\mathbb{Z})}^{\text{reg}} R_{2-2k}^n(\mathbb{P}_{1-k, -D, Q_0 \circ M}).$$

The result then follows from Lemma 5.1, using that $\tau_{Q_0 \circ M} = M^{-1}\tau_{Q_0}$. \square

5.2. Proof of Theorem 1.2 (2). We now have the necessary pieces to show modularity of $\mathcal{F}_{\mathcal{A}}$ and its relation to $f_{\mathcal{A}}$ under differential operators.

Proof of Theorem 1.2 (2). We first show that

$$R_0^k(G_k(z, \tau_{Q_0})) = -2^k(k-1)!\omega_{\tau_{Q_0}} f_{\mathcal{A}}(z). \quad (5.8)$$

For this, we plug $j = k$ into (5.2), which implies that $R_0^k(g_k^{\mathbb{H}}(z, \mathfrak{z}))$ equals

$$-\frac{(k-1)!(\bar{z}-\mathfrak{z})^k(\bar{z}-\bar{\mathfrak{z}})^k}{2y^{2k}\bar{y}^k \cosh(d(z, \mathfrak{z}))^{2k}} {}_2F_1 \left(k + \frac{1}{2}, k; k + \frac{1}{2}; \frac{1}{\cosh(d(z, \mathfrak{z}))^2} \right). \quad (5.9)$$

Now by (2.7) and (2.8), (5.9) becomes

$$-\frac{(k-1)!(\bar{z}-\mathfrak{z})^k(\bar{z}-\bar{\mathfrak{z}})^k}{2y^{2k}\bar{y}^k (\cosh(d(z, \mathfrak{z}))^2 - 1)^k}.$$

Taking $\mathfrak{z} = \tau_Q$, using (2.5) and (2.1), and plugging in termwise gives (5.8).

The statement for the ξ -operator now follows from Corollary 5.2 using (2.18) and (5.8).

We next compute the image of \mathcal{F}_A under \mathcal{D}^{2k-1} . Using Bol's identity (2.17), we have, by Corollary 5.2,

$$\mathcal{D}^{2k-1}(\mathcal{F}_A(z)) = -\frac{1}{(4\pi)^{2k-1}} \frac{a_{k,2k-1}}{2\omega_{\tau_{Q_0}}} R_0^k(G_k(z, \tau_{Q_0})).$$

Using (5.8) then gives the claim, employing the duplication formula.

Finally, Corollary 5.2 immediately implies that \mathcal{F}_A is modular of weight $2 - 2k$ and $\Delta_{2-2k} = -\xi_{2k} \circ \xi_{2-2k}$ together with (1.8) give that \mathcal{F}_A is annihilated by Δ_{2-2k} . From this one concludes that \mathcal{F}_A is a polar harmonic Maass form. \square

5.3. Elliptic expansion of \mathcal{F}_A . Before stating the elliptic expansion of \mathcal{F}_A , we first give a general formula for functions annihilated by the Laplace operator.

Lemma 5.3. *Suppose that $\mathcal{M} : \mathbb{H} \rightarrow \mathbb{C}$ satisfies $\Delta_{2-2k}(\mathcal{M}) = 0$ and has a singularity of finite order around $\varrho \in \mathbb{H}$. Then, for $0 \leq r_\varrho(z) \ll_\varrho 1$, we have*

$$\left(\mathcal{M}_\varrho^+ - \mathbb{P}_{\mathcal{M},\varrho}^+\right)(z) = (2i\eta)^{2-2k} (z - \bar{\varrho})^{2k-2} \sum_{n \geq 0} \frac{\eta^n}{n!} R_{2-2k}^n(\mathcal{M} - \mathbb{P}_{\mathcal{M},\varrho})(\varrho) X_\varrho(z)^n. \quad (5.10)$$

In particular, if \mathcal{M} does not have a singularity at ϱ , then for $0 \leq r_\varrho(z) \ll_\varrho 1$, we have

$$\mathcal{M}_\varrho^+(z) = (2i\eta)^{2-2k} (z - \bar{\varrho})^{2k-2} \sum_{n \geq 0} \frac{\eta^n}{n!} R_{2-2k}^n(\mathcal{M}(\varrho)) X_\varrho(z)^n. \quad (5.11)$$

Remarks.

- (1) In (5.10), one first needs to act on an independent variable and then plug in ϱ because $\mathcal{M}_\varrho - \mathbb{P}_{\mathcal{M},\varrho}$ depends on ϱ . In (5.11), one may directly apply raising.
- (2) A similar statement is true for the non-meromorphic part \mathcal{M}_ϱ^- where raising is instead applied to the conjugate of \mathcal{M} . However, we do not work out the details here.

Proof of Lemma 5.3. Since $\mathcal{M}_\varrho^+ - \mathbb{P}_{\mathcal{M},\varrho}^+$ is holomorphic in some region around ϱ , we have by Lemma 2.2, for z in some neighborhood of ϱ

$$\left(\mathcal{M}_\varrho^+ - \mathbb{P}_{\mathcal{M},\varrho}^+\right)(z) = (2i\eta)^{2-2k} (z - \bar{\varrho})^{2k-2} \sum_{n \geq 0} \frac{\eta^n}{n!} R_{2-2k}^n\left(\mathcal{M}_\varrho^+ - \mathbb{P}_{\mathcal{M},\varrho}^+\right)(\varrho) X_\varrho(z)^n.$$

The claim hence follows once we prove that

$$R_{2-2k}^n\left(\mathcal{M}_\varrho^+ - \mathbb{P}_{\mathcal{M},\varrho}^+\right)(\varrho) = R_{2-2k}^n(\mathcal{M} - \mathbb{P}_{\mathcal{M},\varrho})(\varrho).$$

Noting which terms in the expansion (2.16) grow as $z \rightarrow \varrho$, it suffices to show that for all $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$

$$\left[R_{2-2k,z}^n\left((z - \bar{\varrho})^{2k-2} \beta_0(1 - r_\varrho(z))^2; 2k - 1, m\right) X_\varrho(z)^{-m} \right]_{z=\varrho} = 0. \quad (5.12)$$

To see (5.12), we first use (2.15) to rewrite

$$\begin{aligned} (z - \bar{\varrho})^{2k-2} \beta_0(1 - r_\varrho(z))^2; 2k - 1, m\right) X_\varrho(z)^{-m} \\ = \sum_{0 \leq j \leq 2k-2} \binom{2k-2}{j} \frac{(-1)^{j+1}}{j+m} (z - \bar{\varrho})^{2k-2} X_\varrho(z)^j \overline{X_\varrho(z)^{j+m}}. \end{aligned} \quad (5.13)$$

We next apply $R_{2-2k,z}^n$ to (5.13). All of the factors other than $(z - \bar{\varrho})^{2k-2} X_\varrho(z)^j$ are annihilated by differentiation in z . Note moreover that $\overline{X_\varrho(z)^{j+m}}$ vanishes at $z = \varrho$ and also that the limit of

$R_{2-2k,z}^n((z-\bar{\varrho})^{2k-2}X_\varrho(z)^j)$ as $z \rightarrow \varrho$ exists because $j \geq 0$ and the resulting function is a polynomial (of degree at most $2k-2$) in z with coefficients depending on y and ϱ . Therefore (5.12) follows. \square

We next describe the principal part of the elliptic expansion of \mathcal{F}_A around ϱ and relate the coefficients of its expansion to higher Green's functions.

Lemma 5.4. *The principal part of \mathcal{F}_A around $\varrho \in \mathbb{H}$ is*

$$\mathbb{P}_{\mathcal{F}_A, \varrho}(z) = \delta_{\varrho=\tau_Q} \mathbb{P}_{1-k, -D, Q}(z),$$

where here by $\delta_{\varrho=\tau_Q}$ we mean that $\varrho = \tau_Q$ as points in \mathbb{H} instead of $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ as used throughout the paper. The elliptic coefficients of the meromorphic part of \mathcal{F}_A are given by

$$c_{\mathcal{F}_A, \varrho}^+(n) = \frac{b_{k,n}}{\omega_{\tau_{Q_0}}} \begin{cases} \eta^{2-2k+n} R_{0,\varrho}^{n+1-k} (G_k^{\mathrm{reg}}(\varrho, \tau_{Q_0})) & \text{if } n \geq k-1, \\ \eta^{-n} R_{0,\varrho}^{k-1-n} (G_k^{\mathrm{reg}}(\varrho, \tau_{Q_0})) & \text{if } n < k-1, \end{cases}$$

where $b_{k,n}$ is defined in (1.11).

Proof. First note that since $0 < D \leq Q_z^2$ with $D = Q_z^2$ if and only if $Q(z, 1) = 0$ (by (2.3)) and $\beta(x; a, b)$ only possibly has singularities at $x = 0$, $x = 1$, and $x \rightarrow \infty$, the only terms contributing singularities are those with $\varrho = \tau_Q$. Since Q is entirely determined by τ_Q and D , it remains to show that $\mathbb{P}_{1-k, -D, Q}$ is precisely a principal part (i.e., that its elliptic expansion (2.16) only contains terms which grow towards ϱ). The claim hence follows once we show that

$$\mathbb{P}_{1-k, -D, Q}(z) = 2^{2-3k} (z - \overline{\tau_Q})^{2k-2} v_Q^{1-k} \beta_0(1 - r_{\tau_Q}(z)^2; 2k-1, 1-k) X_{\tau_Q}(z)^{k-1}. \quad (5.14)$$

To obtain (5.14), note that it is not hard to see that $\mathcal{C}_{2k-1, 1-k} = 0$ and thus

$$\beta_0(Z; 2k-1, 1-k) = \beta(Z; 2k-1, 1-k).$$

By (2.2), the right-hand side of (5.14) thus equals

$$2^{1-2k} D^{\frac{1-k}{2}} Q(z, 1)^{k-1} \beta(1 - r_{\tau_Q}(z)^2; 2k-1, 1-k).$$

We then use (2.12), noting that (2.6) and (2.3) imply that

$$-\frac{(1 - r_{\tau_Q}(z)^2)^2}{4r_{\tau_Q}(z)^2} = -\frac{Dy^2}{|Q(z, 1)|^2},$$

to obtain

$$2^{-1} i (-1)^k D^{\frac{1-k}{2}} Q(z, 1)^{k-1} \beta\left(\frac{-Dy^2}{|Q(z, 1)|^2}; k - \frac{1}{2}, \frac{1}{2}\right).$$

Recalling the definition (1.6) of $\mathbb{P}_{1-k, -D, Q}$, this yields the statement for the principal part.

We next evaluate the elliptic coefficients of the meromorphic part. Using Lemma 5.3, for $n \in \mathbb{N}_0$ we rewrite

$$c_{\mathcal{F}_A, \varrho}^+(n) = \frac{(2i)^{2-2k}}{n!} \eta^{2-2k+n} R_{2-2k}^n (\mathcal{F}_A - \mathbb{P}_{\mathcal{F}_A, \varrho})(\varrho).$$

Using (5.14) and acting termwise yields

$$c_{\mathcal{F}_A, \varrho}^+(n) = \frac{(2i)^{2-2k}}{n!} \eta^{2-2k+n} R_{2-2k}^n (\mathcal{F}_A^{\mathrm{reg}}(\varrho)).$$

The result then follows by Corollary 5.2. \square

6. PROOF OF THEOREM 1.4 AND COROLLARY 1.5

In this section, we rewrite the inner product $\langle f, f_{\mathcal{A}} \rangle$ in terms of the elliptic coefficients (1.10) of f and of $\mathcal{F}_{\mathcal{A}}$ (see Lemma 5.4 for their explicit evaluation), proving Corollary 1.5.

Theorem 6.1. *If $f \in \mathbb{S}_{2k}$ has its poles at $\mathfrak{z}_1, \dots, \mathfrak{z}_r$ in $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$, then*

$$\langle f, f_{\mathcal{A}} \rangle = \pi \sum_{\ell=1}^r \frac{1}{y_{\ell} \omega_{\mathfrak{z}_{\ell}}} \sum_{n \geq 1} c_{f, \mathfrak{z}_{\ell}}(-n) c_{\mathcal{F}_{\mathcal{A}}, \mathfrak{z}_{\ell}}^+(n-1).$$

Proof. Since the functions $\{\Psi_{2k, m}(\cdot, \mathfrak{z}) : \mathfrak{z} \in \mathbb{H}, m \in \mathbb{Z}\}$ span \mathbb{S}_{2k} , by linearity we may assume that $f(z) = (2\omega_{\mathfrak{z}})^{-1} \Psi_{2k, m}(z, \mathfrak{z})$ for $m \in \mathbb{Z}, \mathfrak{z} \in \mathbb{H}$. We use a trick employed by many authors (cf. [4, 11, 17]) for rewriting the inner product. By (1.8), we obtain

$$\langle f, f_{\mathcal{A}} \rangle = \langle f, \xi_{2-2k}(\mathcal{F}_{\mathcal{A}}) \rangle. \quad (6.1)$$

We take the implied integral in (6.1) over the cut-off fundamental domain F_T^* , consisting of those $z \in F^*$ for which z is equivalent to a point in F_T under the action of $\mathrm{SL}_2(\mathbb{Z})$, and then let $T \rightarrow \infty$. We require a few additional properties of F^* . Firstly, we may assume, without loss of generality, that $\tau_{Q_0}, \mathfrak{z} \in F^*$. We also claim that since there are no poles of f or $f_{\mathcal{A}}$ for $y \gg 0$, F^* may be constructed so that for $T \gg 0$, the boundary of F_T^* includes the line from $-1/2 + iT$ to $1/2 + iT$. Indeed one can explicitly build F^* from the standard fundamental domain F by successively removing partial balls $\mathcal{B}_{\delta}(\mathfrak{z}_{\ell}) \cap F$ around each pole $\mathfrak{z}_{\ell} \in \partial F$ and moving them to the other side (with respect to the imaginary axis) of the fundamental domain to combine with other partial balls around equivalent points $\gamma \mathfrak{z}_{\ell} \in \partial F$ to form entire balls $\mathcal{B}_{\delta}(\gamma \mathfrak{z}_{\ell})$ for some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Since the part of the fundamental domain with $y \gg 0$ remains unchanged, the boundary of F_T^* is as desired. Moreover, we may choose $\delta > 0$ sufficiently small such that $\mathcal{B}_{\delta}(\mathfrak{z}_{\ell})$ is contained inside $\Gamma_{\mathfrak{z}_{\ell}} F^*$ and balls around different points are disjoint. By Stokes Theorem, using the meromorphicity of f and the vanishing of $f(z)h_{s, \varrho}(z)$ at $z = \varrho$ for $s \in \mathbb{C}$ with $\sigma \gg 0$, (6.1) equals

$$- \mathrm{CT}_{s=0} \left(\int_{F^*} f(z) \overline{\xi_{2k}(h_{s_1, \mathfrak{z}}(z) h_{s_2, \tau_{Q_0}}(z))} \mathcal{F}_{\mathcal{A}}(z) y^{2-2k} \frac{dx dy}{y^2} + \lim_{T \rightarrow \infty} \int_{\partial F_T^*} f(z) h_{s_1, \mathfrak{z}}(z) h_{s_2, \tau_{Q_0}}(z) \mathcal{F}_{\mathcal{A}}(z) dz \right). \quad (6.2)$$

Due to modularity, the exponential decay of f towards $i\infty$, and the polynomial growth of the other factors, the second term vanishes. Using the invariance of the integrand under the action of $\Gamma_{\tau_{Q_0}}$ and $\Gamma_{\mathfrak{z}}$, we then rewrite (6.2) as

$$- \frac{1}{\omega_{\tau_{Q_0}} \omega_{\mathfrak{z}}} \mathrm{CT}_{s=0} \left(\sum_{\gamma_1 \in \Gamma_{\mathfrak{z}}} \sum_{\gamma_2 \in \Gamma_{\tau_{Q_0}}} \int_{\gamma_1 \gamma_2 F^*} f(z) \overline{\xi_{2k}(h_{s_1, \mathfrak{z}}(z) h_{s_2, \tau_{Q_0}}(z))} \mathcal{F}_{\mathcal{A}}(z) y^{2-2k} \frac{dx dy}{y^2} \right). \quad (6.3)$$

Precisely one element of $\Gamma_{\varrho} F^*$ is equivalent to ϱ for every $\varrho \in F^*$ and $r_{\varrho}(Mz) = r_{\varrho}(z)$ for every $M \in \Gamma_{\varrho}$ by (3.9). Hence, for every $z \in \Gamma_{\varrho} F^*$, we have that $h_{s_j, \varrho}(z) = r_{\varrho}(z)^{2s_j}$. Therefore, for $(\varrho, s_0) \in \{(\mathfrak{z}, s_1), (\tau_{Q_0}, s_2)\}$, we may compute

$$\overline{\xi_{2k}(h_{s_0, \varrho}(z))} = -4s_0 y^{2k} \eta r_{\varrho}(z)^{2s_0-2} \frac{X_{\varrho}(z)}{(\bar{z} - \varrho)^2}. \quad (6.4)$$

Thus $1/(s_1 s_2)$ times the first integral in (6.2) restricted to $z \notin \mathcal{B}_{\delta}(\mathfrak{z}) \cup \mathcal{B}_{\delta}(\tau_{Q_0})$ converges absolutely and locally uniformly in s , and hence the corresponding contribution to the integral is analytic and vanishes at $s = 0$. To evaluate the remaining part of (6.3), we first compute $\xi_{2k}(h_{s_1, \mathfrak{z}}(z) h_{s_2, \tau_{Q_0}}(z))$ for

$z \in \mathcal{B}_\delta(\mathfrak{z}) \cap F^*$ using the product rule. The term coming from differentiating $h_{s_2, \tau_{Q_0}}$ vanishes by (6.4). We then use (3.8) and (6.3) to show that the first integral in (6.2) over $\mathcal{B}_\delta(\mathfrak{z}) \cap F^*$ equals

$$-\frac{1}{\omega_3} \text{CT}_{s_1=0} \left(\int_{\mathcal{B}_\delta(\mathfrak{z})} f(z) \overline{\xi_{2k,z}(h_{s_1, \mathfrak{z}}(z))} \mathcal{F}_A(z) y^{2-2k} \frac{dx dy}{y^2} \right).$$

Plugging in (6.4) and repeating the argument for τ_{Q_0} if $\tau_{Q_0} \neq \mathfrak{z}$, (6.2) becomes

$$\frac{4v_{Q_0}}{\omega_{\tau_{Q_0}}} \mathcal{J}(\tau_{Q_0}) + \delta_{\mathfrak{z} \neq \tau_{Q_0}} \frac{4y}{\omega_3} \mathcal{J}(\mathfrak{z}), \quad (6.5)$$

where

$$\mathcal{J}(\varrho) := \text{CT}_{s_0=0} \left(s_0 \int_{\mathcal{B}_\delta(\varrho)} f(z) r_\varrho(z)^{2s_0-2} \frac{X_\varrho(z)}{(\bar{z} - \varrho)^2} \mathcal{F}_A(z) dx dy \right). \quad (6.6)$$

To evaluate $\mathcal{J}(\varrho)$, we insert the elliptic expansion (2.23) of f around ϱ and the expansion of \mathcal{F}_A using the explicit principal part given in Lemma 5.4 (rewritten as in (5.14)), to see that the integral in (6.6) equals

$$\begin{aligned} & \frac{1}{\eta^2} \int_{\mathcal{B}_\delta(\varrho)} \frac{\eta^2}{|z - \bar{\varrho}|^4} \left(\delta_{\varrho=\mathfrak{z}} X_\varrho(z)^m + \sum_{n \geq 0} c(n) X_\varrho(z)^n \right) r_\varrho(z)^{2s_0-2} X_\varrho(z) \\ & \times \left(2^{2-3k} v_{Q_0}^{1-k} \delta_{\varrho=\tau_{Q_0}} \beta_0 \left(1 - r_{\tau_{Q_0}}(z)^2; 2k-1, 1-k \right) X_{\tau_{Q_0}}(z)^{k-1} + \sum_{\ell \geq 0} c_{\mathcal{F}_A, \varrho}^+(\ell) X_\varrho(z)^\ell \right. \\ & \left. + \sum_{\ell < 0} c_{\mathcal{F}_A, \varrho}^-(\ell) \beta_0 \left(1 - r_\varrho(z)^2; 2k-1, -\ell \right) X_\varrho(z)^\ell \right) dx dy. \quad (6.7) \end{aligned}$$

Making the change of variables $X_\varrho(z) = R e^{i\theta}$ and noting that $\frac{\eta^2}{|z - \bar{\varrho}|^4} dx dy = \frac{R}{4} d\theta dR$ and $r_\varrho(z) = R$, we may rewrite (6.7) as

$$\begin{aligned} & \frac{1}{4\eta^2} \int_0^\delta \int_0^{2\pi} \left(\delta_{\varrho=\mathfrak{z}} R^m e^{im\theta} + \sum_{n \geq 0} c(n) R^n e^{in\theta} \right) \left(\delta_{\varrho=\tau_{Q_0}} \frac{R^{k-1+2s_0} e^{ik\theta}}{2^{3k-2} v_{Q_0}^{k-1}} \beta_0 \left(1 - R^2; 2k-1, 1-k \right) \right. \\ & \left. + \sum_{\ell \geq 0} c_{\mathcal{F}_A, \varrho}^+(\ell) R^{\ell+2s_0} e^{i(\ell+1)\theta} + \sum_{\ell < 0} c_{\mathcal{F}_A, \varrho}^-(\ell) \beta_0 \left(1 - R^2; 2k-1, -\ell \right) R^{\ell+2s_0} e^{i(\ell+1)\theta} \right) d\theta dR. \end{aligned}$$

Expanding, the integral over θ vanishes unless the power of $e^{i\theta}$ is zero. The integral thus equals

$$\begin{aligned} & \frac{\pi}{2\eta^2} \int_0^\delta \left(\frac{\delta_{m=-k} \delta_{\varrho=\tau_{Q_0}=\mathfrak{z}}}{2^{3k-2} v_{Q_0}^{k-1}} \beta_0 \left(1 - R^2; 2k-1, 1-k \right) + \delta_{\varrho=\mathfrak{z}} c_{\mathcal{F}_A, \varrho}^+(-m-1) \right. \\ & \left. + \sum_{n \geq 0} (c(n) + \delta_{n=m} \delta_{\varrho=\mathfrak{z}}) c_{\mathcal{F}_A, \varrho}^-(n-1) \beta_0 \left(1 - R^2; 2k-1, n+1 \right) \right) R^{2s_0-1} dR. \quad (6.8) \end{aligned}$$

To determine the residue of (6.8) at $s_0 = 0$, we use (2.15) with $a = 2k-1$ and $b = -\ell$ to expand $\beta_0(1 - R^2; 2k-1, -\ell)$. For $\sigma_0 \gg 0$, multiplying the first term in (2.15) by R^{2s_0-1} and integrating then yields

$$\sum_{\substack{0 \leq j \leq 2k-2 \\ j \neq \ell}} \binom{2k-2}{j} \frac{(-1)^{j+1}}{j-\ell} \int_0^\delta R^{2(j-\ell+s_0)-1} dR = \sum_{\substack{0 \leq j \leq 2k-2 \\ j \neq \ell}} \binom{2k-2}{j} \frac{(-1)^{j+1} \delta^{2(j-\ell+s_0)}}{2(j-\ell)(j-\ell+s_0)},$$

which is holomorphic at $s_0 = 0$, so the corresponding terms in (6.8) give no residue. Hence

$$\mathcal{J}(\varrho) = \frac{\pi \delta_{\varrho=\mathfrak{z}}}{2\eta^2} \text{CT}_{s_0=0} \left(-s_0 \frac{\delta_{m=-k} \delta_{\varrho=\tau_{Q_0}}}{(-8v_{Q_0})^{k-1}} \binom{2k-2}{k-1} \int_0^\delta \log(R) R^{2s_0-1} dR + c_{\mathcal{F}_A, \varrho}^+ (-m-1) \frac{\delta^{2s_0}}{2} \right). \quad (6.9)$$

Using integration by parts, for the first summand in (6.9) we obtain a meromorphic continuation with no constant term, as

$$s_0 \int_0^\delta \log(R) R^{2s_0-1} dR = \frac{\delta^{2s_0}}{2} \log(\delta) - \frac{1}{4s_0} \delta^{2s_0} = -\frac{1}{4s_0} + O(s_0).$$

Therefore

$$\mathcal{J}(\varrho) = \frac{\pi \delta_{\varrho=\mathfrak{z}}}{4\eta^2} c_{\mathcal{F}_A, \varrho}^+ (-m-1).$$

Plugging back into (6.5) and recalling that this equals (6.1) then gives

$$\begin{aligned} \langle f, f_A \rangle &= \langle f, \xi_{2-2k}(\mathcal{F}_A) \rangle = \frac{4v_{Q_0}}{\omega_{\tau_{Q_0}}} \mathcal{J}(\tau_{Q_0}) + \frac{4y \delta_{\mathfrak{z} \neq \tau_{Q_0}}}{\omega_{\mathfrak{z}}} \mathcal{J}(\mathfrak{z}) \\ &= \frac{\pi}{v_{Q_0} \omega_{\tau_{Q_0}}} \delta_{\mathfrak{z}=\tau_{Q_0}} c_{\mathcal{F}_A, \tau_{Q_0}}^+ (-m-1) + \frac{\pi}{y \omega_{\mathfrak{z}}} \delta_{\mathfrak{z} \neq \tau_{Q_0}} c_{\mathcal{F}_A, \mathfrak{z}}^+ (-m-1) = \frac{\pi}{y \omega_{\mathfrak{z}}} c_{\mathcal{F}_A, \mathfrak{z}}^+ (-m-1). \end{aligned}$$

□

The following Theorem generalizes Theorem 1.4 to also allow poles at τ_{Q_0} .

Theorem 6.2. *If $Q_0 \in \mathcal{A} \in \mathcal{Q}_{-D} \setminus \text{SL}_2(\mathbb{Z})$ and $f \in \mathbb{S}_{2k}$ with poles in $\text{SL}_2(\mathbb{Z}) \setminus \mathbb{H}$ at $\mathfrak{z}_1, \dots, \mathfrak{z}_r$, then*

$$\begin{aligned} \langle f, f_A \rangle &= \frac{\pi}{\omega_{\tau_{Q_0}}} \sum_{\ell=1}^r \frac{1}{\omega_{\mathfrak{z}_\ell}} \left(\sum_{n \geq k} b_{k, n-1} y_\ell^{-2k+n} c_{f, \mathfrak{z}_\ell}(-n) R_0^{n-k} (G_k^{\text{reg}}(z, \tau_{Q_0})) \right. \\ &\quad \left. + \sum_{n=1}^{k-1} b_{k, n-1} y_\ell^{-n} c_{f, \mathfrak{z}_\ell}(-n) \overline{R_0^{k-n} (G_k^{\text{reg}}(z, \tau_{Q_0}))} \right). \end{aligned}$$

Proof. The result follows directly by plugging Lemma 5.4 into the statement of Theorem 6.1. □

We finally prove Corollary 1.5.

Proof of Corollary 1.5. This follows immediately from Theorem 1.4 and Lemma 2.3. □

7. FUTURE QUESTIONS

We conclude the paper by discussing some possible future directions that one could pursue:

- (1) Note that by Theorem 1.4, f_A is orthogonal to cusp forms, which was also proven by Petersson [38]. Combining the regularizations for growth towards the cusps and towards points in \mathbb{H} , one can further prove that f_A is orthogonal to weakly holomorphic modular forms, but we do not carry out the details here.
- (2) Images of lifts between integral and half-integral weight weak Maass forms have Fourier expansions which can be written as CM-traces for negative discriminants and cycle integrals for positive discriminants [9, 12, 16]. Thus the appearance of CM-values of $\text{SL}_2(\mathbb{Z})$ -invariant functions in Theorem 6.2 is natural. Since the generating function of Zagier's cusp forms for positive discriminants yields the kernel function for the first Shintani lift, one may ask whether there is a connection between CM-traces and the generating function of the $f_{k, -D}$. However, the naive generating function diverges and furthermore would have a dense set of poles in the

upper half-plane. It hence might be interesting to instead investigate whether the generating function for the regularized $f_{k,-D}^{\text{reg}}$ has any connection to CM-traces.

- (3) In light of the connection in Corollary 1.5, it would be interesting to investigate Conjecture 4.4 of [22], concerning G_k evaluated at CM-points. Moreover, since $G_k^{\text{reg}}(\tau_{Q_0}, \tau_{Q_0})$ naturally appears when computing $\langle f_{\mathcal{A}}, f_{\mathcal{A}} \rangle$, one can probably use the regularized higher Green's functions to reformulate the conjecture to include the case when the CM-points agree. Given the connections to heights and geometry in [22] and [45], it would also be interesting to see if the identity in Corollary 1.5 holds for $k = 1$ and higher level in this case.
- (4) In Conjecture 4.4 of [22], Gross and Zagier took linear combinations of G_k acted on by Hecke operators and conjectured that these linear combinations evaluated at CM-points are essentially logarithms of algebraic numbers whenever the linear combinations satisfy certain relations. These relations are determined by linear equations defined by the Fourier coefficients of weight $2k$ cusp forms. Note that by Corollary 5.2, $G_k(z, \tau_Q)$ is essentially $R_0^{k-1}(\mathcal{F}_{\mathcal{A}}(z))$, while $\mathcal{F}_{\mathcal{A}}$ is naturally related to $f_{\mathcal{A}}$ via differential operators in Theorem 1.2 (2). Translating the condition of Gross and Zagier into a condition on polar harmonic Maass forms might be enlightening in two directions. On the one hand, it might carve out a natural subspace of weight $2k$ meromorphic modular forms (corresponding to image of those forms satisfying these conditions) which may satisfy other interesting properties. On the other side, by applying the theory of harmonic Maass forms, one may be able to loosen the conditions and investigate what happens for general linear combinations.

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