

The false theta functions of Rodgers and their modularity

Kathrin Bringmann

ABSTRACT

In this survey article, we explain how false theta functions can be embedded into a modular framework and show some of the applications of this modularity.

1. Introduction and statement of results

As false theta functions are “obstructed” modular forms, we first explain what modular forms are. The following quote is often attributed to Eichler, and shows the important role that modular forms play:

“There are five fundamental operations of arithmetic: addition, subtraction, multiplication, division and modular forms.”

Modular forms generalize the classical trigonometric functions because they are periodic; however, they have more symmetries. They play a central role in many areas including algebraic topology, arithmetic geometry, combinatorics, mathematical physics, mirror symmetry, monstrous moonshine, number theory, representation theory, and occurred, for example, in progress toward the Birch and Swinnerton–Dyer conjecture, and in the proof of Fermat’s last theorem.

To formally define modular forms, let $\mathbb{H} := \{\tau = \tau_1 + i\tau_2 \in \mathbb{C} : \tau_2 > 0\}$ be the complex upper half-plane. A meromorphic function $f : \mathbb{H} \rightarrow \mathbb{C}$ is a *modular form of weight* $k \in 2\mathbb{Z}$ if, for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$

$$f \Big|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau) - f(\tau) := (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right) - f(\tau) = 0. \quad (1.1)$$

Moreover, f is required to be “meromorphic at the cusps.” The left-hand side of (1.1) is called the *obstruction to modularity* of f . Note that there are generalizations of (1.1) to include half-integral weights (that is, $k \in \mathbb{Z} + \frac{1}{2}$) and multipliers. Modular forms have Fourier expansions of the shape $f(\tau) = \sum_{n \gg -\infty} c_f(n)q^n$ ($q := e^{2\pi i\tau}$ throughout). The modularity of f has applications to its Fourier coefficients, $c_f(n)$, for which it can be viewed as a *generating function*. As an example, consider the generating function of divisor sums ($k \geq 4$ an even integer)

$$\sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n, \quad \text{where} \quad \sigma_k(n) := \sum_{d|n} d^k,$$

Received 19 March 2021; revised 7 April 2021; published online 2 July 2021.

2020 *Mathematics Subject Classification* 11F12, 11F20, 11F27, 11F30, 11F37, 11F50, 11P82 (primary).

The author is supported by the Deutsche Forschungsgemeinschaft (DFG) Grant No. BR 4082/5-1.

© 2021 The Authors. *Bulletin of the London Mathematical Society* is copyright © London Mathematical Society. This is an open access article under the terms of the [Creative Commons Attribution](https://creativecommons.org/licenses/by/4.0/) License, which permits use, distribution and reproduction in any medium, provided the original work is properly cited.

which is basically (up to a constant term) a modular form of weight k . This can be used to show identities including

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{1 \leq m \leq n-1} \sigma_3(m)\sigma_3(n-m).$$

To give another example,

$$\Theta(\tau) := \sum_{n \in \mathbb{Z}} q^{n^2}$$

is a modular form of weight $\frac{1}{2}$. This function and its modularity can be used to prove identities involving sums of squares and has many applications in the theory of quadratic forms.

There are also important cases in which the obstruction to modularity is not zero but explicit and “nice.” One such example is given by the weight two Eisenstein series

$$E_2(\tau) := 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)q^n.$$

Its obstruction to modularity is

$$E_2\left(-\frac{1}{\tau}\right) - \tau^2 E_2(\tau) = \frac{6\tau}{\pi i}.$$

Even more important examples are given by Ramanujan’s mock theta functions, a list of q -series that are reminiscent of modular forms and that were introduced by Ramanujan in his last letter to Hardy. The letter contained a list of 17 examples, including the q -hypergeometric series

$$f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q; q)_n^2},$$

where $(a; q)_n := \prod_{j=0}^{n-1} (1 - aq^j)$ for $n \in \mathbb{N}_0 \cup \{\infty\}$. Sander Zwegers in his PhD thesis [46] viewed the mock theta functions as pieces of real-analytic modular forms. Ken Ono and the author then extended these results and embedded these functions into the framework of harmonic Maass forms [14], which are real-analytic generalizations of modular forms. To be more precise, one “completes” the functions by adding non-holomorphic integrals of the shape

$$\int_{-\bar{\tau}}^{i\infty} \frac{\theta(w)}{\sqrt{-i(w + \tau)}} dw$$

to the mock theta functions, where θ is a weight $\frac{3}{2}$ (modular) theta function.

False *theta functions* are similar to theta functions, but with wrong sign-factors that prevent them from being modular forms. For example,

$$\sum_{\substack{n \in \mathbb{Z} \\ n \equiv 1 \pmod{3}}} \text{sgn}(n)q^{n^2}$$

is a false theta function; deleting the sign-factor gives a modular form. Here we use the usual convention that $\text{sgn}(x) := \frac{|x|}{x}$ for $x \neq 0$ and $\text{sgn}(0) := 0$. We see in Section 3 how one can complete false theta functions following the guide of the mock theta functions.

False theta functions also possess some modular transformation properties on the rationals. Note that (1.1) cannot hold for all $\tau \in \mathbb{Q}$ as $\text{SL}_2(\mathbb{Z})$ acts transitively on \mathbb{Q} and thus f would be zero if (1.1) would hold for all $\tau \in \mathbb{Q}$. That is why it is natural to require the obstruction to modularity to have better properties than the original function. A *quantum modular form* [43] is a function $f : \mathbb{Q} \rightarrow \mathbb{C}$ whose obstruction to modularity is “nice.” In our setting “nice”

means that it extends to a real-analytic function on the reals except for a finite set of points. An interesting and important source of examples is given via quantum invariants of knots and 3-manifolds [33]. Further examples are given by (many) false theta functions (taking vertical limits in the upper half-plane to rational numbers).

The paper is organized as follows. In Section 2 we recall some basic facts on modular forms and Jacobi forms. In Section 3 we complete false theta functions and prove modularity properties for these completions. As a first application of our completion, we explain in Section 4 how to determine the asymptotic behavior of coefficients of functions involving false theta functions. As a second application, quantum modularity of certain false theta functions is discussed in Section 5. In Section 6 we then explain how coefficients of meromorphic Jacobi forms relate to false theta functions. Finally, in Section 7 we treat higher dimensional false theta functions.

2. Preliminaries

2.1. Modular forms and theta functions

We start by recalling classical modular forms. In the simplest case a holomorphic function $f : \mathbb{H} \rightarrow \mathbb{C}$ is called a *modular form of weight $k \in \frac{1}{2}\mathbb{Z}$* if for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ (if $k \in \mathbb{Z} + \frac{1}{2}$, then we require that $\gamma \in \Gamma_0(4)$, where $\Gamma_0(N) := \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \}$)

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = \left(\frac{c}{d}\right)^{2k} \varepsilon_d^{-2k} (c\tau + d)^k f(\tau),$$

where (\cdot) is the extended Legendre symbol and $\varepsilon_d := 1$ if $d \equiv 1 \pmod{4}$ and $\varepsilon_d := i$ if $d \equiv 3 \pmod{4}$. Moreover f is required to be “holomorphic at the cusps.” We now explain the meaning of this in the integral-weight case. Since f is holomorphic and $\begin{pmatrix} 1 & \\ 0 & 1 \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$, f has a Fourier expansion of the shape

$$f(\tau) = \sum_{n \in \mathbb{Z}} c_f(n) q^n.$$

The condition that f is “holomorphic at $i\infty$ ” means that $c_f(n) = 0$ for $n < 0$. We call f a *cusp form* if it is a holomorphic modular form that vanishes at $i\infty$ (that is, $c_f(0) = 0$). If f is allowed to have a finite number of Fourier coefficients that are supported on negative q -exponents, then f is called *weakly holomorphic*. We denote the space of weakly holomorphic modular forms of weight k by $M_k^!$. For a weakly holomorphic modular form f , we call

$$P_f(q) := \sum_{n < 0} c_f(n) q^n$$

the *principal part of f* (at $i\infty$).

A specific example of a modular form of weight $\frac{1}{2}$ is given by the *Dedekind’s eta-function*

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).$$

To be more precise we have

$$\eta(\tau + 1) = e^{\frac{\pi i}{12}} \eta(\tau), \quad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau} \eta(\tau).$$

Throughout we define the square root using the principal branch of the logarithm.

We also require certain theta functions and their transformation properties. These were studied, for example, by Shimura [39]. For $\nu \in \{0, 1\}$, $h \in \mathbb{Z}$, $N, A \in \mathbb{N}$, with $A|N$, $N|hA$, define

$$\Theta_\nu(A, h, N; \tau) := \sum_{\substack{m \in \mathbb{Z} \\ m \equiv h \pmod{N}}} m^\nu q^{\frac{Am^2}{2N^2}}.$$

We have the transformations

$$\Theta_\nu\left(A, h, N; -\frac{1}{\tau}\right) = (-i)^\nu A^{-\frac{1}{2}} (-i\tau)^{\nu+\frac{1}{2}} \sum_{\substack{k \pmod{N} \\ Ak \equiv 0 \pmod{N}}} e\left(\frac{Ahk}{N^2}\right) \Theta_\nu(A, k, N; \tau), \tag{2.1}$$

$$\Theta_\nu(A, h, N; \tau + 2) = e\left(\frac{Ah^2}{N^2}\right) \Theta_\nu(A, h, N; \tau),$$

$$\Theta_\nu(A, h, N; \tau) = \sum_{\substack{g \pmod{cN} \\ g \equiv h \pmod{N}}} \Theta_\nu(cA, g, cN; c\tau) \quad (c \in \mathbb{N}),$$

where $e(x) := e^{2\pi ix}$. From this one can conclude that for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2N)$ with $2|b$, we have

$$\Theta_\nu(A, h, N; M\tau) = e\left(\frac{abAh^2}{2N^2}\right) \left(\frac{2Ac}{d}\right) \varepsilon_d^{-1} (c\tau + d)^{\nu+\frac{1}{2}} \Theta_\nu(A, ah, N; \tau).$$

Also note that if $h_1 \equiv h_2 \pmod{N}$, then we have

$$\Theta_\nu(A, h_1, N; \tau) = \Theta_\nu(A, h_2, N; \tau), \quad \Theta_\nu(A, -h, N; \tau) = (-1)^\nu \Theta_\nu(A, h, N; \tau).$$

2.2. *Jacobi forms*

Jacobi forms were first systematically studied by Eichler and Zagier [23]. They play an important role in number theory and other areas, including the theory of Siegel modular forms (note that there is an important lift, the so-called Saito–Kurokawa lift which maps modular forms to Siegel modular forms; this lift can be constructed using Jacobi forms, see [1, 36, 41]), the study of central L -values and derivatives of twisted elliptic curves [25], and in the theory of umbral moonshine [18], just to name a few. Roughly speaking, a *Jacobi form* is a function $\phi: \mathbb{C} \times \mathbb{H} \rightarrow \mathbb{C}$, which satisfies two transformations similar to the transformations of elliptic functions and of modular forms. To give a more precise definition, let $k, m \in \mathbb{N}$. A *holomorphic Jacobi form* of weight k and index m on $\text{SL}_2(\mathbb{Z})$ is a holomorphic function $\phi: \mathbb{C} \times \mathbb{H} \rightarrow \mathbb{C}$ satisfying.

(1) For all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$, we have that

$$\phi\left(\frac{z}{c\tau + d}; \frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k e\left(\frac{cmz^2}{c\tau + d}\right) \phi(z; \tau). \tag{2.2}$$

(2) For all $\lambda, \mu \in \mathbb{Z}$, we have that

$$\phi(z + \lambda\tau + \mu; \tau) = e(-m(\lambda^2\tau + 2\lambda z)) \phi(z; \tau). \tag{2.3}$$

(3) The function ϕ has a Fourier expansion of the form ($\zeta := e^{2\pi iz}$)

$$\phi(z; \tau) = \sum_{\substack{n, r \in \mathbb{Z} \\ 4mn \geq r^2}} c_\phi(n, r) q^n \zeta^r.$$

Note that one can generalize this definition to include half-integral weight and/or half-integral index and/or multipliers. A particular example of a Jacobi form (of weight and index $\frac{1}{2}$) is given by the *Jacobi theta function*

$$\vartheta(z; \tau) := i \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}(n+\frac{1}{2})^2} \zeta^{n+\frac{1}{2}}. \tag{2.4}$$

To be more precise it satisfies

$$\begin{aligned} \vartheta(z + 1; \tau) &= -\vartheta(z; \tau), & \vartheta(z + \tau; \tau) &= -e^{-\pi i \tau - 2\pi i z} \vartheta(z; \tau), \\ \vartheta(z; \tau + 1) &= e^{\frac{\pi i}{4}} \vartheta(z; \tau), & \vartheta\left(\frac{z}{\tau}; -\frac{1}{\tau}\right) &= -i\sqrt{-i\tau} e^{\frac{\pi i z^2}{\tau}} \vartheta(z; \tau). \end{aligned}$$

Jacobi forms are related to modular forms in various ways, one of which we recall here. It is a classical result that Fourier coefficients (in z) of holomorphic Jacobi forms are modular forms [23]. This follows from the so-called theta decomposition. To state it, define for $a \in \mathbb{Z}$ and $m \in \mathbb{N}$

$$\vartheta_{m,a}(z; \tau) := \sum_{\substack{r \in \mathbb{Z} \\ r \equiv a \pmod{2m}}} q^{\frac{r^2}{4m}} \zeta^r.$$

These theta functions are examples of weight $\frac{1}{2}$ and index m Jacobi forms. In particular, we have

$$\vartheta_{m,a}(z; \tau + 1) = e\left(\frac{a^2}{4m}\right) \vartheta_{m,a}(z; \tau), \tag{2.5}$$

$$\vartheta_{m,a}\left(\frac{z}{\tau}; -\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{2mi}} e\left(\frac{mz^2}{\tau}\right) \sum_{b \pmod{2m}} e\left(-\frac{ab}{2m}\right) \vartheta_{m,b}(z; \tau). \tag{2.6}$$

Now recall that the Fourier coefficients $c_\phi(n, r)$ of a Jacobi form ϕ only depend on $4mn - r^2$ and $r \pmod{2m}$ (which follows from (2.3)). Thus, one can define for $N \in \mathbb{N}_0$ and for any $r \in \mathbb{Z}$ satisfying $r \equiv a \pmod{2m}$,

$$c_a(N) := c_\phi\left(\frac{N + r^2}{4m}, r\right).$$

Here, we set $c_a(N) := 0$ if $N \not\equiv -a^2 \pmod{4m}$. We then define for $a \in \mathbb{Z}/2m\mathbb{Z}$ the generating functions

$$h_a(\tau) := \sum_{N=0}^{\infty} c_a(N) q^{\frac{N}{4m}}.$$

Then ϕ has the *theta decomposition*,

$$\phi(z; \tau) = \sum_{a \pmod{2m}} h_a(\tau) \vartheta_{m,a}(z; \tau). \tag{2.7}$$

Using (2.2) as well as (2.5) and (2.6), one may show that the functions h_a transform like vector-valued modular forms. To be more precise, we have

$$h_a(\tau + 1) = e\left(\frac{-a^2}{4m}\right) h_a(\tau), \quad h_a\left(-\frac{1}{\tau}\right) = \frac{\tau^k}{\sqrt{-2im\tau}} \sum_{b \pmod{2m}} e\left(\frac{ab}{2m}\right) h_b(\tau). \tag{2.8}$$

The decomposition (2.7) gives an isomorphism between Jacobi forms of weight k and index m and vector-valued modular forms satisfying (2.8).

3. False theta functions and their completions

In this section we describe the modular completions found in [13] for the false theta functions. These parallel the situation with mock theta functions in contrast to what one may predict from the above quote by Ramanujan. For simplicity, I describe our results in a special case. That is, we consider the following false Jacobi theta function:

$$\psi(z; \tau) := i \sum_{n \in \mathbb{Z}} \operatorname{sgn}\left(n + \frac{1}{2}\right) (-1)^n q^{\frac{1}{2}(n+\frac{1}{2})^2} \zeta^{n+\frac{1}{2}}. \tag{3.1}$$

Although the function ψ is invariant under $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, it does not transform invariantly under $S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Note that removing the sign-factor yields the Jacobi theta function (2.4), thus a Jacobi form.

To repair the broken modularity in the case of $\psi(z; \tau)$, we define, for $\tau, w \in \mathbb{H}$ and $z \in \mathbb{C}$ with

$$\widehat{\psi}(z; \tau, w) := i \sum_{n \in \mathbb{Z}} \operatorname{erf}\left(-i\sqrt{\pi i(w - \tau)}\left(n + \frac{1}{2} + \frac{z_2}{\tau_2}\right)\right) (-1)^n q^{\frac{1}{2}(n+\frac{1}{2})^2} \zeta^{n+\frac{1}{2}},$$

where for $w \in \mathbb{C}$, let $w_2 := \operatorname{Im}(w)$, and where $\operatorname{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-w^2} dw$ denotes the error function.

Note that

$$\lim_{t \rightarrow \infty} \widehat{\psi}(z; \tau, \tau + it + \varepsilon) = \psi(z; \tau)$$

if $-\frac{1}{2} < \frac{z_2}{\tau_2} < \frac{1}{2}$ and $\varepsilon > 0$ arbitrary. In that sense $\widehat{\psi}$ may be viewed as completion of ψ . The following theorem from [13] gives the modular properties of $\widehat{\psi}$.

THEOREM 3.1. *The function $\widehat{\psi}$ transforms like a Jacobi form. To be more precise, we have for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{SL}_2(\mathbb{Z})$ and $m, r \in \mathbb{Z}$*

$$\widehat{\psi}\left(\frac{z}{c\tau + d}; \frac{a\tau + b}{c\tau + d}, \frac{aw + b}{cw + d}\right) = \chi_{\tau, w}(M) \nu_{\eta}(M)^3 (c\tau + d)^{\frac{1}{2}} e^{\frac{\pi i c z^2}{c\tau + d}} \widehat{\psi}(z; \tau, w),$$

$$\widehat{\psi}(z + m\tau + r; \tau, w) = (-1)^{m+r} q^{-\frac{m^2}{2}} \zeta^{-m} \widehat{\psi}(z; \tau, w).$$

Sketch of proof. Define

$$\widetilde{\psi}(z; \tau, w) := \sqrt{i(w - \tau)} \widehat{\psi}(z; \tau, w).$$

One can show that the series defining $\widetilde{\psi}$ converges to a holomorphic function of w . We next write

$$\widetilde{\psi}(z; \tau, w) = e^{-\frac{\pi i z^2 \tau}{\tau_2^2}} \sum_{n \in \mathbb{Z} + \frac{1}{2}} F_{\tau, w}\left(n + \frac{z_2}{\tau_2}\right) e^{2\pi i n \left(-\frac{\operatorname{Im}(z\bar{\tau})}{\tau_2} + \frac{1}{2}\right)}, \tag{3.2}$$

where

$$F_{\tau, w}(x) := \sqrt{i(w - \tau)} \operatorname{erf}\left(-i\sqrt{\pi i(w - \tau)}x\right) e^{\pi i x \tau}.$$

Changing variables $(z, \tau, w) \mapsto \left(\frac{z}{\tau}, -\frac{1}{\tau}, -\frac{1}{w}\right)$ in (3.2), we obtain

$$\widetilde{\psi}\left(\frac{z}{\tau}; -\frac{1}{\tau}, \frac{w}{\tau}\right) = e^{\frac{\pi i (\operatorname{Im}(z\bar{\tau}))^2}{\tau \tau_2^2} + \pi i \left(\frac{z_2}{\tau_2} + \frac{1}{2}\right)} \sum_{r \in \mathbb{Z}} F_{-\frac{1}{\tau}, -\frac{1}{w}}\left(r + \frac{1}{2} + \frac{\operatorname{Im}(z\bar{\tau})}{\tau_2}\right) e^{2\pi i r \left(\frac{z_2}{\tau_2} + \frac{1}{2}\right)}. \tag{3.3}$$

Let \mathcal{F} be the Fourier transform of $f : \mathbb{R} \rightarrow \mathbb{C}$, defined by

$$\mathcal{F}(f)(x) := \int_{\mathbb{R}} f(y)e^{-2\pi ixy} dy.$$

Then it is easy to verify that

$$\mathcal{F}(F_{\tau,w})(x) = (-i)^{-\frac{1}{2}}\sqrt{w}F_{-\frac{1}{\tau},-\frac{1}{w}}(x).$$

Using this and elementary properties of Fourier transforms, we obtain that (3.3) equals

$$\frac{(-i)^{-\frac{1}{2}}}{\sqrt{w}}e^{\frac{\pi i(\text{Im}(z\bar{\tau}))^2}{\tau\tau_2} + \pi i\left(\frac{z_2}{\tau_2} + \frac{1}{2}\right)} \sum_{r \in \mathbb{Z}} F_{\tau,w}\left(-r + \frac{z_2}{\tau_2} + \frac{1}{2}\right) e^{2\pi i\left(r - \frac{z_2}{\tau_2} - \frac{1}{2}\right)\left(\frac{1}{2} + \frac{\text{Im}(z\bar{\tau})}{\tau_2}\right)}.$$

Changing $r \mapsto -r$ and simplifying gives the claim. □

4. Asymptotics of mixed false theta functions

We call linear combinations of false theta functions multiplied by modular forms *mixed false theta functions*. Before describing how to determine the asymptotic behavior of such functions, let me recall the situation for weakly holomorphic modular forms using an explicit example. A *partition* of $n \in \mathbb{N}_0$ is a non-increasing sequence of positive integers, which sum to n . We let $p(n)$ denote the number of partitions of n and set $p(0) := 1$. For example, the partitions of 5 are given by

$$5, \quad 4 + 1, \quad 3 + 2, \quad 3 + 1 + 1, \quad 2 + 2 + 1, \quad 2 + 1 + 1 + 1, \quad 1 + 1 + 1 + 1 + 1,$$

so that $p(5) = 7$. Euler proved that the partition generating function has a nice representation as infinite product. Namely, we have that

$$p(q) := \sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1 - q^n} = \frac{q^{\frac{1}{24}}}{\eta(\tau)}, \tag{4.1}$$

so (up to a q -power) a weakly holomorphic modular form of weight $-\frac{1}{2}$. This product representation has many important consequences. For example, as Euler deduced from it a recurrence formula for $p(n)$ which enables one to compute much higher values of $p(n)$ than by more naive methods. To be more precise, using the Pentagonal Number Theorem

$$(q; q)_{\infty} = \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{n(3n+1)}{2}}$$

we have that

$$p(n) = \sum_{k=1}^n (-1)^{k+1} \left(p\left(n - \frac{k(3k-1)}{2}\right) + p\left(n - \frac{k(3k+1)}{2}\right) \right).$$

Even better, Rademacher [37] used the modularity of (4.1) to give an exact, infinite summation formula for $p(n)$, as in the following theorem. Before stating the result, define the Kloosterman-type sums

$$A_k(n) := \sum_{\substack{0 \leq h < k \\ \gcd(h,k)=1}} \omega_{h,k} e^{-\frac{2\pi i n h}{k}},$$

where

$$\omega_{h,k} := \exp(\pi i s(h, k)).$$

Here

$$s(h, k) := \sum_{\mu \pmod k} \left(\left(\frac{\mu}{k} \right) \right) \left(\left(\frac{h\mu}{k} \right) \right)$$

is the usual *Dedekind sum*, where for $x \in \mathbb{R}$ we set

$$\left(\left(x \right) \right) := \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

Moreover, we define the *modified Bessel function* (of the first kind)

$$I_\kappa(x) := \sum_{m=0}^\infty \frac{1}{m! \Gamma(m + \kappa + 1)} \left(\frac{x}{2} \right)^{2m + \kappa},$$

where Γ denotes the Gamma-function.

THEOREM 4.1 (Rademacher). *If $n \in \mathbb{N}$, then*

$$p(n) = \frac{2\pi}{(24n - 1)^{\frac{3}{4}}} \sum_{k=1}^\infty \frac{A_k(n)}{k} I_{\frac{3}{2}} \left(\frac{\pi \sqrt{24n - 1}}{6k} \right).$$

In particular, as $n \rightarrow \infty$ we have that

$$p(n) \sim \frac{1}{4\sqrt{3}n} e^{\pi \sqrt{\frac{2n}{3}}}.$$

Sketch of proof. The proof uses the so-called Circle Method [28]. To summarize the idea, suppose that one is interested in the asymptotic behavior of some sequence $\{a(n)\}$ as $n \rightarrow \infty$. One builds a generating function out of this sequence

$$A(q) := \sum_{n=0}^\infty a(n)q^n,$$

which is supposed to be scaled so that A has radius of convergence equal to one. Cauchy’s theorem then gives, for $n \in \mathbb{N}$, the formula

$$a(n) = \frac{1}{2\pi i} \int_C \frac{A(q)}{q^{n+1}} dq, \tag{4.2}$$

where C is an arbitrary path inside the unit disk that loops around zero in the counterclockwise direction exactly once.

If one takes $\{a(n)\} = \{p(n)\}$ (and thus $A(q) = P(q)$), one cannot take the integral in (4.2) over the whole unit circle due to the singularities at roots of unity of the product formula (4.1). For many interesting sequences $\{a(n)\}$, including the case of the sequence $\{p(n)\}$ of partition numbers, the singularities of the generating function A on the unit circle are well understood and occur at roots of unity q . One can often find nice approximations of A near these points. For example, in the case of $\{p(n)\}$, one can use the modularity of $P(q)$ to approximate it toward roots of unity by its principal parts. The integrals of these principal parts may then be evaluated.

Using the asymptotic behavior of the I -Bessel function

$$I_\alpha(x) \sim \frac{e^x}{\sqrt{2\pi x}} \quad (\text{as } x \rightarrow \infty) \tag{4.3}$$

gives the asymptotics. □

There are general formulas for Fourier coefficients of weakly holomorphic modular forms. Rademacher and Zuckerman [38, 44, 45] obtained the following result. For this, define the Kloosterman sums

$$K_k(m, n; c) := \begin{cases} \sum_{d \pmod{c}^*} e\left(\frac{m\bar{d} + nd}{c}\right) & \text{if } k \in \mathbb{Z}, \\ \sum_{d \pmod{c}^*} \left(\frac{c}{d}\right) \varepsilon_d^{2k} e\left(\frac{m\bar{d} + nd}{c}\right) & \text{if } k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}. \end{cases}$$

Here, for $d \in \mathbb{Z}$, \bar{d} is such that $d\bar{d} \equiv 1 \pmod{c}$ and the sum only runs over those d which are coprime to c .

THEOREM 4.2. *If $f \in M_k^!$ with $k \in -\mathbb{N}_0$, then, for $n \in \mathbb{N}$, we have*

$$c_f(n) = 2\pi i^k \sum_{m < 0} c_f(m) \left(\frac{|m|}{n}\right)^{\frac{1-k}{2}} \sum_{c=1}^{\infty} \frac{K_k(m, n; c)}{c} I_{1-k}\left(\frac{4\pi\sqrt{|mn|}}{c}\right).$$

In particular, as $n \rightarrow \infty$, we have

$$c_f(n) \sim c_f(n_0) \frac{|n_0|^{\frac{1}{4} - \frac{k}{2}}}{2\sqrt{2}\pi} n^{\frac{k}{2} - \frac{3}{4}} e^{4\pi\sqrt{|n_0|n}},$$

where n_0 is the smallest negative integer with $c_f(n_0) \neq 0$.

To describe an example involving false theta functions, we recall that a finite sequence of positive integers $\{a_j\}_{j=1}^s$ is called a *unimodal sequence of size n* if there exists $k \in \mathbb{N}$ such that $a_1 \leq a_2 \leq \dots \leq a_k \geq a_{k+1} \geq \dots \geq a_s$ and $a_1 + \dots + a_s = n$. Let $u(n)$ denote the number of unimodal sequences of size n . Then (see, for example, [3])

$$U(q) := \sum_{n=0}^{\infty} u(n)q^n = \frac{1}{(q; q)_{\infty}^2} \sum_{n=1}^{\infty} (-1)^{n+1} q^{\frac{n(n+1)}{2}}.$$

Note that we may write

$$U(q) = \frac{i}{2} q^{-\frac{1}{24}} \frac{\psi(\tau)}{\eta(\tau)^2} + \frac{q^{\frac{1}{12}}}{\eta(\tau)^2},$$

where $\psi(\tau) := \psi(0; \tau)$ (with $\psi(z; \tau)$ defined in (3.1)). The asymptotic main term of $u(n)$ was determined by Auluck [3] as

$$u(n) = \frac{1}{8 \cdot 3^{\frac{3}{4}} n^{\frac{5}{4}}} e^{2\pi\sqrt{\frac{n}{3}}} \left(1 + O\left(n^{-\frac{1}{2}}\right)\right). \tag{4.4}$$

This result was then generalized by Wright [40], who gave the asymptotic expansion to all orders of n for the leading exponential term. Using the modularity of $U(q)$, Nazaroglu and the author proved an exact formula for $u(n)$ [13]. Our expression is analogous to the exact formula in Theorem 4.1. To state the exact formula for $u(n)$, we define for $n, r \in \mathbb{Z}$ and $k \in \mathbb{N}$ the Kloosterman sums

$$K_k(n) := i \sum_{\substack{0 \leq h < k \\ \gcd(h, k) = 1}} \nu_{\eta}(M_{h, k})^2 \zeta_{12k}^{-(12n-1)h-h'},$$

$$K_k(n, r) := e^{\frac{3\pi i}{4}} (-1)^r \sum_{\substack{0 \leq h < k \\ \gcd(h, k) = 1}} \nu_{\eta}(M_{h, k})^{-1} \zeta_{24k}^{-(24n+1)h + (12r^2 + 12r + 1)h'},$$

where h' is a solution of $hh' \equiv -1 \pmod{k}$, $M_{h,k} := \begin{pmatrix} h' & -\frac{hh'+1}{k} \\ k & -h' \end{pmatrix}$, $\zeta_\ell := e^{\frac{2\pi i}{\ell}}$ for $\ell \in \mathbb{N}$ and ν_η is the multiplier system for η . In particular, for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c > 0$ it is given by (see [2, Theorem 3.4])

$$\nu_\eta(M) := \exp\left(\pi i \left(\frac{a+d}{12c} - \frac{1}{4} + s(-d, c)\right)\right), \quad \text{where } s(h, k) := \sum_{r=1}^{k-1} \frac{r}{k} \left(\frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor - \frac{1}{2}\right).$$

We then have the following expression for $u(n)$.

THEOREM 4.3. *We have*

$$\begin{aligned} u(n) &= \frac{2\pi}{12n-1} \sum_{k \geq 1} \frac{K_k(n)}{k} I_2\left(\frac{\pi}{3k} \sqrt{12n-1}\right) - \frac{\pi}{2^{\frac{3}{4}} \sqrt{3} (24n+1)^{\frac{3}{4}}} \sum_{k \geq 1} \sum_{r \pmod{2k}} \frac{K_k(n, r)}{k^2} \\ &\quad \times \int_{-1}^1 (1-x^2)^{\frac{3}{4}} \cot\left(\frac{\pi}{2k} \left(\frac{x}{\sqrt{6}} - r - \frac{1}{2}\right)\right) I_{\frac{3}{2}}\left(\frac{\pi}{3\sqrt{2}k} \sqrt{(24n+1)(1-x^2)}\right) dx. \end{aligned}$$

Sketch of proof. Define for $\varrho \in \mathbb{Q}$

$$f(\tau) := -\frac{i}{2} \frac{\psi(\tau)}{\eta(\tau)^2}, \quad g(\tau) := \frac{1}{\eta(\tau)^2}, \quad \text{and } \mathcal{E}_\varrho(\tau) := \int_\varrho^{\tau+i\infty+\varepsilon} \frac{\eta(\mathfrak{z})^3}{\sqrt{i(\mathfrak{z}-\tau)}} d\mathfrak{z},$$

where the integration path avoids the branch-cut. To apply the Circle Method we first determine the “false” modular behavior of f . Using Theorem 3.1, one may show that \mathcal{E}_ϱ , for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ with $c > 0$,

$$f(\tau) = e^{\frac{\pi i}{4}} \nu_\eta(M)^{-1} \sqrt{-i(c\tau+d)} \left(f\left(\frac{a\tau+b}{c\tau+d}\right) - \frac{1}{2} g\left(\frac{a\tau+b}{c\tau+d}\right) \mathcal{E}_\varrho^{\frac{a}{c}}\left(\frac{a\tau+b}{c\tau+d}\right) \right).$$

We next rewrite the error integrals \mathcal{E}_ϱ as “Mordell-type integrals.” One can prove that, for $V \in \mathbb{C}$ with $\text{Re}(V) > 0$,

$$\mathcal{E}_\varrho(\varrho + iV) = -\frac{i}{\pi} \sum_{n \in \mathbb{Z}} (-1)^n e^{\pi i(n+\frac{1}{2})^2 \varrho} \lim_{\varepsilon \rightarrow 0^+} \int_{-\infty}^{\infty} \frac{e^{-\pi V x^2}}{x - (n+\frac{1}{2})(1+i\varepsilon)} dx.$$

Write $\varrho = \frac{h'}{k}$ where h', k are integers satisfying $\text{gcd}(h', k) = 1$ and $k > 0$. Given a real number d with $0 \leq d < \frac{1}{8}$ we split

$$e^{2\pi d V} \mathcal{E}_{\frac{h'}{k}}\left(\frac{h'}{k} + iV\right) = \mathcal{E}_{\frac{h'}{k}, d}^*\left(\frac{h'}{k} + iV\right) + \mathcal{E}_{\frac{h'}{k}, d}^e\left(\frac{h'}{k} + iV\right),$$

where

$$\begin{aligned} \mathcal{E}_{\frac{h'}{k}, d}^*\left(\frac{h'}{k} + iV\right) &:= \frac{e^{2\pi d V}}{\pi i} \sum_{n \in \mathbb{Z}} (-1)^n e^{\pi i(n+\frac{1}{2})^2 \frac{h'}{k}} \lim_{\varepsilon \rightarrow 0^+} \int_{-\sqrt{2d}}^{\sqrt{2d}} \frac{e^{-\pi V x^2}}{x - (n+\frac{1}{2})(1+i\varepsilon)} dx, \\ \mathcal{E}_{\frac{h'}{k}, d}^e\left(\frac{h'}{k} + iV\right) &:= \frac{e^{2\pi d V}}{\pi i} \sum_{n \in \mathbb{Z}} (-1)^n e^{\pi i(n+\frac{1}{2})^2 \frac{h'}{k}} \lim_{\varepsilon \rightarrow 0^+} \int_{|x| \geq \sqrt{2d}} \frac{e^{-\pi V x^2}}{x - (n+\frac{1}{2})(1+i\varepsilon)} dx. \end{aligned}$$

One can show that for $0 \leq d < \frac{1}{8}$, $h', k \in \mathbb{Z}$, $\text{gcd}(h', k) = 1$, $k > 0$, and $\text{Re}(V) \geq 1$ we have

$$\mathcal{E}_{\frac{h'}{k}, d}^e\left(\frac{h'}{k} + iV\right) = O(\log(k)),$$

where the bound is independent of h' and V . One can then use the Circle Method; the main difficulty is that one has to approximate the additional integrals $\mathcal{E}_{\frac{h'}{d},d}^*$. \square

Using (4.3), the contribution from $k = 1$ in Theorem 4.3 gives the asymptotic main term.

COROLLARY 4.4. *The asymptotic (4.4) holds.*

5. Quantum modular forms

Another consequence of our completions is the proof of quantum modularity of false theta functions. Recall that a *quantum modular form* of weight $k \in \frac{1}{2}\mathbb{Z}$ is a function $f : \mathbb{P}^1(\mathbb{Q}) \setminus S \rightarrow \mathbb{C}$ for some discrete subsets S of \mathbb{Q} such that for all $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ (some congruence subgroup of $\text{SL}_2(\mathbb{Z})$), the function

$$f(x) - \chi^{-1}(M)(cx + d)^{-k} f\left(\frac{ax + b}{cx + d}\right)$$

for certain multipliers χ , satisfies a suitable property of continuity or analyticity in \mathbb{R} .

For simplicity we consider a special family studied by Milas and the author [12]. Define, for $j \in \mathbb{Z}$ and $N \in \mathbb{N}_{>1}$,

$$F_{j,N}(\tau) := \sum_{\substack{n \in \mathbb{Z} \\ n \equiv j \pmod{2N}}} \text{sgn}(n) q^{\frac{n^2}{4N}}.$$

In fact, we can further restrict to $1 \leq j \leq N - 1$, because $F_{j,N} = -F_{-j,N}$ and $F_{j+2N,N} = F_{j,N}$. We first describe the classical proof of quantum modularity. In [12] (following [42]) it was shown (using the Euler–Maclaurin summation formula) that $F_{j,N}$ has an asymptotic expansion of the shape

$$F_{j,N}\left(it + \frac{h}{k}\right) \sim \sum_{m \geq 0} a_{h,k}(m) t^m \quad (t \rightarrow 0^+).$$

Now one can find the following “companion” for $F_{j,N}$:

$$F_{j,N}^*(\tau) := \frac{1}{\sqrt{\pi}} \sum_{\substack{n \in \mathbb{Z} \\ n \equiv j \pmod{2N}}} \text{sgn}(n) \Gamma\left(\frac{1}{2}, \frac{\pi n^2 \tau_2}{N}\right) q^{-\frac{n^2}{4N}}.$$

To be more precise

$$F_{j,N}^*\left(it - \frac{h}{k}\right) \sim \sum_{m \geq 0} a_{h,k}(m) (-t)^m \quad (t \rightarrow 0^+).$$

Using the weight $\frac{3}{2}$ unary theta functions

$$f_{j,N}(\tau) := \frac{1}{2N} \sum_{\substack{n \in \mathbb{Z} \\ n \equiv j \pmod{2N}}} n q^{\frac{n^2}{4N}},$$

one can write

$$F_{j,N}^*(\tau) = -i\sqrt{2N} \int_{-\bar{\tau}}^{i\infty} \frac{f_{j,N}(w)}{\sqrt{-i(w + \tau)}} dw.$$

Quantum modularity now follows from the modularity properties of $f_{j,N}$ (which is implied from (2.1)). In particular, we have

$$f_{j,N}(\tau) = \sqrt{\frac{2}{N}}(-i\tau)^{-\frac{3}{2}} \sum_{k=1}^{N-1} \sin\left(\frac{\pi jk}{N}\right) f_{k,N}\left(-\frac{1}{\tau}\right).$$

From this one may conclude that

$$F_{j,N}^*(\tau) - \frac{1}{\sqrt{-i\tau}} \sqrt{\frac{2}{N}} \sum_{k=1}^{N-1} \sin\left(\frac{\pi jk}{N}\right) F_{k,N}^*\left(-\frac{1}{\tau}\right) = i\sqrt{2} \int_0^{i\infty} \frac{f_{j,N}(w)}{\sqrt{-i(w+\tau)}} dw.$$

In [7] a two-dimensional example was considered. The proof was, however, very technical and the hard part was to find the corresponding companion. Therefore we developed a more systematic approach in [13] which used the transformation behavior of false theta functions on the upper-half plane. To state this, define for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$,

$$\psi_{j,r}(M) := \begin{cases} e^{2\pi i ab \frac{j^2}{4N}} e^{-\frac{\pi i}{4}(1-\text{sgn}(d))} \delta_{j,r} & \text{if } c = 0, \\ e^{-\frac{3\pi i}{4} \text{sgn}(c)} \sqrt{\frac{2}{N|c|}} \sum_{k=0}^{|c|-1} e^{\frac{\pi i}{2Nc}(a(2Nk+j)^2 + dr^2)} \sin\left(\frac{\pi r(2Nk+j)}{N|c|}\right) & \text{if } c \neq 0, \end{cases}$$

where $\delta_{j,r} = 1$ if $j = r$ and 0 otherwise. Using Theorem 3.1, one can show the following theorem.

THEOREM 5.1. *For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$, we have*

$$F_{j,N}(\tau) - \text{sgn}(c\tau_1 + d)(c\tau + d)^{-\frac{1}{2}} \sum_{r=1}^{N-1} \psi_{j,r}(M^{-1}) F_{r,N}\left(\frac{a\tau + b}{c\tau + d}\right) = -i\sqrt{2N} \int_{-\frac{d}{c}}^{i\infty} \frac{f_{j,N}(\mathfrak{z})}{\sqrt{-i(\mathfrak{z} - \tau)}} d\mathfrak{z},$$

where the integration path avoids the branch cut defined by $\sqrt{-i(\mathfrak{z} - \tau)}$.

As a corollary we obtain quantum modular properties of $F_{j,N}$.

COROLLARY 5.2. *The functions $F_{j,N}$ are vector-valued quantum modular forms with quantum set \mathbb{Q} .*

6. Fourier coefficients of meromorphic Jacobi forms

Recall the theta decomposition for holomorphic Jacobi forms given in (2.7). This implies that Fourier coefficients of holomorphic Jacobi forms are modular forms. If ϕ has poles in the elliptic variable, the story becomes more complicated. In this case, the Fourier coefficients (see (6.3)) depend on the choice of range of z and are not modular. Such coefficients showed up in the study of the mock theta functions of Ramanujan in [46]. Meromorphic Jacobi forms also played a key role in the study of Kac–Wakimoto characters [31]. Kac and Wakimoto asked for modularity properties of these characters; this question was answered in [6, 34], when the meromorphic Jacobi form has positive index.

In [5], Creutzig, Rolin, and the author considered Kac–Wakimoto characters for negative index which are, for $M, N \in \mathbb{N}_0$ and after a change of variables, given by

$$\phi_{M,N}(z) = \phi_{M,N}(z; \tau) := \frac{\vartheta\left(z + \frac{1}{2}; \tau\right)^M}{\vartheta(z; \tau)^N}.$$

In [15], we offered a completely general picture for negative index Jacobi forms. To describe our results, we let $m \in -\frac{1}{2}\mathbb{N}$ and $\varepsilon \in \{0, 1\}$, and consider meromorphic functions $\phi : \mathbb{C} \rightarrow \mathbb{C}$ that satisfy the elliptic transformation law (extending (2.3) and suppressing τ)

$$\phi(z + \lambda\tau + \mu) = (-1)^{2m\mu + \varepsilon\lambda} e^{-2\pi im(\lambda^2\tau + 2\lambda z)} \phi(z). \tag{6.1}$$

For example, $\phi_{M,N}$ transforms according to (6.1) with $m = \frac{M-N}{2}$ and $\varepsilon = \varepsilon(N) \in \{0, 1\}$ with $\varepsilon(N) \equiv N \pmod{2}$. Note that a Jacobi form also satisfies a modular transformation law (in the suppressed variable τ), but for our main result, only (6.1) has to be assumed.

We now define $\mathcal{D}_x := \frac{1}{2\pi i} \frac{\partial}{\partial x}$ for a general variable x , and consider the level $2M$ Appell–Lerch sum given for $M \in \frac{1}{2}\mathbb{N}$ and $z_1, z_2 \in \mathbb{C}$ by $(\zeta_j := e^{2\pi iz_j})^\dagger$

$$F_{M,\varepsilon}(z_1, z_2) = F_{M,\varepsilon}(z_1, z_2; \tau) := (\zeta_1 \zeta_2^{-1})^M \sum_{n \in \mathbb{Z}} \frac{(-1)^{\varepsilon n} \zeta_2^{-2Mn} q^{Mn(n+1)}}{1 - \zeta_1 \zeta_2^{-1} q^n}.$$

We have the elliptic transformation property

$$F_{M,\varepsilon}(z_1, z_2 + \lambda\tau + \mu) = (-1)^{2M\mu + \varepsilon\lambda} e^{-2\pi iM(\lambda^2\tau + 2\lambda z_2)} F_{M,\varepsilon}(z_1, z_2), \tag{6.2}$$

for all $\lambda, \mu \in \mathbb{Z}$. Furthermore, $z_2 \mapsto F_{M,\varepsilon}(z_1, z_2)$ is a meromorphic function having only simple poles in $\mathbb{Z}\tau + \mathbb{Z} + z$ and residue $\frac{1}{2\pi i}$ in $z_2 = z$. Let $D_{j,w} = D_{j,w}(\tau)$ be the $-j$ th Laurent coefficient of ϕ around $z_1 = w$, $s_{z_0,\tau}$ gives the locations of a set of representatives of the poles of ϕ , and P_{z_0} is a fundamental parallelogram for the lattice $\mathbb{Z}\tau + \mathbb{Z}$. Further note that in the following theorem, although the dependence on τ is suppressed, both sides of (6.1) depend on τ .

THEOREM 6.1. *Let $m \in -\frac{1}{2}\mathbb{N}$ and $\varepsilon \in \{0, 1\}$, and suppose that z_0 is chosen so that ϕ has no poles on ∂P_{z_0} . If ϕ is a meromorphic function satisfying (6.1), then*

$$\phi(z) = - \sum_{z_1 \in s_{z_0,\tau}} \sum_{n \in \mathbb{N}} \frac{D_{n,z_1}}{(n-1)!} [\mathcal{D}_{z_2}^{n-1} (F_{-m,\varepsilon}(z, z_2))]_{z_2=z_1}.$$

REMARK. As ϕ is a meromorphic function, there are only finitely many non-zero terms in the sum over n in the right-hand side in Theorem 6.1.

Proof of Theorem 6.1. Let $z \in \mathbb{C}$ be such that ϕ is holomorphic in z . Furthermore let $z_0 \in \mathbb{C}$ be such that without loss of generality $z \in P_{z_0}$ and that ϕ has no poles on the boundary of P_{z_0} . We consider the integral

$$\int_{\partial P_{z_0}} \phi(w) F_{-m,\varepsilon}(z, w) dw,$$

which we compute in two different ways: on the one hand, the integral vanishes since by equations (6.1) and (6.2) the integrand is both 1- and τ -periodic. On the other hand, by the Residue Theorem,

$$\int_{\partial P_{z_0}} \phi(w) F_{-m,\varepsilon}(z, w) dw = \phi(z) + 2\pi i \sum_{z_1 \in s_{z_0,\tau}} \operatorname{Res}_{z_2=z_1} (\phi(z_2) F_{-m,\varepsilon}(z, z_2))$$

and thus, we get

$$\phi(z) = -2\pi i \sum_{z_1 \in s_{z_0,\tau}} \operatorname{Res}_{z_2=z_1} (\phi(z_2) F_{-m,\varepsilon}(z, z_2)).$$

[†]Note that now $z_1, z_2 \in \mathbb{C}$ and are not to be confused with the real and imaginary part of z in the previous sections.

The theorem now follows immediately by inserting the definition of the Laurent coefficients D_{n,z_1} . □

Applying this result to the Kac–Wakimoto characters $\phi_{M,N}$ yields the following.

COROLLARY 6.2. *For $M \in \mathbb{N}_0$ and $N \in \mathbb{N}$ with $M < N$, we have the decomposition*

$$\phi_{M,N}(z) = - \sum_{n=1}^N \frac{D_{n,0}}{(n-1)!} \left[\mathcal{D}_{z_2}^{n-1} \left(F_{\frac{N-M}{2}, \varepsilon(N)}(z, z_2) \right) \right]_{z_2=0}.$$

We next find an explicit description for the Fourier coefficients of meromorphic Jacobi forms of negative index. For this, define for $z_0 \in \mathbb{C}$ and ϕ a function satisfying the transformation in (6.1) with $m \in \frac{1}{2}\mathbb{Z}$ and $\varepsilon \in \{0, 1\}$, the (slightly modified) Fourier coefficients by

$$h_\ell(\tau) = h_{\ell,z_0}(\tau) := q^{-\frac{\ell^2}{4m}} \int_{z_0}^{z_0+1} \phi(z; \tau) e^{-2\pi i \ell z} dz, \tag{6.3}$$

where $\ell \in \mathbb{Z} + m$. Here, the path of integration is the straight line connecting z_0 and $z_0 + 1$ if there are no poles on this line. If there is a pole on the line which is not an endpoint, then we define the path to be the average of the paths deformed to pass above and below the pole. Finally, if there is a pole at an endpoint, note that the integral (6.3) only depends on the imaginary part of z_0 . Then we replace the path $[z_0, z_0 + 1]$ with $[z_0 - \delta, z_0 + 1 - \delta]$ for $\delta > 0$ sufficiently small such that there is no pole at an endpoint, and then define the integral as above if there is a pole in the interior of the line. We also require the following partial theta functions defined for $z \in \mathbb{C}$, $\tau \in \mathbb{H}$, $M \in \frac{1}{2}\mathbb{N}$, and $\ell \in \mathbb{Z} + M$ by

$$\vartheta_{\ell,\varepsilon,M}^+(z) = \vartheta_{\ell,\varepsilon,M}^+(z; \tau) := \sum_{n=0}^{\infty} (-1)^{n\varepsilon} q^{\frac{(2Mn-\ell)^2}{4M}} \zeta^{2Mn-\ell}.$$

REMARK. Note that

$$\vartheta_{\ell,\varepsilon,M}^+(z) = \frac{1}{2} \sum_{n \in \mathbb{Z}} \left(\operatorname{sgn} \left(n + \frac{1}{2} \right) + 1 \right) (-1)^{n\varepsilon} q^{\frac{(2Mn-\ell)^2}{4M}} \zeta^{2Mn-\ell}.$$

So we may write $\vartheta_{\ell,\varepsilon,M}^+$ as sum of a false Jacobi theta function and an ordinary Jacobi theta function.

THEOREM 6.3. *Let $m \in -\frac{1}{2}\mathbb{N}$ and ϕ be a meromorphic function satisfying (6.1) with $\varepsilon \in \{0, 1\}$. If $z_0 \in \mathbb{C}$ is chosen so that ϕ has no poles on ∂P_{z_0} , then we have for any $\ell \in \mathbb{Z} + m$ that*

$$h_{\ell,z_0}(\tau) = \sum_{z_1 \in S_{z_0,\tau}} \sum_{n \geq 1} \frac{D_{n,z_1}(\tau)}{(n-1)!} \left[\mathcal{D}_z^{n-1} \left(\vartheta_{\ell,\varepsilon,-m}^+(z; \tau) \right) \right]_{z=z_1}.$$

REMARK. Proceeding as in Section 3 (and turning \mathcal{D}_z into certain invariant operators) one can find completions of the Fourier coefficients $h_{\ell,z_0}(\tau)$.

Before giving the proof of Theorem 6.3, we require the following properties of the partial theta functions, which follow from a direct calculation.

LEMMA 6.4. (1) *For $\lambda, \mu \in \mathbb{Z}$, $\ell \in \frac{1}{2}\mathbb{Z}$, and $M \in \frac{1}{2}\mathbb{N}$, we have*

$$(-1)^{2\ell\mu} q^{M\lambda^2} \zeta^{2M\lambda} \vartheta_{\ell,\varepsilon,M}^+(z + \lambda\tau + \mu) = \vartheta_{\ell-2M\lambda,\varepsilon,M}^+(z).$$

(2) We have

$$\vartheta_{\ell,\varepsilon,M}^+(z) - (-1)^\varepsilon q^M \zeta^{2M} \vartheta_{\ell,\varepsilon,M}^+(z + \tau) = q^{\frac{\ell^2}{4M}} \zeta^{-\ell}.$$

Proof of Theorem 6.3. By the Residue Theorem, we have

$$\int_{\partial P_{z_0}} \phi(w) \vartheta_{\ell,\varepsilon,-m}^+(w) dw = 2\pi i \sum_{w \in s_{z_0,\tau}} \operatorname{Res}_{z=w} \left(\phi(z) \vartheta_{\ell,\varepsilon,-m}^+(z) \right). \tag{6.4}$$

On the other hand, we can compute the integral directly. Since $\phi \vartheta_{\ell,\varepsilon,-m}^+$ is one-periodic, we find, using Lemma 6.4, that

$$\begin{aligned} \int_{\partial P_{z_0}} \phi(w) \vartheta_{\ell,\varepsilon,-m}^+(w) dw &= \int_{z_0}^{z_0+1} \phi(w) \vartheta_{\ell,\varepsilon,-m}^+(w) dw - \int_{z_0+\tau}^{z_0+\tau+1} \phi(w) \vartheta_{\ell,\varepsilon,-m}^+(w) dw \\ &= \int_{z_0}^{z_0+1} (\phi(w) \vartheta_{\ell,\varepsilon,-m}^+(w) - \phi(w + \tau) \vartheta_{\ell,\varepsilon,-m}^+(w + \tau)) dw \\ &= \int_{z_0}^{z_0+1} \phi(w) (\vartheta_{\ell,\varepsilon,-m}^+(w) - (-1)^\varepsilon e^{-2\pi i m(\tau+2w)} \vartheta_{\ell,\varepsilon,-m}^+(w + \tau)) dw \\ &= e^{-\frac{\pi i \ell^2 \tau}{2m}} \int_{z_0}^{z_0+1} \phi(w) e^{-2\pi i \ell w} dw = h_{\ell,z_0}(\tau). \end{aligned}$$

Comparing with (6.4) and inserting the definition of the Laurent coefficients of ϕ give the claim. □

7. Higher dimensional false theta functions

We next turn to higher dimensional false theta functions. In [8], we expressed under quite general conditions, rank two false theta functions as iterated, holomorphic, Eichler-type integrals. This provided a new method for examining their modular properties and we applied it in a variety of situations where rank two false theta functions arose.

A key step in our proof is the following sign-lemma.

LEMMA 7.1. For $\ell_1, \ell_2 \in \mathbb{R}, \kappa \in \mathbb{R}$, with $(\ell_1, \ell_2 + \kappa \ell_1) \neq (0, 0)$, we have

$$\begin{aligned} \operatorname{sgn}(\ell_1) \operatorname{sgn}(\ell_2 + \kappa \ell_1) q^{\frac{\ell_1^2}{2} + \frac{\ell_2^2}{2}} &= \int_{\tau}^{\tau+i\infty} \frac{\ell_1 e^{\pi i \ell_1^2 w_1}}{\sqrt{i(w_1 - \tau)}} \int_{\tau}^{w_1} \frac{\ell_2 e^{\pi i \ell_2^2 w_2}}{\sqrt{i(w_2 - \tau)}} dw_2 dw_1 \\ &+ \int_{\tau}^{\tau+i\infty} \frac{m_1 e^{\pi i m_1^2 w_1}}{\sqrt{i(w_1 - \tau)}} \int_{\tau}^{w_1} \frac{m_2 e^{\pi i m_2^2 w_2}}{\sqrt{i(w_2 - \tau)}} dw_2 dw_1 + \frac{2}{\pi} \arctan(\kappa) q^{\frac{\ell_1^2}{2} + \frac{\ell_2^2}{2}}, \end{aligned}$$

where $m_1 := \frac{\ell_2 + \kappa \ell_1}{\sqrt{1 + \kappa^2}}$ and $m_2 := \frac{\ell_1 - \kappa \ell_2}{\sqrt{1 + \kappa^2}}$.

Summing over a shifted lattice then yields two-dimensional Eichler integrals.

We next explain natural occurrences of (higher dimensional) false theta functions. As already seen in Section 6 (for the one-dimensional case), a rich source for false theta functions is through the Fourier coefficients of meromorphic Jacobi forms with negative index or their multivariable generalizations [10, 15]. In vertex algebra theory, important examples of meromorphic Jacobi forms come from characters of irreducible modules for the simple vertex operator algebra $V_k(\mathfrak{g})$ at an admissible level k . At a boundary admissible level [30], these characters admit a particularly elegant infinite product form. Modular properties of their Fourier coefficients are

understood only for $\mathfrak{g} = \mathfrak{sl}_2$ and $V_{-\frac{3}{2}}(\mathfrak{sl}_3)$. For the latter, the Fourier coefficients are essentially rank two false theta functions (see [10] for more details). On the other extreme, if the level is generic, the character of $V_k(\mathfrak{g})$ is given by

$$\text{ch}[V_k(\mathfrak{g})](\zeta; q) = \frac{q^{-\frac{\dim(\mathfrak{g})k}{24(k+h^\vee)}}}{(q; q)_\infty^n \prod_{\alpha \in \Delta_+} (\zeta^\alpha q; q)_\infty \prod_{\alpha \in \Delta_+} (\zeta^{-\alpha} q; q)_\infty}, \tag{7.1}$$

where h^\vee is the dual Coxeter number, and ζ are variables parametrizing the set of positive roots Δ_+ of \mathfrak{g} . Although (7.1) is not a Jacobi form, a slight modification in the Weyl denominator gives a Jacobi form of negative index. The Fourier coefficients of (7.1) are important because they are essentially characters for the parafermion vertex algebra $N_k(\mathfrak{g})$ [21, 22, 29], whose character is given by

$$(q; q)_\infty^n \text{CT}_{[\zeta]}(\text{ch}[V_k(\mathfrak{g})](\zeta; q)). \tag{7.2}$$

This character can be expressed as a linear combination of coefficients of Jacobi forms. As one example we studied modular properties of (7.2) for types A_2 and B_2 , which leads us to the following result; note that a more precise version of this result was given in [10].

THEOREM 7.2. *Characters of the parafermion vertex algebras of type A_2 and B_2 can be written as linear combinations of (quasi)-modular forms and false theta functions of rank one and two. The rank two pieces in these decompositions can be written as iterated, holomorphic, Eichler-type integrals, which yields the modular transformation properties of these functions.*

Meromorphic Jacobi forms closely related to characters of affine Lie algebras at boundary admissible levels also show up in the computation of the Schur index $\mathcal{I}(q)$ of $4d \mathcal{N} = 2$ SCFTs [4, 16]. If refined by flavor symmetries, the Schur index is denoted by $\mathcal{I}(q, z_1, \dots, z_n)$. In [10], we were only interested in the Schur index of some specific superconformal field theorem (SCFTs), called Argyres–Douglas theories of type (A_1, D_{2k+2}) , whose index with two flavors was first computed in [16] (see also [19]) and later identified with certain vertex algebra characters in [20]. In particular, for $k = 1$ the index coincides with the character of the aforementioned vertex algebra $V_{-\frac{3}{2}}(\mathfrak{sl}_3)$. Our second main result in [8] dealt with modularity of Fourier coefficients of these indices.

THEOREM 7.3. *The Fourier coefficients of the Schur indices of Argyres–Douglas theories of type (A_1, D_{2k+2}) are essentially rank two false theta functions. Moreover, its constant term can be expressed as a double Eichler-type integral.*

The third main result in [8] concerned the \hat{Z} -invariants called homological blocks of plumbed 3-invariants introduced recently by Gukov, Pei, Putrov, and Vafa [27] and further studied from several viewpoints in [11, 17, 24, 26, 27, 32, 35]. For Seifert homology spheres, it is well known that they can be expressed as linear combinations of derivatives of unary false theta functions. Further computations of \hat{Z} -invariants for certain non-Seifert integral homology spheres were given in [11].

THEOREM 7.4. *Let M be a plumbed 3-manifold obtained from a unimodular \mathbb{H} -graph as in [11]. Then the \hat{Z} -invariant of M has a representation*

$$\hat{Z}(\tau) = \int_\tau^{\tau+i\infty} \int_\tau^{w_1} \frac{\Theta_1(w_1, w_2)}{\sqrt{i(w_1 - \tau)}\sqrt{i(w_2 - \tau)}} dw_2 dw_1 + \Theta_2(\tau),$$

where $\Theta_1(w_1, w_2)$ is a linear combination of products of derivatives of unary theta functions in w_1 and w_2 and $\Theta_2(\tau)$ is a rank two theta function. Moreover, there is a completion $\widehat{Z}(\tau, w)$ of \widehat{Z} that transforms like a weight one modular form[†].

Currently, in [9] we are working on building a theory of general false theta functions; this is work in progress.

Acknowledgement. The author thanks Walter Bridges, Caner Nazaroglu, and the referee for helpful comments on an earlier version of this paper.

References

1. A. ANDRIANOV, ‘Modular descent and the Saito–Kurokawa conjecture’, *Invent. Math.* 53 (1979) 267–280.
2. T. APOSTOL, *Modular functions and Dirichlet series in number theory*, Graduate Texts in Mathematics 41 (Springer, New York, NY, 1990).
3. F. AULUCK, ‘On some new types of partitions associated with generalized Ferrers graphs’, *Math. Proc. Cambridge Philos. Soc.* 47 (1951) 679–686.
4. C. BEEM, M. LEMOS, P. LIENDO, W. PEELAERS, L. RASTELLI and B. VAN REES, ‘Infinite chiral symmetry in four dimensions’, *Commun. Math. Phys.* 336 (2015) 1359–1433.
5. K. BRINGMANN, T. CREUTZIG and L. ROLEN, ‘Negative index Jacobi forms and quantum modular forms’, *Res. Math. Sci.* 1 (2014).
6. K. BRINGMANN and A. FOLSOM, ‘Almost harmonic Maass forms and Kac–Wakimoto characters’, *J. Reine Angew. Math.* 694 (2014) 179–202.
7. K. BRINGMANN, J. KASZIAN and A. MILAS, ‘Higher depth quantum modular forms, multiple Eichler integrals, and \mathfrak{sl}_3 false theta functions’, *Res. Math. Sci.* 6 (2019) 1–41.
8. K. BRINGMANN, J. KASZIAN, A. MILAS and C. NAZAROGLU, ‘Integral representations of rank two false theta functions and their modularity properties’, submitted for publication.
9. K. BRINGMANN, J. KASZIAN, A. MILAS and C. NAZAROGLU, ‘Higher depth false modular forms’, in preparation.
10. K. BRINGMANN, J. KASZIAN, A. MILAS and S. ZWEGERS, ‘Rank two false theta functions and Jacobi forms of negative definite matrix index’, *Adv. Appl. Math.* 112 (2020) 101946.
11. K. BRINGMANN, K. MAHLBURG and A. MILAS, ‘Higher depth quantum modular forms and plumbed 3-manifolds’, *Lett. Math. Phys.* 110 (2020) 2675–2702.
12. K. BRINGMANN and A. MILAS, ‘ W -algebras, false theta functions, and quantum modular forms, I’, *Int. Math. Res. Not.* 21 (2015) 11351–11387.
13. K. BRINGMANN and C. NAZAROGLU, ‘A framework for modular properties of false theta functions’, *Res. Math. Sci.* 6 (2019) 1–23.
14. K. BRINGMANN and K. ONO, ‘Dyson’s rank and Maass forms’, *Ann. Math.* 171 (2010) 419–449.
15. K. BRINGMANN, L. ROLEN and S. ZWEGERS, ‘On the Fourier coefficients of negative index meromorphic Jacobi forms’, *Res. Math. Sci.* 3 (2016) 1–9.
16. M. BUICAN and T. NISHINAKA, ‘On the superconformal index of Argyres–Douglas theories’, *J. Phys. A* 49 (2016) 015401.
17. M. CHENG, S. CHUN, F. FERRARI, S. GUKOV and S. HARRISON, ‘3d modularity’, *J. High Energy Phys.* 10 (2019) 10.
18. M. CHENG, J. DUNCAN and J. HARVEY, ‘Umbral moonshine and the Niemeier lattices’, *Res. Math. Sci.* 1 (2014) 1–81.
19. C. CORDOVA and S.-H. SHAO, ‘Schur indices, BPS particles, and Argyres–Douglas theories’, *J. High Energy Phys.* 1 (2016) 40.
20. T. CREUTZIG, ‘ W -algebras for Argyres–Douglas theories’, *Eur. J. Math.* 3 (2017) 659–690.
21. C. DONG and J. LEPOWSKY, *Generalized vertex algebras and relative vertex operators*, Vol. 112 (Springer, Cham, 2012).
22. C. DONG and Q. WANG, ‘The structure of parafermion vertex operator algebras: general case’, *Commun. Math. Phys.* 299 (2010) 783–792.
23. M. EICHLER and D. ZAGIER, *The theory of Jacobi forms*, Progress in Mathematics 55 (Birkhäuser Boston, Inc., Boston, MA, 1985).
24. T. EKHOLM, A. GRUEN, S. GUKOV, P. KUCHARSKI, S. PARK and P. SULKOWSKI, ‘ \widehat{Z} at large N : from curve counts to quantum modularity’, Preprint, 2020, arXiv:2005.13349.
25. B. GROSS, W. KOHNEN and D. ZAGIER, ‘Heegner points and derivatives of L-series II’, *Math. Ann.* 278 (1987) 497–562.
26. S. GUKOV and C. MANOLESCU, ‘A two-variable series for knot complements’, Preprint, 2019, arXiv:1904.06057.

[†]In order to distinguish homological blocks from their modular completion, we use a smaller hat as in \widehat{Z} to denote homological blocks.

27. S. GUKOV, D. PEI, P. PUTROV and C. VAFA, ‘BPS spectra and 3-manifold invariants’, Preprint, 2017, arXiv:1701.06567.
28. G. HARDY and S. RAMANUJAN, ‘Asymptotic formulæ for the distribution of integers of various types’, *Proc. London Math. Soc.* (2) 16 (1917) 112–132.
29. V. KAC and D. PETERSON, ‘Infinite-dimensional Lie algebras, theta functions and modular forms’, *Adv. Math.* 53 (1984) 125–264.
30. V. KAC and M. WAKIMOTO, ‘A remark on boundary level admissible representations’, *C.R. Math.* 355 (2017) 128–132.
31. V. KAC and M. WAKIMOTO, ‘Integrable highest weight modules over affine superalgebras and Appell’s function’, *Commun. Math. Phys.* 215 (2001) 631–682.
32. P. KUCHARSKI, ‘ \hat{Z} invariants at rational τ ’, *J. High Energy Phys.* 9 (2019) 92.
33. R. LAWRENCE and D. ZAGIER, ‘Modular forms and quantum invariants of 3-manifolds’, *Asian J. Math.* 3 (1999) 93–107.
34. R. OLIVETTO, ‘On the Fourier coefficients of meromorphic Jacobi forms’, *Int. J. Number Theory* 10 (2014) 1519–1540.
35. S. PARK, ‘Higher Rank \hat{Z} and F_K ’, *Symmetry Integrability Geom. Methods Appl.* 16 (2020) 044.
36. I. PYATETSKII-SHAPIO, ‘On the Saito–Kurokawa lifting’, *Invent. Math.* 71 (1983) 309–338.
37. H. RADEMACHER, ‘On the partition function $p(n)$ ’, *Proc. London Math. Soc.* 43 (1937) 241–254.
38. H. RADEMACHER and H. ZUCKERMAN, ‘On the Fourier coefficients of certain modular forms of positive dimension’, *Ann. Math.* (2) 39 (1938) 433–462.
39. G. SHIMURA, ‘On modular forms of half-integral weight’, *Ann. Math.* 97 (1973) 440–481.
40. E. WRIGHT, ‘Stacks (II)’, *Q. J. Math. Oxford* (2) 22 (1971) 107–116.
41. D. ZAGIER, ‘Sur la conjecture de Saito–Kurokawa (d’après Maass)’, *Seminar on Number Theory*, Paris, 1979–80, 371–394, *Progress in Mathematics* 12 (Birkhäuser, Boston, MA, 1987).
42. D. ZAGIER, ‘Vassiliev invariants and a strange identity related to the Dedekind eta-function’, *Topology* 40 (2001) 945–960.
43. D. ZAGIER, *Quantum modular forms*, Clay Mathematics Proceedings 11 (American Mathematical Society, Providence, RI, 2010).
44. H. ZUCKERMAN, ‘On the coefficients of certain modular forms belonging to subgroups of the modular group’, *Trans Amer. Math. Soc.* 45 (1939) 298–321.
45. H. ZUCKERMAN, ‘On the expansions of certain modular forms of positive dimension’, *Amer. J. Math.* 62 (1940) 127–152.
46. S. ZWEGERS, Mock theta functions, PhD Thesis, Universiteit Utrecht, 2002.

Kathrin Bringmann
 Department of Mathematics and Computer Science
 University of Cologne
 Weyertal 86-90
 Cologne 50931
 Germany

kbringma@math.uni-koeln.de