



Conjectures of Sun About Sums of Polygonal Numbers

Kathrin Bringmann¹  · Ben Kane² 

Received: 15 March 2022 / Revised: 10 June 2022 / Accepted: 15 June 2022
© The Author(s) 2022

Abstract

In this paper, we consider representations of positive integers as sums of generalized m -gonal numbers, which extend the formula for the number of dots needed to make up a regular m -gon. We mainly restrict to the case where the sums contain at most four distinct generalized m -gonal numbers, with the second m -gonal number repeated twice, the third repeated four times, and the last is repeated eight times. For a number of small choices of m , Sun conjectured that every positive integer may be written in this form. By obtaining explicit quantitative bounds for Fourier coefficients related to theta functions which encode the number of such representations, we verify that Sun's conjecture is true for sufficiently large positive integers. Since there are only finitely many choices of m appearing in Sun's conjecture, this reduces Sun's conjecture to a verification of finitely many cases. Moreover, the bound beyond which we prove that Sun's conjecture holds is explicit.

Keywords Polygonal numbers · Theta functions · Modular forms · Quadratic forms

1 Introduction and Statement of Results

For $m \in \mathbb{N}_{\geq 3}$ and $\ell \in \mathbb{Z}$, let $p_m(\ell)$ be the ℓ -th (generalized) m -gonal number

$$p_m(\ell) := \frac{1}{2}(m-2)\ell^2 - \frac{1}{2}(m-4)\ell.$$

The first author has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (Grant Agreement No. 101001179). The research of the second author was supported by a grant from the Research Grants Council of the Hong Kong SAR, China (Project Number HKU 17303618).

✉ Kathrin Bringmann
kbringma@math.uni-koeln.de

¹ Department of Mathematics and Computer Science, Division of Mathematics, University of Cologne, Weyertal 86-90, 50931 Cologne, Germany

² Department of Mathematics, University of Hong Kong, Pokfulam, Hong Kong

For $\ell \in \mathbb{N}$, these count the number of dots contained in a regular polygon with m sides having ℓ dots on each side. For example, the special case $m = 3$ corresponds to triangular numbers, $m = 4$ gives squares, and $m = 5$ corresponds to pentagonal numbers. There are several conjectures related to sums of polygonal numbers. Specifically, for $\alpha \in \mathbb{N}^d$,¹ we are interested in the solvability of the Diophantine equation

$$\sum_{1 \leq j \leq d} \alpha_j p_m(\ell_j) = n \quad (1.1)$$

with $\ell_j \in \mathbb{N}_0$ or $\ell_j \in \mathbb{Z}$. We call such a sum *universal* if it is solvable for every $n \in \mathbb{N}$. Fermat stated (his claimed proof was not found in his writings) that every positive integer is the sum of three triangular number, four squares, five pentagonal numbers, and in general at most m m -gonal numbers. In other words, he claimed that the sum $\sum_{1 \leq j \leq m} p_m(\ell_j)$ is universal. His claim for squares ($m = 4$) was proven by Lagrange in 1770, the claim for triangular numbers ($m = 3$) was shown by Gauss in 1796, and the full conjecture was proven by Cauchy in 1813. Going in another direction, Ramanujan fixed $m = 4$ and conjectured a full list of choices of $\alpha \in \mathbb{N}^4$ for which the resulting sum is universal; this conjecture was later proven by Dickson [11]. Following this, the classification of universal quadratic forms was a central area of study throughout the twentieth century, culminating in the Conway–Schneeberger 15-theorem [3, 8] and the 290-theorem [4], which state that arbitrary quadratic forms whose cross terms are even (resp. are allowed to be odd) are universal if and only if they represent every integer up to 15 (resp. 290). Theorems of this type are now known as *finiteness theorems*. Namely, given an infinite set $S \subseteq \mathbb{N}$, one determines a finite subset S_0 of S such that a solution to (1.1) exists for every $n \in S$ if and only if it exists for every $n \in S_0$. Taking $S = \mathbb{N}$, one obtains a condition for universality of a given sum of polygonal numbers. For example, choosing $m = 3$ or $m = 6$, (1.1) is solvable with $\ell \in \mathbb{Z}^d$ for all $n \in \mathbb{N}$ if and only if it is solvable for every $n \leq 8$ [6], for $m = 5$ it is solvable with $\ell \in \mathbb{Z}^d$ for all $n \in \mathbb{N}$ if and only if it is solvable for every $n \leq 109$ [13], while it is solvable with $\ell \in \mathbb{N}_0^d$ for all $n \in \mathbb{N}$ if and only if it is solvable for every $n \leq 63$ [14] and for $m = 8$ it is solvable for $\ell \in \mathbb{Z}^d$ for all $n \in \mathbb{N}$ if and only if it is solvable for every $n \leq 60$ [15].

Here, we consider the question of universality in the case $\alpha = (1, 2, 4, 8)$ as one varies m . Specifically, we have the following conjecture of Sun (see [21, Conjecture 5.4]).

Conjecture 1.1 For $m \in \{7, 9, 10, 11, 12, 13, 14\}$ and $\ell \in \mathbb{Z}^4$, the equation

$$p_m(\ell_1) + 2p_m(\ell_2) + 4p_m(\ell_3) + 8p_m(\ell_4) = n \quad (1.2)$$

is solvable for every $n \in \mathbb{N}$.

Remark A proof of Conjecture 1.1 would give a classification of those m for which the sum (1.1) is universal in the case $\alpha = (1, 2, 4, 8)$. By direct computation, one sees that $p_m(\ell) \in \{0, 1\}$ or $p_m(\ell) \geq m - 3$. Using this, one obtains that (1.2) is not

¹ We denote vectors like α in bold and the j -th component of a vector α we write as α_j throughout.

solvable for $n = 16$ for every $m \geq 20$. For $m = 16, m = 17, m = 18,$ and $m = 19,$ one finds directly that there is no solution for $n = 29, n = 30, n = 16,$ and $n = 17,$ respectively. Moreover, for $m \in \{3, 6\},$ it is known by work of Liouville [18] that the sum is universal, for $m = 4,$ it was conjectured by Ramanujan and proven by Dickson [11] that the sum is universal, while for $m = 5$ (resp. $m = 8$), it was shown by Sun in [20] (resp. [21]) to be universal.

In this paper, we prove that Conjecture 1.1 is true for n sufficiently large.

Theorem 1.2 *For $m \in \{7, 9, 10, 11, 12, 13, 14\},$ there exists an explicit constant C_m (defined in Table 2) such that (1.2) is solvable with $\ell \in \mathbb{Z}^4$ for every $n \in \mathbb{N}_{\geq C_m},$ with the restriction that $n \not\equiv 4 \pmod{16}$ if $m = 12.$*

Remark To prove Conjecture 1.1 for $m = 12,$ it suffices to show that (1.2) holds for all $n \in \mathbb{N}$ with $n \not\equiv 4 \pmod{16}$ (see Lemma 5.1). Hence, the restriction in Theorem 1.2 is natural.

By completing the square, one easily sees that representations of integers as sums of polygonal numbers are closely related to sums of squares with congruence conditions. In particular, setting

$$r_{m,\alpha}(n) := \# \left\{ \ell \in \mathbb{Z}^4 : \sum_{1 \leq j \leq 4} \alpha_j p_m(\ell_j) = n \right\},$$

$$s_{r,M,\alpha}(n) := \# \left\{ \mathbf{x} \in \mathbb{Z}^4 : \sum_{1 \leq j \leq 4} \alpha_j x_j^2 = n, x_j \equiv r \pmod{M} \right\},$$

we have

$$r_{m,(1,2,4,8)}(n) = s_{m,2(m-2),(1,2,4,8)} \left(8(m-2)n + 15(m-4)^2 \right). \tag{1.3}$$

Hence, since Conjecture 1.1 is equivalent to proving that $r_{m,(1,2,4,8)}(n) > 0$ for every $n \in \mathbb{N}$ and $m \in \{7, 9, 10, 11, 12, 13, 14\},$ the conjecture is equivalent to showing that for every $n \in \mathbb{N},$ we have

$$s_{m,2(m-2),(1,2,4,8)} \left(8(m-2)n + 15(m-4)^2 \right) > 0. \tag{1.4}$$

We investigate the numbers $s_{r,M,\alpha}(n)$ by forming the generating function (setting $q := e^{2\pi i\tau}$)

$$\Theta_{r,M,\alpha}(\tau) := \sum_{n \geq 0} s_{r,M,\alpha}(n) q^n.$$

It is well known that these functions are modular forms (see Lemma 2.1 for the precise statement). By the theory of modular forms, there is a natural decomposition

$$\Theta_{r,M,\alpha} = E_{r,M,\alpha} + f_{r,M,\alpha}, \tag{1.5}$$

where $E_{r,M,\alpha}$ lies in the space spanned by Eisenstein series and $f_{r,M,\alpha}$ is a cusp form. In order to prove Theorem 1.2, we obtain in the special case $r = m$, $M = 2(m - 2)$, and $\alpha = (1, 2, 4, 8)$ an explicit lower bound for the n -th Fourier coefficient $a_{r,M,\alpha}(n)$ of $E_{r,M,\alpha}$ in Corollary 4.2 and an explicit upper bound on the absolute value of the n -th Fourier coefficient $b_{r,M,\alpha}(n)$ of $f_{r,M,\alpha}$ in the proof of Theorem 1.2.

The paper is organized as follows. In Sect. 2, we recall properties of the theta functions $\Theta_{r,M,\alpha}$, the actions of certain operators on modular forms, the decomposition of modular forms into Eisenstein series and cusp forms, and evaluate certain Gauss sums. In Sect. 3, we investigate the growth of the theta functions toward all cusps and use this to compute the Eisenstein series component of the decomposition (1.5). The Fourier coefficients of the Eisenstein series components are then explicitly computed and lower bounds are obtained in Sect. 4. We complete the paper by obtaining upper bounds on the coefficients of the cuspidal part of the decomposition (1.5) and prove Theorem 1.2 in Sect. 5.

2 Setup and Preliminaries

2.1 Modularity of the Generating Functions

In this subsection, we consider the modularity properties of the theta functions $\Theta_{r,M,\alpha}$. To set notation, for $\Gamma_1(N) \subseteq \Gamma \subseteq \text{SL}_2(\mathbb{Z})$ ($N \in \mathbb{N}$) and a character χ modulo N , let $M_k(\Gamma, \chi)$ be the space of modular forms of weight k with character χ . In particular, an element f in this space satisfies, for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

$$f|_k \gamma(\tau) := (c\tau + d)^{-k} f(\gamma\tau) = \chi(d) f(\tau).$$

Setting $\Gamma_{N,L} := \Gamma_0(N) \cap \Gamma_1(L)$, by [7, Theorem 2.4], we have the following.

Lemma 2.1 *For $\alpha \in \mathbb{N}^4$, we have*

$$\Theta_{r,M,\alpha} \in M_2 \left(\Gamma_{4\text{lcm}(\alpha)M^2,M}, \left(\frac{\prod_{j=1}^4 \alpha_j}{\cdot} \right) \right).$$

2.2 Operators on Non-holomorphic Modular Forms

For a translation-invariant function f with Fourier expansion (denoting $\tau = u + iv \in \mathbb{H}$)

$$f(\tau) = \sum_{n \geq 0} c_{f,v}(n) q^n,$$

we define the *sieving operator* $(M, m \in \mathbb{N})$

$$f|S_{M,m}(\tau) := \sum_{\substack{n \geq 0 \\ n \equiv m \pmod{M}}} c_{f,v}(n) q^n.$$

As usual, we also define the V -operator ($\delta \in \mathbb{N}$) by

$$f|V_\delta(\tau) := \sum_{n \geq 0} c_{f,\delta v}(n)q^{\delta n}.$$

We require the modularity properties of (non-holomorphic) modular forms under the operators $S_{M,m}$ and V_d . Arguing via commutator relations for matrices, a standard argument (for example, see the proof of [17, Lemma 2]), one obtains the following.

Lemma 2.2 *Suppose that $k \in \mathbb{Z}$, $L, N \in \mathbb{N}$ with $L \mid N$, and f satisfies weight k modularity on $\Gamma_{N,L}$.*

- (1) *For $d \in \mathbb{N}$, the function $f|V_d$ satisfies weight k modularity on $\Gamma_{\text{lcm}(4, Nd), L}$.*
- (2) *For $m \in \mathbb{Z}$ and $M \in \mathbb{N}$, the function $f|S_{M,m}$ satisfies weight k modularity on $\Gamma_1(NM^2)$.*

It is useful to determine the commutator relations between the V -operator and sieving.

Lemma 2.3 *Let $m \in \mathbb{Z}$ and $M_1, M_2 \in \mathbb{N}$ be given and set $d := \text{gcd}(M_1, M_2)$ and $\mu_j := \frac{M_j}{d}$. Then for any translation-invariant function f , we have*

$$f|V_{M_1}|S_{M_2,m} = \begin{cases} f|S_{\mu_2, \bar{\mu}_1 \frac{m}{d}}|V_{M_1} & \text{if } d \mid m, \\ 0 & \text{otherwise,} \end{cases}$$

where $\bar{\mu}_1$ is the inverse of $\mu_1 \pmod{\mu_2}$.

Proof Recall that

$$\begin{aligned} f|S_{M_1,m}|V_{M_2}(\tau) &= \sum_{n \equiv m \pmod{M_1}} c_{f, M_2 v}(n)q^{M_2 n}, \\ f|V_{M_1}|S_{M_2,m}(\tau) &= \sum_n c_{f, M_1 v}(n)q^{M_1 n} \Big| S_{M_2,m}(\tau) = \sum_{n \equiv m \pmod{M_2}} \tilde{c}_{f,v}(n)q^n \\ &= \sum_{\substack{n \equiv m \pmod{M_2} \\ M_1 \mid n}} c_{f, M_1 v} \left(\frac{n}{M_1} \right) q^n, \end{aligned}$$

where

$$\tilde{c}_{f,v}(n) := \begin{cases} c_{f, M_1 v} \left(\frac{n}{M_1} \right) & \text{if } M_1 \mid n, \\ 0 & \text{otherwise.} \end{cases}$$

If $d = \text{gcd}(M_1, M_2) \nmid m$, then $n \equiv m \pmod{M_2}$ and $M_1 \mid n$ are not consistent, and hence $f|V_{M_1}|S_{M_2,m}$ vanishes identically.

We may hence assume that $d \mid m$ and we note that $\gcd(\mu_1, \mu_2) = 1$, obtaining

$$\begin{aligned} f \mid V_{M_1} \mid S_{M_2, m}(\tau) &= \sum_{\substack{n \equiv \frac{m}{d} \pmod{\mu_2} \\ M_1 \mid n}} c_{f, M_1 v} \left(\frac{n}{M_1} \right) q^n \\ &\stackrel{n \rightarrow M_1 n}{=} \sum_{\substack{n \equiv \bar{\mu}_1 \frac{m}{d} \pmod{\mu_2}}} c_{f, M_1 v}(n) q^{M_1 n} \\ &= f \mid S_{\mu_2, \bar{\mu}_1 \frac{m}{d}} \mid V_{M_1}(\tau). \end{aligned} \quad \square$$

2.3 Decomposition Into Eisenstein Series and Cusp Forms

Comparing Fourier coefficients on both sides of (1.5), we have

$$s_{r, M, \alpha}(n) = a_{r, M, \alpha}(n) + b_{r, M, \alpha}(n). \tag{2.1}$$

Theorem 1.2 is equivalent to showing (1.4) for n sufficiently large (with the restriction that $n \not\equiv 4 \pmod{16}$ for $m = 12$). Roughly speaking, the approach in this paper to proving (1.4) is to prove that for n sufficiently large with $n \equiv 15(m - 4)^2 \pmod{8(m - 2)}$ (noting the congruence conditions in (1.4))

$$a_{r, M, \alpha}(n) > |b_{r, M, \alpha}(n)|.$$

To obtain an upper bound for $|b_{r, M, \alpha}(n)|$, we recall that Deligne [9] proved that for a normalized newform $f(\tau) = \sum_{n \geq 1} c_f(n)q^n$ of weight k on $\Gamma_0(N)$ with Nebentypus character χ (normalized so that $c_f(1) = 1$), we have

$$|c_f(n)| \leq \sigma_0(n)n^{\frac{k-1}{2}}, \tag{2.2}$$

where $\sigma_k(n) := \sum_{d \mid n} d^k$. To obtain an explicit bound for $|c_f(n)|$ for arbitrary $f \in S_k(\Gamma_1(N))$, we combine (2.2) with a trick implemented by Blomer [5] and Duke [12]. For cusp forms $f, g \in S_k(\Gamma)$, we define the *Petersson inner product* by

$$\langle f, g \rangle := \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]} \int_{\Gamma \backslash \mathbb{H}} f(\tau) \overline{g(\tau)} v^k \frac{dudv}{v^2}.$$

Letting $\|f\| := \sqrt{\langle f, f \rangle}$ denote the *Petersson norm* of $f \in S_k(\Gamma)$, a bound for $|c_f(n)|$ in terms of $\|f\|$ may be obtained. Specifically, suppose that f is a cusp form f of weight $k \in \mathbb{N}$ on $\Gamma_{N, L}$ (with $L \mid N$) and character χ modulo N . Using Blomer’s method from [5], an explicit bound is obtained in [1, Lemma 4.1] for $|c_f(n)|$ as a function of N, L , and the Petersson norm $\|f\|$. Denoting by φ Euler’s totient function, we recall a bound from the case $k = 2$ below (see [1, (4.4)]).

Lemma 2.4 *Suppose that $f \in S_2(\Gamma_{N,L}, \chi)$ with $L \mid N$ and χ a character modulo N . Then, we have the inequality*

$$|c_f(n)| \leq 6.95 \cdot 10^{18} \cdot \|f\| N^{1+2.5 \cdot 10^{-6}} \prod_{p \mid N} \left(1 + \frac{1}{p}\right)^{\frac{1}{2}} \varphi(L)n^{\frac{3}{5}}.$$

By Lemma 2.4, in order to obtain an explicit bound for $|b_{r,M,\alpha}(n)|$, it remains to estimate $\|f_{r,M,\alpha}\|$, where $f_{r,M,\alpha}$ is the cusp form appearing in the decomposition in (1.5). An explicit bound for $\|f_{r,M,\alpha}\|$ was obtained in [16, Lemma 3.2]. To state the result, let $\alpha \in \mathbb{Z}^\ell$. For the quadratic form $Q = Q_\alpha$ given by

$$Q_\alpha(\mathbf{x}) := \sum_{j=1}^{\ell} \alpha_j x_j^2,$$

we define the *level* and the *discriminant* of Q_α as

$$N_\alpha = 4 \operatorname{lcm}(\alpha), \quad D_\alpha = 2^\ell \prod_{j=1}^{\ell} \alpha_j.$$

Lemma 2.5 *Let $\ell \geq 4$ be even, $\alpha \in \mathbb{N}^\ell$, $r \in \mathbb{Z}$, and $M \in \mathbb{N}$. Then,*

$$\begin{aligned} \|f_{r,M,\alpha}\|^2 &\leq \frac{3^{2\ell-2} \left(\frac{\ell}{2} - 2\right)!}{2^{\frac{\ell}{2}-3} \pi^\ell} \frac{M^{2\ell-4} N_\alpha^{\ell-2}}{\prod_{p \mid M^2 N_\alpha} (1 - p^{-2})} \\ &\times \sum_{\delta \mid M^2 N_\alpha} \varphi\left(\frac{M^2 N_\alpha}{\delta}\right) \varphi(\delta) \frac{M^2 N_\alpha}{\delta} \left(\frac{\gcd(M^2, \delta)}{M^2}\right)^\ell \\ &\times \sum_{m=0}^{\frac{\ell}{2}-2} \frac{(2\pi)^{-m}}{\left(\frac{\ell}{2} - 2 - m\right)!} (\ell - m - 2)! \left(\frac{9}{D_\alpha} (\ell - m - 1) \frac{M^2 N_\alpha}{\pi} + \ell^2\right). \end{aligned}$$

2.4 Gauss Sums

Define the *generalized quadratic Gauss sum* ($a, b \in \mathbb{Z}, c \in \mathbb{N}$)

$$G(a, b; c) := \sum_{\ell \pmod{c}} e^{\frac{2\pi i}{c} (a\ell^2 + b\ell)}.$$

Background information and many properties of these sums may be found in [2]. To state the properties that we require, for d odd, we define

$$\varepsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod{4}, \\ i & \text{if } d \equiv 3 \pmod{4}, \end{cases}$$

and we write $[a]_b$ for the inverse of a modulo b if $\gcd(a, b) = 1$.

Lemma 2.6 *For $a, b \in \mathbb{Z}$ and $c, d \in \mathbb{N}$, the following hold.*

(1) *If $\gcd(a, c) \nmid b$, then $G(a, b; c) = 0$, while if $\gcd(a, c) \mid b$ then*

$$G(a, b; c) = \gcd(a, c)G\left(\frac{a}{\gcd(a, c)}, \frac{b}{\gcd(a, c)}; \frac{c}{\gcd(a, c)}\right).$$

(2) *If $\gcd(a, c) = 1$ and c is odd, then*

$$G(a, b; c) = \varepsilon_c \sqrt{c} \left(\frac{a}{c}\right) e^{-\frac{2\pi i [4a]_c b^2}{c}}.$$

(3) *If $\gcd(c, d) = 1$, then*

$$G(a, b; cd) = G(ac, b; d)G(ad, b; c).$$

(4) *If $\gcd(a, c) = 1, 4 \mid c$, and b is odd, then $G(a, b; c) = 0$.*

(5) *If a is odd, b is even, and $r \in \mathbb{N}_{\geq 2}$, then*

$$G(a, b; 2^r) = 2^{\frac{r}{2}}(1+i) \left(\frac{-2^r}{a}\right) \varepsilon_a e^{\frac{2\pi i}{2^r}(-[a]_{2^r} \frac{b^2}{4})}.$$

We require an explicit evaluation of certain Gauss sums that naturally occur in the study of theta functions (see Lemma 3.1 below). Throughout the paper, for $k, M \in \mathbb{N}$, and $r \in \mathbb{Z}$ with $\text{ord}_2(r) \leq \text{ord}_2(M)$, we write $M = 2^\mu M_0, r = 2^\ell r_0$ (with $\varrho \leq \mu$), and $k = 2^\kappa k_0$ with M_0, r_0 , and k_0 odd. We furthermore set $g_0 := \gcd(M_0, k_0)$ and $g_1 := \gcd(g_0, \frac{k_0}{g_0})$.

Lemma 2.7 *Suppose that $h \in \mathbb{Z}, k \in \mathbb{N}$ with $\gcd(h, k) = 1, \ell \in \mathbb{N}_0, r \in \mathbb{Z}, M \in \mathbb{N}$ with $\gcd(M, r) \in \{1, 2, 4\}$, and $\varrho \leq \mu$.*

(1) *If $g_1 \neq 1$ or $\varrho < \min(\mu, \kappa - \ell - \mu) - 1$, then*

$$G\left(2^\ell h M^2, 2^{\ell+1} h r M; k\right) = 0.$$

(2) *Suppose that $g_1 = 1$ and $\varrho \geq \min(\mu, \kappa - \ell - \mu) - 1$. Setting $\delta := \min(\ell + 2\varrho, \kappa)$, we then have*

$$\begin{aligned}
 & e^{\frac{2\pi i 2^\ell h r^2}{k}} G\left(2^\ell h M^2, 2^{\ell+1} h r M; k\right) \\
 &= \sqrt{k g_0} \begin{cases} 2^{\frac{\delta}{2}} \varepsilon_{\frac{k_0}{g_0}} \left(\frac{2^{\ell+\kappa} h g_0}{\frac{k_0}{g_0}}\right) e^{\frac{2\pi i h r^2}{2^{\kappa-\delta} g_0} 2^{\ell+2\varrho-\delta} \left[\frac{k_0}{g_0}\right]_{2^{\kappa-\delta} g_0}} & \text{if } \kappa \leq \ell + 2\mu \\ & \text{or } \kappa = \ell + 2\mu + 1 \text{ and } \varrho = \mu - 1, \\ 2^{\frac{\ell+2\mu}{2}} (1+i) \varepsilon_{h g_0} \left(\frac{-2^{\ell+\kappa} \frac{k_0}{g_0}}{h g_0}\right) e^{\frac{2\pi i h r^2}{g_0} \left[2^{\kappa-\ell-2\mu} \frac{k_0}{g_0}\right]_{g_0}} & \text{if } \kappa \geq \ell + 2\mu + 2 \text{ and } \varrho = \mu, \\ 0 & \text{otherwise.} \end{cases}
 \end{aligned}$$

Proof We evaluate $G(a, b; c)$ for $a := 2^\ell h M^2$, $b := 2^{\ell+1} h r M$, and $c := k$. By Lemma 2.6 (1), $G(a, b; c) = 0$ unless $\gcd(a, c) \mid b$. Hence, we first compute, using the fact that $\gcd(h, k) = 1$, $\gcd\left(\frac{M_0}{g_0}, \frac{k_0}{g_0}\right) = 1$, and $\frac{k_0}{g_0}$ is odd,

$$\gcd(a, c) = 2^{\min(\ell+2\mu, \kappa)} g_0 g_2, \tag{2.3}$$

where $g_2 := \gcd(M_0, \frac{k_0}{g_0})$.

- (1) A direct calculation gives that $\gcd(a, c) \mid b$ if and only if $g_1 = 1$ and $\varrho \geq \min(\mu, \kappa - \ell - \mu) - 1$, which implies the claim by Lemma 2.6 (1).
- (2) Set $\gamma := \min(\ell + 2\mu, \kappa)$. Note that $\gamma \leq \ell + \mu + \varrho + 1$. From the calculation yielding (1), we see that $g_1 = 1$ implies $g_2 = 1$. Plugging $g_1 = g_2 = 1$ into (2.3) yields $\gcd(a, c) = 2^\gamma g_0$ and it is not hard to see that $\gcd(a, c) \mid b$. Therefore, Lemma 2.6 (1),(3) implies that

$$\begin{aligned}
 G(a, b; c) &= 2^\gamma g_0 G\left(2^{\ell+2\mu+\kappa-2\gamma} h M_0 \frac{M_0}{g_0}, 2^{\ell+\mu+\varrho+1-\gamma} h r_0 \frac{M_0}{g_0}; \frac{k_0}{g_0}\right) \\
 &\quad \times G\left(2^{\ell+2\mu-\gamma} h M_0 \frac{M_0}{g_0} \frac{k_0}{g_0}, 2^{\ell+\mu+\varrho+1-\gamma} h r_0 \frac{M_0}{g_0}; 2^{\kappa-\gamma}\right).
 \end{aligned}$$

Since $\frac{k_0}{g_0}$ is odd, we use Lemma 2.6 (2) to evaluate the first Gauss sum, yielding, after simplification,

$$\begin{aligned}
 G(a, b; c) &= 2^\gamma \varepsilon_{\frac{k_0}{g_0}} \sqrt{k_0 g_0} \left(\frac{2^{\ell+\kappa} h g_0}{\frac{k_0}{g_0}}\right) e^{-\frac{2\pi i h r^2}{\frac{k_0}{g_0}} 2^{\ell+2\varrho} [2^\kappa g_0]_{\frac{k_0}{g_0}}} \\
 &\quad \times G\left(2^{\ell+2\mu-\gamma} h M_0 \frac{M_0}{g_0} \frac{k_0}{g_0}, 2^{\ell+\mu+\varrho+1-\gamma} h r_0 \frac{M_0}{g_0}; 2^{\kappa-\gamma}\right). \tag{2.4}
 \end{aligned}$$

It remains to evaluate the final Gauss sum in (2.4). We use Lemma 2.6 (4) and Lemma 2.6 (5) to obtain

$$\begin{aligned}
 &G\left(2^{\ell+2\mu-\gamma} h M_0 \frac{M_0}{g_0} \frac{k_0}{g_0}, 2^{\ell+\mu+\varrho+1-\gamma} h r_0 \frac{M_0}{g_0}, 2^{\kappa-\gamma}\right) \\
 &= \begin{cases} 1 & \text{if } \kappa \leq \ell + 2\mu, \\ 2 & \text{if } \kappa = \ell + 2\mu + 1, \varrho = \mu - 1, \\ 2^{\frac{\kappa-\ell-2\mu}{2}} (1+i) \left(\frac{-2^{\ell+\kappa}}{h M_0 \frac{M_0}{g_0} \frac{k_0}{g_0}}\right) \varepsilon_{h M_0 \frac{M_0}{g_0} \frac{k_0}{g_0}} e^{-\frac{2\pi i h r_0^2}{2^{\kappa-\ell-2\mu}} [k_0]_{2^{\kappa-\ell-2\mu}}} & \text{if } \kappa \geq \ell + 2\mu + 2, \varrho = \mu, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}
 \end{aligned}$$

Plugging (2.5) into (2.4) and then simplifying yields that $G(a, b; c)$ equals

$$\begin{aligned}
 &\varepsilon_{\frac{k_0}{g_0}} \sqrt{k g_0} \left(\frac{h g_0}{\frac{k_0}{g_0}}\right) \begin{cases} 2^{\frac{\kappa}{2}} \left(\frac{2^{\ell+\kappa}}{\frac{k_0}{g_0}}\right) e^{-\frac{2\pi i h r_0^2}{g_0} 2^{\ell+2\varrho} [2^\kappa g_0]_{\frac{k_0}{g_0}}} & \text{if } \kappa \leq \ell + 2\mu \\ & \text{or } \kappa = \ell + 2\mu + 1, \varrho = \mu - 1, \\ 2^{\frac{\ell+2\mu}{2}} (1+i) \varepsilon_{h M_0 \frac{M_0}{g_0} \frac{k_0}{g_0}} \left(\frac{-2^{\ell+\kappa}}{h M_0 \frac{M_0}{g_0} \frac{k_0}{g_0}}\right) \left(\frac{2^{\ell+\kappa}}{\frac{k_0}{g_0}}\right) & \text{if } \kappa \geq \ell + 2\mu + 2, \varrho = \mu, \\ \times e^{-\frac{2\pi i h r_0^2}{g_0} 2^{\ell+2\mu} [2^\kappa g_0]_{\frac{k_0}{g_0}}} e^{-\frac{2\pi i h r_0^2}{2^{\kappa-\ell-2\mu}} [k_0]_{2^{\kappa-\ell-2\mu}}} & \\ 0 & \text{otherwise.} \end{cases}
 \end{aligned}$$

To obtain the claim, we multiply by $e^{\frac{2\pi i 2^\ell h r^2}{\kappa}}$ and simplify by using the Chinese Remainder Theorem to combine the exponentials. For example, if $\kappa \leq \ell + 2\mu$ or ($\kappa = \ell + 2\mu + 1$ and $\varrho = \mu - 1$), then the exponential becomes

$$e^{\frac{2\pi i h r_0^2}{2^{\kappa-\delta} k_0} 2^{\ell+2\varrho-\delta} \left(1 - 2^\kappa g_0 [2^\kappa g_0]_{\frac{k_0}{g_0}}\right)}.$$

Since $\gcd(g_0, \frac{k_0}{g_0}) = g_1 = 1$ and k_0 is odd, to determine $1 - 2^\kappa g_0 [2^\kappa g_0]_{\frac{k_0}{g_0}} \pmod{2^{\kappa-\delta} k_0}$ the Chinese Remainder Theorem implies that it suffices to compute

$$\begin{aligned}
 1 - 2^\kappa g_0 [2^\kappa g_0]_{\frac{k_0}{g_0}} &\equiv 1 \pmod{g_0}, & 1 - 2^\kappa g_0 [2^\kappa g_0]_{\frac{k_0}{g_0}} &\equiv 0 \pmod{\frac{k_0}{g_0}}, \\
 1 - 2^\kappa g_0 [2^\kappa g_0]_{\frac{k_0}{g_0}} &\equiv 1 \pmod{2^{\kappa-\delta}}.
 \end{aligned}$$

Thus,

$$1 - 2^\kappa g_0 [g_0]_{\frac{k_0}{g_0}} \equiv \frac{k_0}{g_0} \left[\frac{k_0}{g_0}\right]_{2^{\kappa-\delta} g_0} \pmod{2^{\kappa-\delta} k_0}.$$

So the exponential simplifies in this case as $e^{\frac{2\pi i h r_0^2}{2^{\kappa-\delta} g_0} 2^{\ell+2\varrho-\delta} [\frac{k_0}{g_0}]_{2^{\kappa-\delta} g_0}}$.

The remaining case $\kappa \geq \ell + 2\mu + 2$ and $\varrho = \mu$ follows by a similar but longer and more tedious calculation. □

3 Growth Toward the Cusps of Certain Modular Forms

In this section, we determine the growth toward the cusps of theta functions $\Theta_{r,M,\alpha}$ and certain (non-holomorphic) Eisenstein series. The purpose of this calculation is to compare the growth in order to determine the unique Eisenstein series $E_{r,M,\alpha}$ in (1.5) whose growth toward the cusps matches that of the theta function.

3.1 Growth of the Theta Functions at the Cusps

In order to obtain the Eisenstein series, we determine the growth of $\Theta_{r,M,\alpha}$ toward all of the cusps, which follows by a straightforward calculation.

Lemma 3.1 *Let $m \in \mathbb{N}_{\geq 3}$ and $\alpha \in \mathbb{N}^4$ be given. For $h \in \mathbb{Z}$ and $k \in \mathbb{N}$ with $\gcd(h, k) = 1$, we have*

$$\begin{aligned}
 & - \lim_{z \rightarrow 0^+} z^2 \Theta_{r,M,\alpha} \left(\frac{h}{k} + \frac{iz}{k} \right) \\
 & = - \frac{1}{4k^2 M^4 \prod_{j=1}^4 \sqrt{\alpha_j}} \prod_{j=1}^4 e^{\frac{2\pi i r^2 h \alpha_j}{k}} G \left(h \alpha_j M^2, 2hr \alpha_j M; k \right).
 \end{aligned}$$

Proof We have

$$\Theta_{r,M,\alpha}(\tau) = \sum_{\substack{\mathbf{x} \in \mathbb{Z}^d \\ x_j \equiv r \pmod{M}}} q^{\sum_{j=1}^4 \alpha_j x_j^2} = \prod_{j=1}^4 \vartheta(r, M; 2M \alpha_j \tau), \tag{3.1}$$

where

$$\vartheta(r, M; \tau) := \sum_{n \equiv r \pmod{M}} q^{\frac{n^2}{2M}}.$$

By definition,

$$\vartheta \left(r, M; 2M \alpha_j \left(\frac{h}{k} + \frac{iz}{k} \right) \right) = \sum_{n \equiv r \pmod{M}} e^{2\pi i \alpha_j n^2 \left(\frac{h}{k} + \frac{iz}{k} \right)}.$$

Write $n = r + M\alpha + Mk\ell$ ($\alpha \pmod{k}$, $\ell \in \mathbb{Z}$) to obtain that this equals

$$\begin{aligned}
 & \sum_{\alpha \pmod{k}} e^{2\pi i \alpha_j (r+M\alpha)^2 \frac{h}{k}} \sum_{\ell \in \mathbb{Z}} e^{-2\pi (r+M\alpha+Mk\ell)^2 \frac{\alpha_j}{k} z} \\
 & = \sum_{\alpha \pmod{k}} e^{2\pi i \alpha_j (r+M\alpha)^2 \frac{h}{k}} \vartheta(r + M\alpha, Mk; 2M \alpha_j i z).
 \end{aligned}$$

We now recall the modular inversion formula (see [19, (2.4)])

$$\vartheta\left(r, M; -\frac{1}{\tau}\right) = M^{-\frac{1}{2}}(-i\tau)^{\frac{1}{2}} \sum_{v \pmod{M}} e^{\frac{2\pi i v k}{M}} \vartheta(v, M; \tau).$$

We use this with $\tau = \frac{i}{2M\alpha_j z}$, $r \mapsto r + M\alpha$, $M \mapsto Mk$ to obtain that

$$\begin{aligned} &\vartheta(r + M\alpha, Mk; 2M\alpha_j i z) \\ &= (Mk)^{-\frac{1}{2}} \left(\frac{1}{2M\alpha_j z}\right)^{\frac{1}{2}} \sum_{v \pmod{Mk}} e^{\frac{2\pi i}{Mk}(r+M\alpha)v} \vartheta\left(v, Mk; \frac{i}{2M\alpha_j z}\right). \end{aligned}$$

Thus,

$$\begin{aligned} \vartheta\left(r, M; 2M\alpha_j\left(\frac{h}{k} + \frac{iz}{k}\right)\right) &= \frac{1}{M} \sqrt{\frac{1}{2k\alpha_j z}} \sum_{\alpha \pmod{k}} e^{2\pi i \alpha_j (r+M\alpha)^2 \frac{h}{k}} \\ &\quad \times \sum_{v \pmod{Mk}} e^{\frac{2\pi i}{Mk}(r+M\alpha)v} \vartheta\left(v, Mk; \frac{i}{2M\alpha_j z}\right). \end{aligned}$$

Now assume that $z \in \mathbb{R}^+$ and let $z \rightarrow 0^+$. The contribution that is not exponentially decaying comes from $v = 0$ and gives

$$\lim_{z \rightarrow 0^+} \sqrt{z} \vartheta\left(r, M; 2M\alpha_j\left(\frac{h}{k} + \frac{iz}{k}\right)\right) = \frac{1}{M} \sqrt{\frac{1}{2k\alpha_j}} \sum_{\alpha \pmod{k}} e^{2\pi i \alpha_j (r+M\alpha)^2 \frac{h}{k}}.$$

Note that

$$\begin{aligned} \sum_{\alpha \pmod{k}} e^{2\pi i \alpha_j (r+M\alpha)^2 \frac{h}{k}} &= e^{\frac{2\pi i \alpha_j r^2 h}{k}} \sum_{\alpha \pmod{k}} e^{\frac{2\pi i}{k}(\alpha_j M^2 \alpha^2 + 2r\alpha_j M\alpha)h} \\ &= e^{\frac{2\pi i \alpha_j r^2 h}{k}} G\left(h\alpha_j M^2, 2hr\alpha_j M; k\right). \end{aligned}$$

Plugging back into (3.1) yields the claim. □

We next use Lemma 2.7 to evaluate the right-hand side of Lemma 3.1. Since the theta function $\Theta_{r, M, \alpha}$ only depends on r modulo M , we may assume without loss of generality that

$$\varrho = \text{ord}_2(r) \leq \text{ord}_2(M) = \mu$$

by replacing r with $r + M$ in Lemma 3.1 if $\varrho > \mu$. A direct calculation gives the following.

Corollary 3.2 *Suppose that $h \in \mathbb{Z}$ and $k \in \mathbb{N}$ with $\gcd(h, k) = 1$, $\alpha = (1, 2, 4, 8)$, $r \in \mathbb{Z}$, and $M \in \mathbb{N}$ with $\gcd(M, r) \in \{1, 2, 4\}$ and $\text{ord}_2(r) \leq \text{ord}_2(M)$. If $g_1 \neq 1$ or $\varrho < \min(\mu, k - \ell - \mu) - 1$, then*

$$-\lim_{z \rightarrow 0^+} z^{2\Theta_{r,M,\alpha}} \left(\frac{h}{k} + \frac{iz}{k} \right) = 0.$$

If $g_1 = 1$ and $\varrho \geq \min(\mu, k - \ell - \mu) - 1$, then, setting $\delta_0 := \min(\kappa, 2\varrho)$,

$$-\lim_{z \rightarrow 0^+} z^{2\Theta_{r,M,\alpha}} \left(\frac{h}{k} + \frac{iz}{k} \right) = \begin{cases} -\frac{2^{2\kappa-4\mu-5}}{M_0^4} g_0^2 e^{\frac{2\pi i h r_0^2}{2^{\kappa-\delta_0} g_0}} 2^{2\varrho-\delta_0} 15 \left[\frac{k_0}{g_0} \right]_{2^{\kappa-\delta_0} g_0} & \text{if } \kappa \leq 2\mu, \\ & \text{or } \kappa = 2\mu + 1 \text{ and } \varrho = \mu - 1, \\ \frac{g_0^2}{M_0^4} e^{\frac{2\pi i h r_0^2}{g_0}} 15 \left[2^{\kappa-2\mu} \frac{k_0}{g_0} \right]_{g_0} & \text{if } \kappa \geq 2\mu + 5 \text{ and } \varrho = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

Remark Although the right-hand side of Corollary 3.2 splits into a number of cases, we obtain an explicit element of the cyclotomic field $\mathbb{Q}(\zeta_{2^j g_0})$ for some $j \in \mathbb{N}_0$, where $\zeta_v := e^{\frac{2\pi i}{v}}$. To use Corollary 3.2 for practical purposes, one can evaluate the right-hand side of Corollary 3.2 with a computer as an element of $\mathbb{Q}(\zeta_v) \cong \mathbb{Q}[x]/\langle f_v \rangle$, where f_v is the minimal polynomial of ζ_v over \mathbb{Q} , which is well known to be

$$f_v(x) = \prod_{\substack{1 \leq k \leq v \\ \gcd(k,v)=1}} (x - \zeta_v^k) = \prod_{d|v} (x^d - 1)^{\mu\left(\frac{v}{d}\right)}.$$

Here, μ denotes the Möbius μ -function.

3.2 Growth of Eisenstein Series Toward the Cusps

The goal of this section is to obtain the growth of certain weight two Eisenstein series toward the cusps. These are formed by applying certain sieving and V -operators to the (non-holomorphic but modular) weight two Eisenstein series

$$\widehat{E}_2(\tau) := E_2(\tau) - \frac{3}{\pi v}, \quad \text{where} \quad E_2(\tau) := 1 - 24 \sum_{n \geq 1} \sigma(n) q^n$$

with $\sigma(n) := \sigma_1(n)$. In light of Lemma 2.3, we may furthermore always assume without loss of generality that sieving is applied before the V -operator. The growth toward the cusps of such functions is given in the following lemma.

Lemma 3.3 *Let $m \in \mathbb{Z}$ and $M_1, M_2 \in \mathbb{N}$. Then, for $h \in \mathbb{Z}$ and $k \in \mathbb{N}$ with $\gcd(h, k) = 1$, we have*

$$\begin{aligned}
 & - \lim_{z \rightarrow 0^+} z^2 \widehat{E}_2 |_{S_{M_1, m}} |_{V_{M_2}} \left(\frac{h}{k} + \frac{iz}{k} \right) \\
 & = \frac{1}{M_1^3 M_2^2} \sum_{j \pmod{M_1}} \gcd(hM_1 M_2 + jk, M_1 k)^2 \zeta_{M_1}^{-jm}.
 \end{aligned}$$

Proof For a translation-invariant function f , we use the presentation

$$f |_{S_{M_1, m}}(\tau) = \frac{1}{M_1} \sum_{j=0}^{M_1-1} \zeta_{M_1}^{-jm} f\left(\tau + \frac{j}{M_1}\right).$$

Applying V_{M_2} to this yields

$$f |_{S_{M_1, m}} |_{V_{M_2}}(\tau) = \frac{1}{M_1} \sum_{j=0}^{M_1-1} \zeta_{M_1}^{-jm} f\left(M_2 \tau + \frac{j}{M_1}\right).$$

Plugging in $f = \widehat{E}_2$ and using the weight two modularity of \widehat{E}_2 , the claim follows by a standard calculation. □

4 Eisenstein Series Component

In this section, we determine the Eisenstein series component $E_{r, M, \alpha}$ in (1.5).

Proposition 4.1 *For $n \in \mathbb{N}$, we have the following.*

- (1) *For $m = 7$, we have $a_{7, 10, (1, 2, 4, 8)}(n) = 0$ unless $n \equiv 15 \pmod{40}$, in which case we have*

$$a_{7, 10, (1, 2, 4, 8)}(n) = \frac{1}{240} \left(\sigma(n) - \sigma\left(\frac{n}{5}\right) \right).$$

- (2) *For $m = 9$, we have $a_{9, 14, (1, 2, 4, 8)}(n) = 0$ unless $n \equiv 39 \pmod{56}$, in which case we have*

$$a_{9, 14, (1, 2, 4, 8)}(n) = \frac{1}{672} \sigma(n).$$

- (3) *For $m = 10$, we have $a_{10, 16, (1, 2, 4, 8)}(n) = 0$ unless $n \equiv 28 \pmod{64}$, in which case we have*

$$a_{10, 16, (1, 2, 4, 8)}(n) = \frac{1}{256} \sigma\left(\frac{n}{4}\right).$$

(4) For $m = 11$, we have $a_{11,18,(1,2,4,8)}(n) = 0$ unless $n \equiv 15 \pmod{72}$, in which case we have

$$a_{11,18,(1,2,4,8)}(n) = \frac{1}{1728}\sigma(n).$$

(5) For $m = 12$, we have $a_{12,20,(1,2,4,8)}(n) = 0$ unless $80 \mid n$, in which case we have

$$\begin{aligned} a_{12,20,(1,2,4,8)}(n) = & \frac{1}{120} \left(\sigma\left(\frac{n}{16}\right) - \sigma\left(\frac{n}{32}\right) - \sigma\left(\frac{n}{80}\right) + \sigma\left(\frac{n}{160}\right) \right. \\ & + 8\sigma\left(\frac{n}{256}\right) - 32\sigma\left(\frac{n}{512}\right) \\ & \left. - 8\sigma\left(\frac{n}{1280}\right) + 32\sigma\left(\frac{n}{2560}\right) \right). \end{aligned}$$

(6) For $m = 13$, we have $a_{13,22,(1,2,4,8)}(n) = 0$ unless $n \equiv 71 \pmod{88}$, in which case we have

$$a_{13,22,(1,2,4,8)}(n) = \frac{1}{2640}\sigma(n).$$

(7) For $m = 14$, we have

$$a_{14,24,(1,2,4,8)}(n) = \begin{cases} \frac{1}{768} \left(\sigma\left(\frac{n}{4}\right) - \sigma\left(\frac{n}{12}\right) \right) & \text{if } n \equiv 60 \pmod{96}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof (1) By comparing Fourier coefficients, we see that the identity is equivalent to

$$E_{7,10,(1,2,4,8)} = -\frac{1}{5760}E_2 \Big| (1 - V_5) \Big| S_{40,15}. \tag{4.1}$$

Lemma 2.1 and (1.5) give that

$$E_{7,10,(1,2,4,8)} \in M_2\left(\Gamma_{3200,10}\right),$$

while Lemma 2.2 implies that

$$E_2 \Big| (1 - V_5) \Big| S_{40,15} \in M_2\left(\Gamma_1(1600)\right).$$

Enumerating the cusps of $\Gamma_1(3200)$ (see [10, Proposition 3.8.3]), we then use a computer together with Lemma 3.3 and Corollary 3.2 to verify that the growth toward every cusp of both sides of (4.1) agrees, yielding the claim.

To see this in more details note that by [10, Proposition 3.8.3], two cusps $\frac{a}{c}$ and $\frac{\alpha}{\gamma}$ are equivalent modulo the action of $\Gamma_1(N)$ if and only if there exists $j \in \mathbb{Z}$ such that $(\alpha, \gamma) \equiv \pm(a + jc, c) \pmod{N}$ (for some choice of \pm). As in [10, p. 102], by taking $d := \gcd(c, N)$, we may write a set of representatives of the inequivalent cusps in the form $\frac{a}{d\gamma}$ with $d \mid N$, a running modulo d with $\gcd(a, d) = 1$, and $1 \leq \gamma \leq \lceil \frac{N}{2d} \rceil$

with $\gcd(\gamma, \frac{N}{d}) = 1$. Since both sides of (4.1) are elements of $M_2(\Gamma_1(3200))$, we thus need to compute the constant term at every cusp $\frac{h}{k}$ with $h, k \in \mathbb{Z}, \gcd(h, k) = 1$, and $k = d\gamma$ with $d \mid 3200$ and $1 \leq \gamma \leq \frac{1600}{d}$ with $\gcd(\gamma, \frac{3200}{d}) = 1$. For each such representative $\frac{h}{k}$ of the cusps of $\Gamma_1(3200)$, we use Lemma 3.3 together with a computer to evaluate

$$-\frac{1}{5760} \lim_{z \rightarrow 0^+} z^2 \left(E_2|S_{8,3}|V_5 \left(\frac{h}{k} + \frac{iz}{k} \right) - E_2|S_{40,15} \left(\frac{h}{k} + \frac{iz}{k} \right) \right)$$

as an element of $\mathbb{Q}(\zeta_{40})$. Comparing this with Corollary 3.2 in the case $r = 7$ and $M = 10$, we then verify with a computer that

$$\begin{aligned} &-\frac{1}{5760} \lim_{z \rightarrow 0^+} z^2 \left(E_2|S_{8,3}|V_5 \left(\frac{h}{k} + \frac{iz}{k} \right) - E_2|S_{40,15} \left(\frac{h}{k} + \frac{iz}{k} \right) \right) \\ &= - \lim_{z \rightarrow 0^+} z^2 \Theta_{7,10,\alpha} \left(\frac{h}{k} + \frac{iz}{k} \right). \end{aligned} \tag{4.2}$$

Since $f_{7,10,(1,2,4,8)}$ is a cusp form, we have

$$-\lim_{z \rightarrow 0^+} z^2 f_{7,10,\alpha} \left(\frac{h}{k} + \frac{iz}{k} \right) = 0$$

and hence

$$-\lim_{z \rightarrow 0^+} z^2 \Theta_{7,10,\alpha} \left(\frac{h}{k} + \frac{iz}{k} \right) = - \lim_{z \rightarrow 0^+} z^2 E_{7,10,\alpha} \left(\frac{h}{k} + \frac{iz}{k} \right).$$

Therefore, (4.2) implies that

$$E_{7,10,(1,2,4,8)} + \frac{1}{5760} E_2|(1 - V_5)S_{40,15}$$

vanishes toward all cusps, and is hence a cusp form. Since it is also in the subspace of Eisenstein series, it is orthogonal to all cusp forms and therefore vanishes, implying (4.1), and hence the claim.

For the remaining cases, the argument is similar, but we provide the identities analogous to (4.1) for the convenience of the reader.

(2) The claim is equivalent to

$$E_{9,14,(1,2,4,8)} = -\frac{1}{16128} E_2|S_{56,39}.$$

(3) The claim is equivalent to

$$E_{10,16,(1,2,4,8)} = -\frac{1}{6144} E_2|S_{16,7}|V_4.$$

(4) The claim is equivalent to

$$E_{11,18,(1,2,4,8)} = -\frac{1}{41472} E_2 | S_{72,15}.$$

(5) The claim is equivalent to

$$E_{12,20,(1,2,4,8)} = -\frac{1}{2880} E_2 | (S_{5,0} - V_5) | (1 - V_2 + 8V_{16} - 32V_{32}) | V_{16}.$$

(6) The claim is equivalent to

$$E_{13,22,(1,2,4,8)} = -\frac{1}{63360} E_2 | S_{88,71}.$$

(7) The claim is equivalent to

$$E_{14,24,(1,2,4,8)} = -\frac{1}{18432} E_2 | (1 - V_3) | S_{24,15} | V_4. \quad \square$$

As a corollary to Proposition 4.1, we obtain explicit lower bounds on the Fourier coefficients $a_{r,M,\alpha}(n)$ in these special cases.

Corollary 4.2 *Let $n \in \mathbb{N}$.*

(1) *If $n \equiv 15 \pmod{40}$, then we have*

$$a_{7,10,(1,2,4,8)}(n) \geq \frac{n}{240}.$$

(2) *If $n \equiv 39 \pmod{56}$, then we have*

$$a_{9,14,(1,2,4,8)}(n) \geq \frac{n}{672}.$$

(3) *If $n \equiv 28 \pmod{64}$, then we have*

$$a_{10,16,(1,2,4,8)}(n) \geq \frac{n}{1024}.$$

(4) *If $n \equiv 15 \pmod{72}$, then we have*

$$a_{11,18,(1,2,4,8)}(n) \geq \frac{n}{1728}.$$

(5) *Assume that $80 \mid n$ and write $n = 2^a 5^b c$ with $\gcd(10, c) = 1$. We have*

$$a_{12,20,(1,2,4,8)}(n) \geq \frac{5^b c}{120} \begin{cases} 2^{a-4} & \text{if } 4 \leq a \leq 7, \\ 24 & \text{if } a \geq 8. \end{cases}$$

(6) If $n \equiv 71 \pmod{88}$, then we have

$$a_{13,22,(1,2,4,8)}(n) \geq \frac{n}{2640}.$$

(7) If $n \equiv 60 \pmod{96}$, then we have

$$a_{14,24,(1,2,4,8)}(n) \geq \frac{n}{3072}.$$

Proof For $m \neq 12$, the claims with the exception of (5) follow directly from Proposition 4.1. For (5), a direct simplification yields that the right-hand side of Proposition 4.1 (5) simplifies as

$$\frac{5^b \sigma(c)}{120} \begin{cases} 2^{a-4} & \text{if } 4 \leq a \leq 7, \\ 24 & \text{if } a \geq 8, \end{cases}$$

which gives the claim. □

5 Proof of Theorem 1.2

In this section, we prove Theorem 1.2. The constants C_m from the theorem statement may be found in Table 2.

Proof of Theorem 1.2 We require the case $\ell = 4$ of Lemma 2.5. Since the inner sum only has a single term namely $m = 0$ in this case, Lemma 2.5 simplifies as

$$\begin{aligned} \|f_{r,M,\alpha}\|^2 &\leq \frac{2 \cdot 3^6}{\pi^4} \frac{M^4 N_\alpha^2}{\prod_{p|M^2 N_\alpha} (1 - p^{-2})} \\ &\times \sum_{\delta|M^2 N_\alpha} \varphi\left(\frac{M^2 N_\alpha}{\delta}\right) \varphi(\delta) \frac{M^2 N_\alpha}{\delta} \left(\frac{\gcd(M^2, \delta)}{M^2}\right)^4 2 \left(\frac{27}{D_\alpha} \frac{M^2 N_\alpha}{\pi} + 16\right). \end{aligned} \tag{5.1}$$

For $\theta_{r,M,(1,2,4,8)}$, we obtain a lower bound for $a_{r,M,(1,2,4,8)}(n)$ (for n in an appropriate congruence class) from Corollary 4.2 (see the third column of Table 1 for a list of the bounds for individual choices of r and M).

Computing the constants in (5.1) explicitly for fixed M yields an upper bound for $\|f_{r,M,(1,2,4,8)}\|^2$ (see the fourth column of Table 1 for the explicit bounds), which plugged into Lemma 2.4 yields an upper bound for $|b_{r,M,(1,2,4,8)}(n)|$ (see the final column of Table 1 for the explicit bounds). Plugging the bounds for $a_{r,M,(1,2,4,8)}(n)$ and $|b_{r,M,(1,2,4,8)}(n)|$ into (2.1), we see that $s_{r,M,(1,2,4,8)}(n) > 0$ for n sufficiently large in an appropriate congruence class (see Table 2 for the explicit constants).

We then conclude that $r_{m,(1,2,4,8)} > 0$ for n sufficiently large by using (1.3), yielding the claim. □

In order to explain why it is sufficient to assume that $n \not\equiv 4 \pmod{16}$ for $m = 12$ in Theorem 1.2, we require the following lemma combined with (1.3).

Table 1 Bounds for $a_{r,M,(1,2,4,8)}$, $\|f_{r,M,(1,2,4,8)}\|$, and $|b_{r,M,(1,2,4,8)}|$

r	M	Bound for $a_{r,M,(1,2,4,8)}$	Bound for $\ f_{r,M,(1,2,4,8)}\ $	Bound for $ b_{r,M,(1,2,4,8)} $
7	10	$\frac{n}{240}$	$8.11 \cdot 10^{14}$	$3.41 \cdot 10^{30}n^{\frac{3}{5}}$
9	14	$\frac{n}{672}$	$1.03 \cdot 10^{16}$	$3.48 \cdot 10^{31}n^{\frac{3}{5}}$
10	16	$\frac{n}{1024}$	$3.2 \cdot 10^{16}$	$9.98 \cdot 10^{31}n^{\frac{3}{5}}$
11	18	$\frac{n}{1728}$	$6.1 \cdot 10^{16}$	$1.52 \cdot 10^{32}n^{\frac{3}{5}}$
12	20	$\frac{n}{1920}$	$1.49 \cdot 10^{17}$	$3.69 \cdot 10^{32}n^{\frac{3}{5}}$
13	22	$\frac{n}{2640}$	$2.55 \cdot 10^{17}$	$6.96 \cdot 10^{32}n^{\frac{3}{5}}$
14	24	$\frac{n}{3072}$	$5.63 \cdot 10^{17}$	$1.09 \cdot 10^{33}n^{\frac{3}{5}}$

Table 2 Bounds on n for $s_{m,2(m-2),(1,2,4,8)}(n) > 0$ and $r_{m,(1,2,4,8)} > 0$

m	Bound for $s_{m,2(m-2),(1,2,4,8)}(n) > 0$	Bound C_m for $r_{m,2(m-2),(1,2,4,8)}(n) > 0$
7	$1.92 \cdot 10^{82}$	$4.8 \cdot 10^{80}$
9	$8.38 \cdot 10^{85}$	$1.5 \cdot 10^{84}$
10	$3.41 \cdot 10^{87}$	$5.33 \cdot 10^{85}$
11	$3.55 \cdot 10^{88}$	$4.93 \cdot 10^{86}$
12	$4.25 \cdot 10^{89}$	$5.31 \cdot 10^{87}$
13	$4.57 \cdot 10^{90}$	$5.19 \cdot 10^{88}$
14	$2.04 \cdot 10^{91}$	$2.13 \cdot 10^{89}$

Lemma 5.1 *Let $n \in \mathbb{N}$ be given. If the equation*

$$x_1^2 + 2x_2^2 + 4x_3^2 + 8x_4^2 = n$$

is solvable with $x_j \equiv 12 \pmod{20}$, then the equation

$$x_1^2 + 2x_2^2 + 4x_3^2 + 8x_4^2 = 256n$$

is also solvable with $x_j \equiv 12 \pmod{20}$.

Acknowledgements We thank the referees for helpful comments on an earlier version of the paper.

Declarations

Conflict of interest On behalf of all authors, the corresponding author, Kathrin Bringmann, states that there is no conflict of interest.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If

material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Banerjee, S., Kane, B.: Finiteness theorems for universal sums of squares of almost prim. Trans. Amer. Math. Soc., to appear, [arxiv:2106.05107](https://arxiv.org/abs/2106.05107)
2. Berndt, B., Evans, R., Williams, K.: Gauss and Jacobi Sums. Wiley, New York (1998)
3. Bhargava, M.: On the Conway–Schneeberger fifteen theorem. In: Quadratic Forms and Their Applications, Contemp. Math. **272** (200), 27–37
4. Bhargava, M., Hanke, J.: Universal quadratic forms and the 290-theorem, Invent. Math., accepted for publication
5. Blomer, V.: Uniform bounds for Fourier coefficients of theta-series with arithmetic applications. Acta Arith. **114**, 1–21 (2004)
6. Bosma, W., Kane, B.: The triangular theorem of eight and representation by quadratic polynomials. Proc. Am. Math. Soc. **141**, 1473–1486 (2013)
7. Cho, B.: On the number of representations of integers by quadratic forms with congruence conditions. J. Math. Anal. Appl. **462**, 999–1013 (2018)
8. Conway, J.: Universal quadratic forms and the fifteen theorem. In: “Quadratic Forms and Their Applications”, Contemp. Math. **272** (200), 23–26
9. Deligne, P.: La conjecture de Weil I. Inst. Hautes Études Sci. Publ. Math. **43**, 273–307 (1974)
10. Diamond, F., Shurman, J.: A First Course in Modular Forms. Springer, New York (2005)
11. Dickson, L.: Integers represented by positive ternary quadratic forms. Bull. Am. Math. Soc. **33**, 63–70 (1927)
12. Duke, W.: On ternary quadratic forms. J. Number Theory **110**, 37–43 (2005)
13. Ju, J.: Universal sums of generalized pentagonal numbers. Ramanujan J. **51**, 479–494 (2020)
14. Ju, J., Kim, D.: The pentagonal theorem of sixty-three and generalizations of Cauchy’s Lemma, preprint, [arxiv:2010.16123](https://arxiv.org/abs/2010.16123)
15. Ju, J., Oh, B.-K.: Universal sums of generalized octagonal numbers. J. Number Theory **190**, 292–302 (2018)
16. Kamaraj, R., Kane, B., Tomiyasu, R.: *Universal sums of heptagonal numbers*, submitted for publication, [arxiv:2203.14678](https://arxiv.org/abs/2203.14678)
17. Li, W.: Newforms and functional equations. Math. Ann. **212**, 285–315 (1975)
18. Liouville, J.: Nouveaux théorèmes concernant les nombres triangulaires. J. Math. Pures Appl. **8**, 73–84 (1863)
19. Shimura, G.: On modular forms of half integral weight. Ann. Math. **97**, 440–481 (1973)
20. Sun, Z.-W.: On universal sums of polygonal numbers. Sci. China Math. **58**, 1367–1396 (2015)
21. Sun, Z.-W.: A result similar to Lagrange’s Theorem. J. Number Theory **162**, 190–211 (2016)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.