ON THE MODULARITY OF THE UNIFIED WRT INVARIANTS OF CERTAIN SEIFERT MANIFOLDS

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For Dennis Stanton on his 60th birthday

Abstract. We compute the unified WRT invariants of the Seifert manifolds $M(2,3,8)$ and $M(2,3,4)$ (arising from ± 2 surgery on the trefoil knot). The first is essentially a mock theta function which is a piece of one of Ramanujan's third order mock theta functions. The second is essentially the sum of a modular form and a false theta function.

1. Introduction and statement of results

Starting with the work of Lawrence and Zagier [16], many mock theta functions have been shown to coincide asymptotically with Witten-Reshetikhin-Turaev (WRT) invariants of Seifert manifolds (see [14]). Recently a family of unified WRT invariants, while technically not mock theta functions, were shown to have Hecke-type expansions closely resembling those of mock theta functions [15]. For example, related to the Poincaré homology sphere (c.f. [12, 17]) is the q-series

$$M_1(q) := \sum_{n\geq 0} q^n (q^n)_n = 1 + q + q^3 + q^7 - q^8 - q^{14} - q^{20} - q^{29} + q^{31} + \cdots , \tag{1.1}$$

which has the Hecke-type expansion [15, Eq. (3.36)]

$$\sum_{n\geq 0} q^n (q^n)_n = \frac{1}{(q)_\infty} \left(\sum_{r,s\geq 0} - \sum_{r,s<0} \right) (-1)^{r+s} q^{r(3r+1)/2+s(3s+1)/2+2rs}. \tag{1.2}$$

Here we have employed the standard q-series notation

$$(a)_n := (a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

The right-hand side of (1.2) resembles expansions for mock theta functions (see [24]), but the quadratic form in the exponent of q is positive definite instead of indefinite. In fact, $M_1(q)$ is a false theta function.

A natural question is then whether any unified WRT invariants will turn out to be genuine mock theta functions. Here we give one such example. We compute the unified WRT invariant of the

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Seifert manifold $M(2,3,8)$ (arising from $+2$ surgery on the trefoil) and show that it is a mock theta function.

Define $\overline{\phi}_0(q)$ by

$$\overline{\phi}_0(q) := \sum_{n\geq 0} q^n(-q)_{2n+1} = 1 + 2q + 2q^2 + 3q^3 + 5q^4 + 6q^5 + 8q^6 + 11q^7 + 13q^8 + \cdots, \tag{1.3}$$

and recall Ramanujan's third order mock theta function $\psi(q)$,

$$\psi(q) := \sum_{n\geq 1} \frac{q^{n^2}}{(q;q^2)_n} = q + q^2 + q^3 + 2q^4 + 2q^5 + 2q^6 + 3q^7 + 3q^8 + \cdots. \tag{1.4}$$

Theorem 1.1. *The unified WRT invariant of $M(2,3,8)$ is*

$$\frac{\sqrt{2}q^{1/4}}{(1-q)}\overline{\phi}_0(-q^{1/2}).$$

Moreover, $\overline{\phi}_0(q)$ is a mock theta function satisfying

$$2q^2\overline{\phi}_0(q^2) = \psi(q) + \psi(-q). \tag{1.5}$$

We also consider the case of -2 surgery on the trefoil. Here we do not encounter a mock theta function, but the sum of a modular form and a false theta function. Define $M(q)$ by

$$M(q) := \sum_{n\geq 0} q^{2n}(-q)_{2n+1} = 1 + q + q^2 + q^3 + 2q^4 + 3q^5 + 3q^6 + 4q^7 + 5q^8 + \cdots. \tag{1.6}$$

Theorem 1.2. *The unified WRT invariant of $M(2,3,4)$ is*

$$\frac{\sqrt{2}q^{1/4}}{(1-q)}M(-q^{1/2}).$$

Moreover, $M(q)$ satisfies,

$$2 + 2q^2 M(q) = (-q)_\infty + \sum_{n\geq 0} q^{n(3n+1)/2}(1-q^{2n+1}). \tag{1.7}$$

2. Proof of Theorem 1.1

The WRT invariant $\tau_N(M)$ for a 3-manifold M is constructed from the colored Jones polynomial $J_N(K;q)$ for the knot K to be surgered [21]. To compute $\tau_N(M)$ explicitly, useful is the cyclotomic expansion of the colored Jones polynomial,

$$J_N(K;q) = \sum_{n=0}^{\infty} C_K(n)(q^{1+N})_n(q^{1-N})_n. \tag{2.1}$$

Here we normalize the colored Jones polynomial to be $J_N(\text{unknot};q) = 1$. We mean $J_1(K;q) = 1$, and the $N = 2$ case corresponds to the Jones polynomial. Habiro proved that for arbitrary K we have $C_K(n) \in \mathbb{Z}[q, q^{-1}]$ [13].

For example, we suppose that the 3-manifold M_2 is constructed by $+2$-surgery on a knot K. Then we have $H_1(M_2;\mathbb{Z}) = \mathbb{Z}_2$, and the SU(2) WRT invariant $\tau_N(M_2)$ for M_2 is computed as [21]

$$(1-\zeta_N)\tau_N(M_2) = -\frac{1}{2\sqrt{2}}(1+(-1)^N)\zeta_N^{3/4}\sum_{n=0}^{\infty} C_K(n)\ \mathcal{L}_{2:N}\left[(q^N)_{n+1}(q^{-N})_{n+1}\right]\Big|_{q=\zeta_N}. \tag{2.2}$$

Here $\zeta_N = e^{2\pi i/N}$, and $\mathcal{L}$ denotes the Laplace transform [4]. Explicitly we have

$$\mathcal{L}_{2:N}\left[(q^N)_{n+1}(q^{-N})_{n+1}\right] = 2(-1)^{n+1}q^{-\frac{n+1}{2}}(1-q^{\frac{1}{2}})(-q;-q^{\frac{1}{2}})_{2n}. \tag{2.3}$$

Thus the unified WRT invariant $I_q(M_2)$ for M_2 is written as

$$(1-q)I_q(M_2) = \sqrt{2}q^{\frac{1}{4}}\sum_{n=0}^{\infty}C_K(n)(-1)^n q^{-\frac{n}{2}}(q^{\frac{1}{2}};-q^{\frac{1}{2}})_{2n+1}. \tag{2.4}$$

We set M to be the Seifert manifold $M(2,3,8)$ (see, *e.g.*, [20]), which is obtained from $+2$-surgery on the trefoil. Applying (2.4) to the colored Jones polynomial for the trefoil [11, 17]

$$J_N(\text{trefoil};q) = \sum_{n=0}^{\infty}q^n(q^{1+N})_n(q^{1-N})_n, \tag{2.5}$$

we obtain

$$(1-q)I_q(M) = \sqrt{2}q^{\frac{1}{4}}\sum_{n=0}^{\infty}(-1)^n q^{\frac{n}{2}}(q^{\frac{1}{2}};-q^{\frac{1}{2}})_{2n+1}. \tag{2.6}$$

This establishes the first part of Theorem 1.1.

Now to prove that $\overline{\phi}_0(q)$ is a mock theta function, one is tempted to apply the results in [24], after showing that

$$\begin{aligned}\overline{\phi}_0(q) &= \frac{(-q)_\infty}{(q)_\infty}\sum_{n\geq 0}\sum_{j=-n-1}^{n}(-1)^j q^{4n^2+7n-3j^2-5j}(1-q^{2n+2}) \\ &= \frac{(-q)_\infty}{(q)_\infty}\left(\sum_{\substack{r,s\geq 0\\ r\not\equiv s \pmod 2}} - \sum_{\substack{r,s<0\\ r\not\equiv s \pmod 2}}\right)(-1)^{(r-s-1)/2}q^{(r+s+1)^2/4+(3r+2)s}\end{aligned} \tag{2.7}$$

using standard Bailey pair techniques in [2]. However, it turns out that we may prove (1.5) directly.

We use a combinatorial argument. Since $n^2 = 1+3+\cdots+(2n-1)$ and $1/(q;q^2)_n$ is the generating function for partitions into odd parts less then $2n+1$, it is clear that $\psi(q)$ is the generating function for partitions into odd parts without gaps, i.e., where all odd parts $< 2n-1$ occur if $2n-1$ occurs. (This was first observed by N. Fine [8, p.57].) Now consider

$$q^2\overline{\phi}_0(q^2) = \sum_{n\geq 0}q^{2n+2}(-q^2;q^2)_{2n+1}.$$

We interpret this graphically. The term $(-q^2;q^2)_{2n+1}$ contributes a partition λ into distinct even parts of size at most $4n+2$. We represent λ as rows of 2's of length at most $2n+1$. The term q^{2n+2} contributes a row of $2n+2$ ones, which we place above λ. For example, if $n=4$ and $\lambda = (16,12,8,4,2)$, we have

1111111111
22222222
222222
2222
22
2

Reading the columns, we obtain a partition into $2n+2$ odd parts without gaps, the fact that there are no gaps coming from the fact that the rows of twos are of unequal length. In the example, we obtain $(11,9,7,7,5,5,3,3,1,1)$. Since the number of columns is even, the number being partitioned is even. Thus $q^2\overline{\phi}_0(q^2)$ is the "even part" of $\psi(q)$, i.e. $(\psi(q)+\psi(-q))/2$. This proves (1.5).

Now using the modern definition of a mock theta function as the holomorphic part of a harmonic Maass form [5, 23, 24], it is clear that if $f(q)$ is a mock theta function, then $f(q)+f(-q)$ is either a mock theta function or a modular form. To finish the proof of Theorem 1.1 we need to verify that $\psi(q)+\psi(-q)$ is not modular. (It is worth pointing out that it can happen that $f(q)+f(-q)$ is modular when $f(q)$ is mock, for example when f is the second order mock theta function $B(q)$ [10, Section 8].) To see this, recall that a mock theta function is modular iff its shadow (see [23] for the definition) is zero. Using [22, p. 65] and [24, Chapter 2], one may compute that $\psi(q)$ has (up to a non-zero constant)

$$\sum_{n\in\mathbb{Z}}(6n+1)q^{n(3n+1)/2}$$

as shadow. From this one can conclude that $\psi(q)+\psi(-q)$ has (up to a non-zero constant) the shadow

$$\sum_{n\in\mathbb{Z}}(6n+1)\left(1+(-1)^{n(3n+1)/2}\right)q^{n(3n+1)/2}. \tag{2.8}$$

One can easily verify that (2.8) is nonzero. □

Before continuing, we wish to make some remarks. First, there is a companion to $\overline{\phi}_0(q)$ which is the "odd part" of $\psi(q)$. Define $\overline{\phi}_1(q)$ by

$$\overline{\phi}_1(q)=\sum_{n\geq 0}q^n(-q)_{2n}=1+q+2q^2+3q^3+4q^4+5q^5+7q^6+9q^7+12q^8+\cdots. \tag{2.9}$$

Arguing as above it is easy to see that

$$2q\overline{\phi}_1(q^2)=\psi(q)-\psi(-q) \tag{2.10}$$

and that $\psi(q)-\psi(-q)$ is not modular, but mock.

Next, while we have opted for the combinatorial argument above, (1.5) and (2.10) can also be deduced from the q-series identity

$$\sum_{n\geq 1}\frac{b^nq^{n(n+1)/2}}{(tq)_n}=bq\sum_{n\geq 0}(-bq/t)_n(tq)^n, \tag{2.11}$$

which follows from [8, Eq. (6.1)] or by counting partitions into distinct parts in two different ways. Namely, if we let $G(b,t;q)$ denote the left-hand side of (2.11), then we have

$$\begin{aligned}\psi(q)+\psi(-q)&=2G(1/q,1/q;q^2)+2G(-1/q,-1/q;q^2)\\&=2\sum_{n\geq 1}q^n(-q^2;q^2)_{n-1}+2\sum_{n\geq 1}(-1)^nq^n(-q^2;q^2)_{n-1}\quad\text{(by (2.11))}\\&=2\sum_{n\geq 1}q^{2n}(-q^2;q^2)_{2n-1}\\&=2q^2\overline{\phi}_0(q^2).\end{aligned}$$

A similar argument yields (2.10).

If we allow more complicated series involving multiple sums and the q-binomial coefficients,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q)_n}{(q)_k (q)_{n-k}}$$

then this kind of argument can be applied to other mock theta functions. For a simple example, consider Ramanujan's fifth order mock theta function

$$F_1(q) := \sum_{n\geq 0} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}}.$$

Define $H(t;q)$ by

$$H(t;q) := \sum_{n\geq 0} \frac{q^{2n^2+2n}}{(tq;q^2)_{n+1}},$$

so that $F_1(q) \pm F_1(-q) = H(1;q) \pm H(-1;q)$. Then by [1, Eq. (3.37)] we have

$$H(t;q) = \sum_{n\geq 0} q^{2n^2+2n} \sum_{m\geq 0} t^m q^m \begin{bmatrix} n+m \\ m \end{bmatrix}_{q^2}.$$

Thus each of

$$\sum_{m,n\geq 0} q^{n^2+n+m} \begin{bmatrix} n+2m \\ 2m \end{bmatrix}_q \tag{2.12}$$

and

$$\sum_{m,n\geq 0} q^{n^2+n+m+1} \begin{bmatrix} n+2m+1 \\ 2m+1 \end{bmatrix}_q \tag{2.13}$$

is either modular or mock. We have verified that they are both indeed mock.

3. Proof of Theorem 1.2

In general when the manifold M_{-2} is obtained from -2 surgery on the knot K, we have

$$(1-\zeta_N)\tau_N(M_{-2}) = \frac{1}{2\sqrt{2}}(1+(-1)^N)\zeta_N^{1/4} \sum_{n=0}^{\infty} C_K(n)\, \mathcal{L}_{-2:N}\left[(q^N)_{n+1}(q^{-N})_{n+1}\right]\Big|_{q=\zeta_N}, \tag{3.1}$$

where the Laplace transformation (2.3) is replaced with [4]

$$\mathcal{L}_{-2:N}\left[(q^N)_{n+1}(q^{-N})_{n+1}\right] = 2(1-q^{\frac{1}{2}})(-q;-q^{\frac{1}{2}})_{2n}. \tag{3.2}$$

Thus the unified WRT invariant is given by

$$(1-q)I_q(M_{-2}) = \sqrt{2}q^{\frac{1}{4}} \sum_{n=0}^{\infty} C_K(n)(q^{\frac{1}{2}};-q^{\frac{1}{2}})_{2n+1}. \tag{3.3}$$

In the case that K is the trefoil, whose colored Jones polynomial is given in (2.5), M_{-2} is the Seifert manifold $M(2,3,4)$, and substituting for $C_K(n)$ gives the first part of Theorem 1.2.

To confirm (1.7) we note that $q^2M(q)$ is the generating function for partitions into distinct parts whose largest part is even. On the other hand, $(-q)_\infty$ is the generating function for all partitions into distinct parts, and by Franklin's involution (see [1]) we have that the final term in (1.7) is the generating function for partitions into distinct parts whose largest part is even minus the number of partitions into distinct parts whose largest part is odd. This completes the proof of Theorem 1.2. □

4. Concluding remarks

It is to be hoped that more unified WRT invariants will give rise to new mock theta functions, but this remains to be seen. These invariants are not easy to compute and when they can be computed they are often only defined at roots of unity, since the coefficients $C_K(n)$ in (2.1) are only guaranteed to be in $\mathbb{Z}\left[q, q^{-1}\right]$. The following describes one instance where (2.4) and (3.3) are convergent q-series.

The colored Jones polynomial for the twist knot $K_{p>0}$ is given by [18]

$$J_N(K_p;q) = \sum_{s_p \geq \cdots \geq s_2 \geq s_1 \geq 0}^{\infty} q^{s_p}(q^{1-N})_{s_p}(q^{1+N})_{s_p} \prod_{i=1}^{p-1} q^{s_i(s_i+1)} \begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix}_q . \tag{4.1}$$

It is known [6] that ± 2 surgery on the twist knot K_p is the Seifert manifold. Applying the Laplace transforms (2.3) and (3.2) to (4.1) we get the q-series related to the unified WRT invariants;

$$A_p(q) = \sum_{s_p \geq \cdots \geq s_1 \geq 0}^{\infty} q^{s_p}(q^{\frac{1}{2}}; -q^{\frac{1}{2}})_{2s_p+1} \prod_{i=1}^{p-1} q^{s_i(s_i+1)} \begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix}_q ,$$

$$B_p(q) = \sum_{s_p \geq \cdots \geq s_1 \geq 0}^{\infty} (-1)^{s_p} q^{\frac{s_p}{2}}(q^{\frac{1}{2}}; -q^{\frac{1}{2}})_{2s_p+1} \prod_{i=1}^{p-1} q^{s_i(s_i+1)} \begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix}_q .$$

For now we have no futher information about these series for $p \geq 2$ which would allow us to deduce any automorphic properties.

Finally, if we would like combinatorial interpretations for the undilated $\overline{\phi}_0(q)$ and $\overline{\phi}_1(q)$ instead of $\overline{\phi}_0(q^2)$ and $\overline{\phi}_1(q^2)$, then we may appeal to overpartitions [7]. Arguing as in Section 2, we find that for $i = 0$ or 1, the series $q\overline{\phi}_i(q)$ is the generating function for overpartitions into odd parts such that (i) the non-overlined parts are without gaps and (ii) the overlined parts are less than $2k - 2i$, where k is the number of non-overlined parts. We point out that two eighth order/second order mock theta functions of Gordon and McIntosh may also be interpreted in terms of overpartitions into odd parts where the non-overlined parts are without gaps. These are [9, 19]

$$V_0(q) := -1 + 2\sum_{n \geq 0} \frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n}$$

and

$$V_1(q) := \sum_{n \geq 0} \frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}}.$$

All of this suggests that further series related to overpartitions where the non-overlined parts are without gaps are well worth investigating.

References

[1] G.E. Andrews, The Theory of Partitions, Cambridge Univ. Press, Cambridge, 1998.
[2] G.E. Andrews, The fifth and seventh order mock theta functions, Trans. Amer. Math. Soc. 293 (1986), 113-134.
[3] G.E. Andrews, F.J. Dyson, and D. Hickerson, Partitions and indefinite quadratic forms, Invent. Math. 91 (1988), 391–407.

[4] A. Beliakova and T. T. Q. Le, Integrality of quantum 3-manifold invariants and a rational surgery formula, Compos. Math. 143 (2007), 1593–1612.
[5] K. Bringmann and K. Ono, Dyson's ranks and Maass forms, Ann. Math. 171 (2010), 419-449.
[6] M. Brittenham and Y.-Q. Wu, The classification of exceptional Dehn surgeries on 2-bridge knots, Comm. Anal. Geom. 9 (2001), 97–113.
[7] S. Corteel and J. Lovejoy, Overpartitions, Trans. Amer. Math. Soc. 356 (2004), 1623-1635.
[8] N. Fine, Basic Hypergeometric Series and Applications, AMS, Prividence, 1988.
[9] B. Gordon and R.J. McIntosh, Some eighth order mock theta functions, J. London Math. Soc. 62 (2000), 321-335.
[10] B. Gordon and R.J. McIntosh, A survey of classical mock theta functions, Ramanujan J., to appear.
[11] K. Habiro, On the colored Jones polynomial of some simple links, RIMS Kokyuroku 1172 (2000), 34–43.
[12] K. Habiro, On the quantum $\mathfrak{sl}_2$ invariants of knots and integral homology spheres, Geometry and Topology Monographs 4 (2002), 55–68.
[13] K. Habiro, A unified Witten–Reshetikhin–Turaev invariant for integral homology sphere, Invent. Math. 171 (2008), 1–81.
[14] K. Hikami, Mock (false) theta functions as quantum invariants, Regul. Chaotic Dyn. 10, No. 4 (2005), 509–530.
[15] K. Hikami, Hecke type formula for unified Witten-Reshetikhin-Turaev invariants as higher-order mock theta functions, Int. Math. Res. Not. (2007), Article ID rnm022.
[16] R. Lawrence and D. Zagier, Modular forms and quantum invariants of 3-manifolds, Asian J. Math. 3 (1999), 93-107.
[17] T. T. Q. Le, Quantum invariants of 3-manifolds: Integrality, splitting, and perturbative expansion, Topology Appl. 127 (2003), 125–152.
[18] G. Masbaum, Skein-theoretical derivation of some formulas of Habiro, Algebr. Geom. Topol. 3 (2003), 537–556.
[19] R.J. McIntosh, Second order mock theta functions, Canad. Math. Bull. 50 (2007), 284-290.
[20] J. Milnor, On the 3-dimensional Brieskorn manifolds $M(p,q,r)$, in "Knots, Groups, and 3-manifolds. Dedicated to the Memory of R. H. Fox", edited by L. P. Neuwirth, Princeton Univ. Press, pp.175–225 (1975).
[21] N. Yu. Reshetikhin and V. G. Turaev, Invariants of 3-manifolds via link polynomials and quantum groups, Invent. Math. 103 (1991), 547–597.
[22] G.N. Watson, The final problem: An account of the mock theta functions, J. London Math. Soc. 11 (1936), 55-80.
[23] D. Zagier, Ramanujan's mock theta functions and their applications [d'aprés Zwegers and Bringmann-Ono], Séminaire Bourbaki, no 986.
[24] S. Zwegers, Mock Theta Functions, PhD Thesis, Universiteit Utrecht (2002).

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