

SPECIAL L -VALUES AND PERIODS OF WEAKLY HOLOMORPHIC MODULAR FORMS

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ABSTRACT. In this paper, we explore a method for associating L -series to weakly holomorphic modular forms, and then proceed to study their L -values. As our main application, we prove a very curious limiting theorem which relates three “periods” of a mock modular form and its shadow to the ratio of their non-critical L -values. Critical L -values are then shown to fit nicely within the framework of period polynomials and an extended Eichler-Shimura theory recently studied by Guerzhoy, Ono, and the first and third author.

1. INTRODUCTION AND STATEMENT OF RESULTS

Special values of L -series have been the subject of intense study in arithmetic algebraic geometry and analytic number theory. In particular, Kohnen and Zagier [13] and many others [6, 11] have pursued and made use of an underlying algebraic theory of special values of L -series associated to (holomorphic) modular forms. On the other hand, such values are central objects in the deep conjectures posed by Birch-Swinnerton-Dyer, Beilinson, and Bloch-Kato (see e.g. [1, 5, 7, 10, 14, 15, 19, 20, 21]).

Here we consider special values of L -series associated to weakly holomorphic modular forms (those meromorphic modular forms whose only poles are supported at the cusps). The very existence of such values has proven elusive in the past due to the fact that the classical definitions in terms of infinite summation or integrals taken along the positive imaginary axis of the complex plane are divergent in this more general setting.

To precisely describe the problem we address in this paper, we briefly introduce the classical setting. Let $f(z) = \sum_{m=1}^{\infty} a_f(m)q^m \in S_k$ ($q := e^{2\pi iz}$ as usual) be a holomorphic cusp form of weight $k \in 2\mathbb{N}$ on $\mathrm{SL}_2(\mathbb{Z})$. Then for $\mathrm{Re}(s) \gg 0$, the L -series associated to f is defined by

$$(1.1) \quad L_f(s) := \sum_{m=1}^{\infty} \frac{a_f(m)}{m^s}.$$

It is known that $L_f(s)$ satisfies a functional equation under $s \mapsto k - s$ and has an analytic continuation to the complex plane (which by abuse of notation we also denote by $L_f(s)$). The *special values* of $L_f(s)$ are those at integral arguments. In particular, the *critical values*, those special values of $L_f(s)$ that lay within the critical strip $0 < \mathrm{Re}(s) < k$, coincide with certain period integrals of f as follows:

$$(1.2) \quad L_f(n) = \frac{(2\pi)^n}{(n-1)!} \int_0^{\infty} f(iy)y^{n-1} dy \quad \text{for } n = 1, 2, \dots, k-1.$$

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Moreover, when f is a Hecke eigenform, Manin's Periods Theorem [16] states that the critical values are essentially algebraic integers. To be more precise, there exist two nonzero complex numbers ω_f^\pm , known as *periods* for f , such that the quotient $L_f^*(n)/\omega_f^{sgn((-1)^{n+1})}$ is an algebraic integer for $n = 1, 2, \dots, k-1$.

Now consider $f(z) = \sum_{m \geq m_0} a_f(m)q^m \in S_k^!$ (with m_0 throughout the smallest integer s.t. $a_f(m_0) \neq 0$), where $S_k^!$ denotes the space of weight k weakly holomorphic cusp forms (i.e., those weakly holomorphic modular forms satisfying $a_f(0) = 0$). In [2], Guerzhoy, Ono, and the first and third authors made many observations regarding critical values of holomorphic cusp forms by extending the classical theory of periods to the much larger space $S_k^!$. Clearly (1.1) and (1.2) are no longer well-defined for general forms in $S_k^!$ due to their possible pole at $i\infty$ and the growth of their coefficients. To get around this issue, the authors of [2] use the formal *Eichler integral* given by

$$(1.3) \quad \mathcal{E}_f(z) := \sum_{m \geq m_0} \frac{a_f(m)}{m^{k-1}} q^m$$

rather than a period integral to develop their theory. In the case when $f \in S_k$, we have that

$$(1.4) \quad \mathcal{E}_f(z) - z^{k-2} \mathcal{E}_f\left(\frac{-1}{z}\right) = \sum_{n=0}^{k-2} \frac{L_f(n+1)}{(k-2-n)!} \cdot (2\pi iz)^{k-2-n}.$$

In general, the left hand side of (1.4) is always defined for $f \in S_k^!$ and always a polynomial in z . Therefore, as an interesting side effect, this suggests the existence of an L -series for weakly holomorphic cusp forms.

Indeed, in §2, we introduce a way to regularize the period integrals in (1.2) so that they are well-defined for all forms in $S_k^!$. In doing so, we are able to define L -series for $f \in S_k^!$ and $t_0 > 0$ by

$$(1.5) \quad \begin{aligned} L_f(s) &:= \frac{(2\pi)^s}{\Gamma(s)} L_f^*(s) \quad \text{with} \\ L_f^*(s) &:= \sum_{m \geq m_0} \frac{a_f(m) \Gamma(s, 2\pi m t_0)}{(2\pi m)^s} + i^k \sum_{m \geq m_0} \frac{a_f(m) \Gamma\left(k - s, \frac{2\pi m}{t_0}\right)}{(2\pi m)^{k-s}}. \end{aligned}$$

Here the incomplete gamma function, $\Gamma(s, z)$ is given by the analytic continuation (to an entire function with respect to s and fixed $z \neq 0$) of $\int_z^\infty e^{-t} t^{s-1} dt$.

Remarks. (1) For cusp forms the L -series $L_f^*(s)$ has been defined in the shape of (1.5) by Kohnen and Zagier [13] in order to carry out numerical calculations.

(2) Absolute convergence of $L_f^*(s)$ is guaranteed since $\Gamma(s, x) \sim x^{s-1} e^{-x}$ as $x \rightarrow \infty$ and the Circle Method implies that

$$(1.6) \quad a_f(m) \ll m^{\frac{1}{2}(k-\frac{3}{2})} e^{4\pi\sqrt{m_0 m}}.$$

(3) Using the modularity of f , it is not hard to see that (1.5) is independent of t_0 . Moreover, we find in Theorem 2.2 that $L_f^*(s)$ satisfies a functional equation under $s \mapsto k-s$ and has an integral representation.

Armed now with the L -series for $f \in S_k^!$, we are in a position to study their special values and their associated periods. In §2, we give a detailed account of regularized integration and how it relates to $L_f(s)$ and the formal Eichler integrals (1.3) of [2].

To state our first main result, we require harmonic weak Maass forms. The theory of harmonic weak Maass forms [3], for example explains Ramanujan's mock theta functions [22, 23, 24]. Every harmonic weak Maass form $\mathcal{F}(z)$ has a unique natural decomposition

$$\mathcal{F}(z) = \mathcal{F}^-(z) + \mathcal{F}^+(z),$$

where \mathcal{F}^- (resp. \mathcal{F}^+) is nonholomorphic (resp. holomorphic) on the complex upper-half plane. The holomorphic part \mathcal{F}^+ has a Fourier expansion

$$\mathcal{F}^+(z) = \sum_{m \geq m_0} c_{\mathcal{F}}^+(m) q^m$$

which, following Zagier, we call a *mock modular form*. The nonholomorphic part has an expansion in incomplete gamma functions. The differential operators D^{k-1} with $D := \frac{1}{2\pi i} \cdot \frac{\partial}{\partial z}$, and $\xi_{2-k} := 2iy^{2-k} \cdot \frac{\partial}{\partial \bar{z}}$, which play a central role in the theory, only see the holomorphic and nonholomorphic parts, respectively, of such forms. Furthermore, in [3, 4] it is shown that they define surjective maps

$$\xi_{2-k} : \begin{cases} H_{2-k} \rightarrow M_k^! \\ H_{2-k}^* \rightarrow S_k, \end{cases} \quad \text{and} \quad D^{k-1} : \begin{cases} H_{2-k} \rightarrow M_k^! \\ H_{2-k}^* \rightarrow S_k^!, \end{cases}$$

where $M_k^!$ denotes the space of weight k weakly holomorphic modular forms, H_{2-k} is the space of harmonic weak Maass forms of weight $2-k$, and $H_{2-k}^* \subseteq H_{2-k}$ is the subspace of harmonic weak Maass forms of weight $2-k$ whose image under ξ_{2-k} maps onto S_k .

Following Proposition 5.1 of [4], we are always able to find a harmonic weak Maass form $\mathcal{F} \in H_{2-k}^*$ which is *good* for a normalized Hecke eigenform $g \in S_k$ (see §3 for the definition). In general, the coefficients of the corresponding \mathcal{F}^+ appear to be transcendental. However, in [9], Guerzhoy, Ono, and the third author proved that the coefficients of the corrected form, $D^{k-1}(\mathcal{F}^+) - c_{\mathcal{F}}^+(1)g$, are in K_g (where $K_g = \mathbb{Q}(a_g(1), a_g(2), \dots)$). This coefficient $c_{\mathcal{F}}^+(1)$ of \mathcal{F}^+ , is considered to be a third *period* associated to the eigenform $g \in S_k$. It turns out (see Corollary 4.4) that there is a nice relation between the L -series from $D^{k-1}(\mathcal{F})$ and $\xi_{2-k}(\mathcal{F})$ inside the critical strip. Numerical computations show that such a result is not true at integers outside the critical strip. However, surprisingly, the quotient of the two L -values has some interesting limiting behavior as $n \rightarrow \infty$ which relates the three periods of an eigenform $g \in S_k$.

Theorem 1.1. *Let $g \in S_k$ be a normalized Hecke eigenform with periods ω_g^{\pm} chosen such that $\langle g, g \rangle = \omega_g^+ \omega_g^-$, where $\langle g, g \rangle$ is the usual Peterson inner product. Let $\mathcal{F} \in H_{2-k}^*$ be good for g . Then for $\delta = 0$ or 1 we have that*

$$\lim_{n \equiv \delta \pmod{2}} \frac{L_{D^{k-1}(\mathcal{F})}(n)}{L_{\xi_{2-k}(\mathcal{F})}(n)} = \left(c_{\mathcal{F}}^+(1) + (-1)^{\delta} c_{\mathcal{F}}^+(-1) \right) \omega_g^+ \omega_g^-.$$

Remarks. (1) By a theorem of Kohnen and Zagier (p. 202 of [12]), we may always choose ω_g^{\pm} such that $\langle g, g \rangle = \omega_g^+ \omega_g^-$ for a normalized Hecke eigenform $g \in S_k$.

(2) One can prove an analogous result for twisted L -series as defined in (1.7).

As our other application, we extend a result of [2] which shows that forms $\mathcal{F} \in H_{2-k}^*$ encode the critical values of L -functions arising from S_k to the weakly holomorphic setting.

We also define twisted L -series associated to forms in $S_k^!$. We first recall the situation for cusp forms $f \in S_k$. For a Dirichlet character χ with conductor c , we define the L -series of f twisted by χ by

$$L_f(\chi, s) := \sum_{m=1}^{\infty} \frac{\chi(m)a_f(m)}{m^s}.$$

We define the associated twisted L -series, for $t_0 > 0$, by

$$L_f(\chi, s) := \frac{(2\pi)^s}{\Gamma(s)} L_f^*(\chi, s)$$

with

$$(1.7) \quad L_f^*(\chi, s) := \sum_{m \geq m_0} \frac{\chi(m)a_f(m)\Gamma(s, 2\pi m t_0)}{(2\pi m)^s} + i^k c^{k-2s} \sum_{m \geq m_0} \frac{\chi(-m)a_f(m)\Gamma\left(k-s, \frac{2\pi m}{c^2 t_0}\right)}{(2\pi m)^{k-s}}.$$

Moreover, for each $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, we define the γ -mock modular period function for \mathcal{F}^+ by

$$(1.8) \quad \mathbb{P}(\mathcal{F}^+, \gamma; z) := \frac{(4\pi)^{k-1}}{(k-2)!} (\mathcal{F}^+ - \mathcal{F}^+|_{2-k}\gamma)(z),$$

where for any function $g : \mathbb{H} \rightarrow \mathbb{C}$, we let $g|_{\ell}\gamma(z) := (cz+d)^{-\ell} g\left(\frac{az+b}{cz+d}\right)$. For a positive integer c , let $\zeta_c := e^{2\pi i/c}$, and for $0 \leq d < c$, let $\gamma_{c,d} \in \mathrm{SL}_2(\mathbb{Z})$ be any matrix satisfying $\gamma_{c,d} := \begin{pmatrix} * & * \\ c' & d' \end{pmatrix}$. Here the integers $0 \leq d' < c'$ are chosen so that $\frac{d}{c} = \frac{d'}{c'}$ in lowest terms. We then obtain

Theorem 1.2. *Let $\mathcal{F} \in H_{2-k}$ such that $f := \xi_{2-k}(\mathcal{F}) \in S_k^!$. Then we have that*

$$\overline{\mathbb{P}(\mathcal{F}^+, \gamma_{1,0}; \bar{z})} = \sum_{n=0}^{k-2} \frac{L_f(n+1)}{(k-2-n)!} (2\pi i z)^{k-2-n}.$$

Moreover, if χ is a Dirichlet character with conductor c , then

$$\frac{1}{c} \sum_{\ell \pmod{c}^*} \chi(\ell) \sum_{d=0}^{c-1} \zeta_c^{\ell d} \cdot \overline{\mathbb{P}\left(\mathcal{F}^+, \gamma_{c,d}; \bar{z} - \frac{d}{c}\right)} = \sum_{n=0}^{k-2} \frac{L_f(\chi, n+1)}{(k-2-n)!} (2\pi i z)^{k-2-n},$$

where the sum on ℓ runs over those integers coprime to c .

Remark. Theorem 1.2 was proven in [2] is the special case when $\xi_{2-k}(\mathcal{F}) \in S_k$.

In §2, we define regularized L -series and show that they have an integral representation. We also show (see Theorem 2.4) that the error of modularity of an Eichler integral may be viewed as the generating function of critical L -values. We also prove certain vanishing results for a subclass of weakly holomorphic forms. In §3, we recall basic facts regarding the theory of harmonic weak Maass forms and the extended Eichler-Shimura theory of [2]. In §4, we discuss various ways to twist the L -series and derive related results including Theorem 1.2. In §5, we prove Theorem 1.1.

2. REGULARIZED L -SERIES FOR $S_k^!$

In this section we describe how to associate L -series to weakly holomorphic cusp forms. Let us start by recalling what is known for classical cusp forms.

Theorem 2.1. *If $f \in S_k$, we have for $s \gg 0$ the integral representation*

$$L_f^*(s) = \int_0^\infty f(iy)y^{s-1}dy.$$

Furthermore L_f^* has a holomorphic continuation to \mathbb{C} and we have the functional equation

$$L_f^*(k-s) = i^k L_f^*(s).$$

To generalize these results to the setting of weakly holomorphic forms, we require certain regularized integrals. For this, consider a continuous function $f : \mathbb{H} \rightarrow \mathbb{C}$. Assume that there is a $c \in \mathbb{R}^+$ such that

$$(2.1) \quad f(z) = O\left(e^{c\operatorname{Im}(z)}\right)$$

uniformly in $\operatorname{Re}(z)$ as $\operatorname{Im}(z) \rightarrow \infty$. Then, for each $z_0 \in \mathbb{H}$, the integral

$$\int_{z_0}^{i\infty} e^{uiv} f(w) dw$$

(where the path of integration lies within a vertical strip) is convergent for $u \in \mathbb{C}$ with $\operatorname{Re}(u) \gg 0$. If this integral has an analytic continuation to $u = 0$, we define the *regularized integral* by

$$R. \int_{z_0}^{i\infty} f(w) dw := \left[\int_{z_0}^{i\infty} e^{uiv} f(w) dw \right]_{u=0},$$

where the right hand side means that we take the value at $u = 0$ of the analytic continuation of the integral. Similarly, we define integrals at other cusps \mathfrak{a} . Specifically, suppose that $\mathfrak{a} = \sigma_{\mathfrak{a}}(i\infty)$ for a scaling matrix $\sigma_{\mathfrak{a}} \in \operatorname{SL}_2(\mathbb{Z})$. If $f(\sigma_{\mathfrak{a}}z)$ satisfies (2.1), then we define

$$R. \int_{z_0}^{\mathfrak{a}} f(w) dw := R. \int_{\sigma_{\mathfrak{a}}^{-1}z_0}^{i\infty} f|_2 \sigma_{\mathfrak{a}}(w) dw.$$

For cusps $\mathfrak{a}, \mathfrak{b}$, we set

$$(2.2) \quad R. \int_{\mathfrak{a}}^{\mathfrak{b}} f(w) dw := R. \int_{z_0}^{\mathfrak{b}} f(w) dw + R. \int_{\mathfrak{a}}^{z_0} f(w) dw$$

for any $z_0 \in \mathbb{H}$. It is not hard to see that this integral is independent of $z_0 \in \mathbb{H}$. We now prove an analogue to Theorem 2.1 which will also appear in the second author's thesis.

Theorem 2.2. *Assume that $f \in S_k^!$. We have the identity*

$$(2.3) \quad L_f^*(s) = R. \int_0^\infty f(iy)y^{s-1}dy.$$

Furthermore, $L_f^*(s)$ satisfies the functional equation

$$L_f^*(k-s) = i^k L_f^*(s).$$

Proof: To prove (2.3), we split the integral into two ranges. Inserting the Fourier expansion of $f(z) = \sum_{m \geq m_0} a_f(m) q^m$ yields that, for arbitrary $t_0 > 0$, we have

$$\begin{aligned} R. \int_{t_0}^{\infty} f(iy) y^{s-1} dy &= \sum_{m \geq m_0} a_f(m) \left[\int_{t_0}^{\infty} e^{-2\pi m y - t y} y^{s-1} dy \right]_{t=0} \\ &= \left[\sum_{m \geq m_0} \frac{a_f(m)}{(t + 2\pi m)^s} \int_{2\pi m t_0 + t t_0}^{\infty} e^{-y} y^{s-1} dy \right]_{t=0} = \sum_{m \geq m_0} \frac{a_f(m)}{(2\pi m)^s} \Gamma(s, 2\pi m t_0). \end{aligned}$$

Similarly we compute that

$$R. \int_0^{t_0} f(iy) y^{s-1} dy = i^{-k} \sum_{m \geq m_0} \frac{a_f(m) \Gamma\left(k - s, \frac{2\pi m}{t_0}\right)}{(2\pi m)^{k-s}}.$$

This directly yields (2.3). Moreover the functional equation follows directly from the independence of t_0 . \square

Recalling the Eichler integral (1.3), we find that if $f \in S_k$ then it is not hard to see that

$$\mathcal{E}_f(z) = c_k^{-1} \int_z^{i\infty} f(\tau) (z - \tau)^{k-2} d\tau,$$

where $c_k := -\frac{(k-2)!}{(2\pi i)^{k-1}}$. We show that a similar result holds when $f \in S_k^!$.

Lemma 2.3. *For $f \in S_k^!$, we have*

$$\mathcal{E}_f(z) = c_k^{-1} R. \int_z^{i\infty} f(\tau) (\tau - z)^{k-2} d\tau.$$

Proof: In the range of integration, the integrand has only a possible pole when $\tau \rightarrow i\infty$. Thus we compute, writing $f(z) = \sum_{m \geq m_0} a_f(m) q^m$,

$$\begin{aligned} R. \int_z^{i\infty} f(\tau) (\tau - z)^{k-2} d\tau &= \left[\int_z^{i\infty} e^{iu\tau} f(\tau) (\tau - z)^{k-2} d\tau \right]_{u=0} \\ &= \sum_{m \geq m_0} a_f(m) \left[\int_z^{i\infty} e^{iu\tau + 2\pi i m \tau} (\tau - z)^{k-2} d\tau \right]_{u=0} = \sum_{m \geq m_0} a_f(m) q^m \left[\int_0^{i\infty} e^{i\tau(u + 2\pi m)} \tau^{k-2} d\tau \right]_{u=0}. \end{aligned}$$

The integral now converges at $u = 0$ and inserting the integral representation of the gamma function yields the claim. \square

We next consider the “error of modularity” of Eichler integrals. To be more precise, let

$$r(f; z) := c_k \left(\mathcal{E}_f(z) - \mathcal{E}_f(z) \Big|_{2-k} S \right)$$

with $S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. It is known [2, 12] that $r(f; z)$ satisfies period relations, i.e.,

$$r(f; z) \in W := \left\{ P \in V_{k-2} : P + P \Big|_{2-k} S = P + P \Big|_{2-k} U + P \Big|_{2-k} U^2 = 0 \right\},$$

where $U := \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, and V_{k-2} denotes the space of polynomials of degree at most $k - 2$. We next prove that the period polynomial $r(f; z)$ represents the generating functions for L -values inside the critical strip (see [2] for the case of classical cusp forms).

Theorem 2.4. *We have that for $f \in S_k^!$*

$$r(f; z) = \sum_{n=0}^{k-2} i^{1-n} \binom{k-2}{n} L_f^*(n+1) z^{k-2-n}.$$

Proof: Making the change of variables $\tau \rightarrow -\frac{1}{\tau}$ in the second summand defining $r(f; z)$ and applying Lemma 2.3 easily gives that

$$r(f; z) = R. \int_0^{i\infty} f(\tau) (\tau - z)^{k-2} d\tau.$$

Now the claim follows using the binomial expansion, (1.5) and (2.3). \square

As an application, we next turn to some simple vanishing results of regularized L-values associated to forms in $D^{k-1}(M_{2-k}^!)$ which play an important role in the Hecke theory for weakly holomorphic forms [19].

Theorem 2.5. *If $f \in D^{k-1}(M_{2-k}^!)$, then we have for $2 \leq n \leq k-2$ that*

$$L_f(n) = 0.$$

Moreover $L_f(1) = L_f(k-1) = 0$ if and only if $f \in D^{k-1}(S_{2-k}^!)$.

Proof: Assume $f = D^{k-1}(F)$ with $F \in M_{2-k}^!$. We have that

$$\mathcal{E}_f = F + c,$$

where c is the negative of the constant term of F . Using that $F \in M_{2-k}^!$ gives that

$$\mathcal{E}_f(z) - z^{k-2} \mathcal{E}_f\left(-\frac{1}{z}\right) = c(1 - z^{k-2}).$$

From this we obtain the claim by Theorem 2.4. \square

3. HARMONIC WEAK MAASS FORMS

We let $z = x + iy \in \mathbb{H}$, with $x, y \in \mathbb{R}$, and suppose throughout that $k \geq 4$ is even. The weight $2-k$ hyperbolic Laplacian is defined by

$$\Delta_{2-k} := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + i(2-k)y \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

A harmonic (weak) Maass form of weight $2-k$ is any smooth function $\mathcal{F} : \mathbb{H} \rightarrow \mathbb{C}$ satisfying:

- (i) $\mathcal{F}(z) = F|_{2-k}\gamma(z)$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$;
- (ii) $\Delta_{2-k}(\mathcal{F}) = 0$;
- (iii) The function \mathcal{F} has at most linear exponential growth at infinity.

We denote the space of such forms by H_{2-k} . Moreover, we let $H_{2-k}^* \subset H_{2-k}$ be the subspace of forms that are mapped to S_k under ξ_{2-k} .

Every harmonic Maass form \mathcal{F} has two natural associated period polynomials, one coming from $\xi_{2-k}(\mathcal{F})$ and one from $D^{k-1}(\mathcal{F})$. The next theorem from [2] shows that these are related.

Theorem 3.1. *If $\mathcal{F} \in H_{2-k}$, then we have that*

$$r(\xi_{2-k}(\mathcal{F}); z) \equiv -\frac{(4\pi)^{k-1}}{(k-2)!} \cdot \overline{r(D^{k-1}(\mathcal{F}); \bar{z})} \pmod{z^{k-2} - 1}$$

where equivalence modulo $z^{k-2} - 1$ means that the difference of the two functions is a constant multiple of $z^{k-2} - 1$. Moreover, there is a function $\widehat{\mathcal{F}} \in H_{2-k}$ for which $\xi_{2-k}(\widehat{\mathcal{F}}) = \xi_{2-k}(\mathcal{F})$ and

$$r(\xi_{2-k}(\mathcal{F}); z) = -\frac{(4\pi)^{k-1}}{(k-2)!} \cdot \overline{r(D^{k-1}(\widehat{\mathcal{F}}); \bar{z})}.$$

This directly relates the L -values associated to the two pieces of a harmonic Maass form.

Corollary 3.2. *If $\mathcal{F} \in H_{2-k}$, then for integers $0 < n < k - 2$, we have that*

$$L_{\xi_{2-k}(\mathcal{F})}(n+1) = (-1)^n \frac{(4\pi)^{k-1}}{(k-2)!} \overline{L_{D^{k-1}(\mathcal{F})}(n+1)}.$$

Finally we need the notation of good. We say that a harmonic Maass form $\mathcal{F} \in H_{2-k}$ is *good* for the normalized Hecke eigenform $g(z) := \sum_{n=1}^{\infty} a_g(n)q^n \in S_k$ if it satisfies the following:

- (1) The principal part of \mathcal{F} at the cusp $i\infty$ belongs to $K_g[q^{-1}]$ where $K_g := \mathbb{Q}(a_g(1), a_g(2), \dots)$.
- (2) We have that $\xi_{2-k}(\mathcal{F}) = \frac{g}{(g, g)}$.

4. TWISTED L -SERIES

Throughout this section, we let χ be a Dirichlet character with conductor c . We require, as an auxiliary function, a regularized L -series associated to $f(z) = \sum_{m \geq m_0} a_f(m)q^m \in S_k^!$ twisted by roots of unity. To be more precise, for $(d, c) = 1$, we define, for $t > 0$

$$L_f(\zeta_c^d, s) := \frac{(2\pi)^s}{\Gamma(s)} L_f^*(\zeta_c^d, s) \quad \text{with}$$

$$L_f^*(\zeta_c^d, s) := \sum_{m \geq m_0} \frac{a_f(m) \zeta_c^{dm} \Gamma(s, 2\pi m t_0)}{(2\pi m)^s} + i^k c^{k-2s} \sum_{m \geq m_0} \frac{a_f(m) \zeta_c^{-am} \Gamma\left(k - s, \frac{2\pi m}{c^2 t_0}\right)}{(2\pi m)^{k-s}},$$

where a is defined by $ad \equiv 1 \pmod{c}$. As before we can see that this definition is independent of t_0 . A straightforward calculation allows us to write the L -series twisted by χ in terms of the above defined L -series.

Lemma 4.1. *We have that*

$$L_f^*(\chi, s) = \frac{1}{c} \sum_{\ell \pmod{c}^*} \chi(\ell) \sum_{d=0}^{c-1} \zeta_c^{-\ell d} L_f^*(\zeta_c^d, s).$$

A calculation completely analogous to that of Theorem 2.2 yields the following

Theorem 4.2. *Assume that $f \in S_k^!$. We have the integral representation*

$$L_f^*(\zeta_c^d, s) = i^{-s} R. \int_{\frac{d}{c}}^{i\infty} f(w) \left(w - \frac{d}{c}\right)^{s-1} dw.$$

Moreover it satisfies the functional equation

$$L_f^*(\zeta_c^d, k-s) = i^k c^{2s-k} L_f^*(\zeta_c^{-a}, s).$$

Clearly, by Lemma 4.1 we also obtain an integral representation and functional equation for the twisted L -series. Our next result concerns a twisted analogue of Corollary 3.2, relating the two L -functions naturally associated to a harmonic Maass form.

Theorem 4.3. *Let $\mathcal{F} \in H_{2-k}$ with $\xi_{2-k}(\mathcal{F}), D^{k-1}(\mathcal{F}) \in S_k^!$. Then we have for $0 < n < k-2$*

$$L_{\xi_{2-k}(\mathcal{F})}(\zeta_c^d, n+1) = (-1)^n \frac{(4\pi)^{k-1}}{(k-2)!} \overline{L_{D^{k-1}(\mathcal{F})}(\zeta_c^d, n+1)}.$$

Combining this theorem with Lemma 4.1 immediately gives us the following

Corollary 4.4. *Let $\mathcal{F} \in H_{2-k}$ with $\xi_{2-k}(\mathcal{F}), D^{k-1}(\mathcal{F}) \in S_k^!$. Then we have for $0 < n < k-2$*

$$L_{\xi_{2-k}(\mathcal{F})}(\chi, n+1) = (-1)^n \frac{(4\pi)^{k-1}}{(k-2)!} \overline{L_{D^{k-1}(\mathcal{F})}(\chi^-, n+1)},$$

where $\chi^-(m) := \chi(-m)$.

Proof of Theorem 4.3: Define

$$\mathcal{G}(z) := -(2i)^{1-k} R. \int_{-\bar{z}}^{i\infty} \xi_{2-k}(\mathcal{F})^c(w)(z+w)^{k-2} dw + c_k^{-1} R. \int_z^{i\infty} D^{k-1}(\mathcal{F})(w)(z-w)^{k-2} dw,$$

where $g^c(z) := \sum_{m \geq m_0} \overline{a_g(m)} q^m$ for $g(z) = \sum_{m \geq m_0} a_g(m) q^m \in S_k^!$. We show that \mathcal{G} and \mathcal{F} are identical up to a constant term. A direct calculation gives us that

$$\xi_{2-k}(\mathcal{G}) = \xi_{2-k}(\mathcal{F}).$$

Moreover, using Lemma 2.3, we obtain that

$$D^{k-1}(\mathcal{G}) = D^{k-1}(\mathcal{E}_{D^{k-1}(\mathcal{F})}) = D^{k-1}(\mathcal{F}).$$

It follows that

$$\mathcal{G} = \mathcal{F} + p,$$

where p is a polynomial of degree at most $k-2$. Because \mathcal{G} and \mathcal{F} are both translation invariant, we have that $p = a_0$ is constant. Using the modularity of \mathcal{F} with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ we have that

$$(4.1) \quad 0 = \mathcal{F} \Big|_{2-k} (1-\gamma) \left(z - \frac{d}{c} \right) = \mathcal{F}_1 \Big|_{2-k} (1-\gamma) \left(z - \frac{d}{c} \right) + \mathcal{F}_2 \Big|_{2-k} (1-\gamma) \left(z - \frac{d}{c} \right) - a_0 \Big|_{2-k} (1-\gamma) \left(z - \frac{d}{c} \right),$$

where

$$\mathcal{F}_1(z) := -(2i)^{1-k} R. \int_{-\bar{z}}^{i\infty} \xi_{2-k}(\mathcal{F})^c(w)(z+w)^{k-2} dw,$$

$$\mathcal{F}_2(z) := c_k^{-1} R. \int_z^{i\infty} D^{k-1}(\mathcal{F})(w)(z-w)^{k-2} dw.$$

Clearly we have

$$a_0 \Big|_{2-k} (1 - \gamma) \left(z - \frac{d}{c} \right) = a_0 \left(1 - (cz)^{k-2} \right).$$

Moreover, a direct calculation shows that

$$\begin{aligned} \mathcal{F}_1(z) \Big|_{2-k} \gamma &= -(2i)^{1-k} R. \int_{-\bar{z}}^{\frac{d}{c}} \xi_{2-k}(\mathcal{F})^c(w) (z+w)^{k-2} dw, \\ \mathcal{F}_2(z) \Big|_{2-k} \gamma &= c_k^{-1} R. \int_z^{-\frac{d}{c}} D^{k-1}(\mathcal{F})(w) (z-w)^{k-2} dw. \end{aligned}$$

Thus

$$\begin{aligned} &(2i)^{1-k} R. \int_{\frac{d}{c}}^{i\infty} \xi_{2-k}(\mathcal{F})^c(w) \left(z - \frac{d}{c} + w \right)^{k-2} dw \\ &\equiv c_k^{-1} R. \int_{-\frac{d}{c}}^{i\infty} D^{k-1}(\mathcal{F})(w) \left(z - \frac{d}{c} - w \right)^{k-2} dw \pmod{1 - (cz)^{k-2}}. \end{aligned}$$

Inserting the binomial expansion and using Theorem 4.2 then gives

$$\begin{aligned} &c_k^{-1} \sum_{n=0}^{k-2} i^{1-n} \binom{k-2}{n} L_{D^{k-1}(\mathcal{F})}^* (\zeta_c^{-d}, n+1) z^{k-2-n} \\ &\equiv (2i)^{1-k} \sum_{n=0}^{k-2} i^{1-n} \binom{k-2}{n} L_{\xi_{2-k}(\mathcal{F})^c}^* (\zeta_c^d, n+1) (-z)^{k-2-n} \pmod{1 - (cz)^{k-2}}. \end{aligned}$$

The claim now follows by comparing coefficients and using the fact that for integers $0 \leq n \leq k-2$ we have

$$(4.2) \quad L_{f^c}^* (\zeta_c^d, n+1) = \overline{L_f^* (\zeta_c^{-d}, n+1)}.$$

□

Proof of Theorem 1.2: Let $\mathcal{F} \in H_{2-k}$ with $f = \xi_{2-k}(\mathcal{F}) \in S_k^!$. By definition

$$\mathbb{P}(\mathcal{F}^+, \gamma_{c,d}; z) = -\frac{(4\pi)^{k-1}}{(k-2)!} (\mathcal{F}^- - \mathcal{F}^-|_{2-k}\gamma_{c,d})(z).$$

As in the proof of Theorem 4.3, we obtain

$$\mathbb{P}\left(\mathcal{F}^+, \gamma_{c,d}; z - \frac{d}{c}\right) = -\frac{(2\pi i)^{k-1}}{(k-2)!} \sum_{n=0}^{k-2} \binom{k-2}{n} i^{n+1} L_{f^c}^* (\zeta_c^d, n+1) z^{k-2-n}.$$

Next we use (4.2) to find that

$$\begin{aligned} \overline{\mathbb{P}\left(\mathcal{F}^+, \gamma_{c,d}; \bar{z} - \frac{d}{c}\right)} &= \frac{(2\pi i)^{k-1}}{(k-2)!} \sum_{n=0}^{k-2} \binom{k-2}{n} i^{-n-1} L_f^* (\zeta_c^{-d}, n+1) z^{k-2-n} \\ &= \sum_{n=0}^{k-2} \frac{L_f (\zeta_c^{-d}, n+1)}{(k-2-n)!} (2\pi i z)^{k-2-n}. \end{aligned}$$

Finally, we apply Lemma 4.1 to obtain the result.

□

5. PROOF OF THEOREM 1.1

We begin by recalling some basic facts regarding the incomplete gamma function (see Section 8 of [17] for more details). For $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 0$, we define the *incomplete gamma function* by

$$(5.1) \quad \Gamma(s, x) := \int_x^\infty e^{-t} t^s \frac{dt}{t}.$$

We note that the usual gamma function is given for $\operatorname{Re}(s) > 0$ by $\Gamma(s) := \Gamma(s, 0)$. Integrations by parts gives the following recurrence relation

$$\Gamma(s + 1, x) = e^{-x} x^s + s \Gamma(s, x).$$

There is an analytic continuation of $\Gamma(s, x)$ to an entire function with respect to s for fixed $x \neq 0$ (see Section 8.2 of [17]). Note that (5.1) also holds for $x > 0$ and $s \in \mathbb{C}$. Moreover for $n \in \mathbb{N}$, we require the following representation

$$(5.2) \quad \frac{\Gamma(n, x)}{(n-1)!} = e^{-x} \sum_{m=0}^{n-1} \frac{x^m}{m!}$$

and

$$(5.3) \quad \Gamma(-n, x) = (-1)^n \frac{\Gamma(0, x)}{n!} + \frac{(-1)^{n+1} e^{-x}}{n! x} \sum_{m=0}^{n-1} \frac{(-1)^m m!}{x^m}.$$

A main step in the proof of Theorem 1.1 is the following.

Proposition 5.1. *Let $\delta = 0$ or 1 , and $f(z) = \sum_{m \geq m_0} a_f(m) q^m \in S_k^!$. Then*

$$\lim_{n \equiv \delta \pmod{2}} \frac{(2\pi m_1)^n}{(n-1)!} L_f^*(n) = a_f(m_1) + (-1)^\delta a_f(-m_1),$$

where $m_1 \in \mathbb{N}$ is minimal with

$$a_f(m_1) + (-1)^\delta a_f(-m_1) \neq 0.$$

Proof: We consider the individual pieces of the L -series (1.5) for f and choose $t_0 = 1$. First we show that

$$\lim_{n \rightarrow \infty} \sum_{m \geq m_0} a_f(m) \frac{\Gamma(k-n, 2\pi m)}{(2\pi m)^{k-n}} = \lim_{n \rightarrow \infty} \sum_{m \geq m_0} a_f(m) \Gamma(-n, 2\pi m) (2\pi m)^n$$

exists. This then clearly gives that the corresponding contribution to the limit is 0 since for all x

$$(5.4) \quad \lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0.$$

Firstly we split off all negative indexed terms. For this, we show that for $|x| > 1$ the function $\Gamma(-n, x)x^n$ is bounded. Using (5.3), we estimate

$$\left| \Gamma(-n, x)x^n \right| \ll \frac{|x|^n}{n!} \sum_{m=0}^{n-1} \frac{m!}{|x|^m} = \sum_{m=0}^{n-1} \binom{n}{m}^{-1} \frac{|x|^{n-m}}{(n-m)!} \leq \sum_{m=1}^n \frac{|x|^m}{m!} < e^x.$$

Here all bounds depend on x .

We next show that

$$(5.5) \quad \lim_{n \rightarrow \infty} \sum_{m \geq 1} a_f(m) \Gamma(-n, 2\pi m) (2\pi m)^n = 0.$$

Using (1.6) gives that the sum on m for some $c > 0$ may be bounded against

$$(5.6) \quad \begin{aligned} &\ll \sum_{m \geq 1} (2\pi m)^{n+\frac{1}{2}(k-\frac{3}{2})} e^{c\sqrt{2\pi m}} \Gamma(-n, 2\pi m) \ll \int_{2\pi}^{\infty} \Gamma(-n, x) x^{n+\frac{1}{2}(k-\frac{3}{2})} e^{c\sqrt{x}} dx \\ &= \int_{2\pi}^{\infty} x^{\frac{1}{2}(k-\frac{3}{2})} e^{c\sqrt{x}} \int_1^{\infty} e^{-tx} t^{-n-1} dt dx. \end{aligned}$$

Using integration by parts, the inner integral may be estimated against

$$\frac{e^{-x}}{n} - \frac{x}{n} \int_1^{\infty} e^{-tx} t^{-n} dt \ll \frac{e^{-x}}{n} + \frac{x e^{-x}}{n} \int_0^{\infty} e^{-tx} dt \ll \frac{e^{-x}}{n}.$$

Thus (5.6) can be bounded by

$$\frac{1}{n} \int_{2\pi}^{\infty} e^{-x+c\sqrt{x}} dx \ll \frac{1}{n} \xrightarrow{n \rightarrow \infty} 0$$

giving (5.5).

We next show that for $M > \frac{2c^2}{\pi}$ we have that

$$(5.7) \quad \sum_{m \geq M} \frac{a_f(m) \Gamma(n, 2\pi m)}{(2\pi m)^n} \ll \frac{(n-2)!}{(\pi M)^n}.$$

Picking $M > 2m_1$ then gives that the corresponding contribution to the limit vanishes. To see (5.7) we use again (1.6) to bound the left-hand side against

$$\begin{aligned} &\ll (2\pi)^{-n} \sum_{m \geq M} m^{\frac{1}{2}(k-\frac{3}{2})-n} e^{c\sqrt{2\pi m}} \int_{2\pi m}^{\infty} e^{-t} t^{n-1} dt \ll (2\pi)^{-n} \sum_{m \geq M} m^{\frac{1}{2}(k-\frac{3}{2})-n} \int_{2\pi m}^{\infty} e^{-\frac{t}{2}} t^{n-1} dt \\ &\ll (2\pi)^{-n} \sum_{m \geq M} m^{\frac{1}{2}(k-\frac{3}{2})-n} \int_0^{\infty} e^{-\frac{t}{2}} t^{n-1} dt \ll \pi^{-n} (n-1)! \sum_{m \geq M} m^{\frac{1}{2}(k-\frac{3}{2})-n} \ll \frac{(n-2)!}{(\pi M)^n} \end{aligned}$$

as claimed.

For the remaining terms, we combine the m -th and the $-m$ -th term. For $n \equiv \delta \pmod{2}$ we have that

$$(5.8) \quad \begin{aligned} &\frac{(2\pi m_1)^n}{(n-1)!} \left(a_f(m) \frac{\Gamma(n, 2\pi m)}{(2\pi m)^n} + a_f(-m) \frac{\Gamma(n, -2\pi m)}{(-2\pi m)^n} \right) \\ &= \left(\frac{m_1}{m} \right)^n \left(a_f(m) \frac{\Gamma(n, 2\pi m)}{(n-1)!} + (-1)^\delta a_f(-m) \frac{\Gamma(n, -2\pi m)}{(n-1)!} \right). \end{aligned}$$

Using the fact (which easily follows from (5.2)) that for all real x ,

$$\lim_{n \rightarrow \infty} \frac{\Gamma(n, x)}{(n-1)!} = 1$$

we then obtain for $m \geq m_0$

$$\begin{aligned} & \lim_{\substack{n \rightarrow \infty \\ n \equiv \delta \pmod{2}}} \frac{(2\pi m_1)^n}{(n-1)!} \left(a_f(m) \frac{\Gamma(n, 2\pi m)}{(2\pi m)^n} + a_f(-m) \frac{\Gamma(n, -2\pi m)}{(-2\pi m)^n} \right) \\ &= \left(a_f(m) + (-1)^\delta a_f(-m) \right) \lim_{\substack{n \rightarrow \infty \\ n \equiv \delta \pmod{2}}} \left(\frac{m_1}{m} \right)^n \\ &= \begin{cases} a_f(m_1) + (-1)^\delta a_f(-m_1) & \text{if } m = m_1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

For $0 < m < m_1$, we have

$$(-1)^\delta a_f(-m) = -a_f(m)$$

by hypothesis. Using this reduces (5.8) to

$$(5.9) \quad a_f(m) \left(\frac{m_1}{m} \right)^n \frac{\Gamma(n, 2\pi m) - \Gamma(n, -2\pi m)}{(n-1)!}.$$

To finish the proof, it suffices to show that (5.9) vanishes as $n \rightarrow \infty$. To see this, we bound for $x > 1$

$$\Gamma(n, x) - \Gamma(n, -x) = \int_x^{-x} e^{-t} t^{n-1} dt \ll x^n.$$

Thus (5.9) may be estimated against

$$\ll \left(\frac{m_1}{m} \right)^n \frac{(2\pi m)^n}{(n-1)!} \xrightarrow{n \rightarrow \infty} 0$$

again using (5.4). □

Making use of Proposition 5.1 and (5.4), we immediately obtain the following three corollaries.

Corollary 5.2. *If $f \in S_k^!$, $f \not\equiv 0$, then*

$$\lim_{n \rightarrow \infty} |L_f^*(n)| = \infty.$$

Corollary 5.3. *For $f \in S_k^!$, $f \not\equiv 0$, at most finitely many values $L_f^*(n)$ vanish for $n \in \mathbb{N}$.*

Corollary 5.4. *Let $f, g \in S_k^!$ with $g \not\equiv 0$ and q -expansions given by $f(z) = \sum_{m \geq m_0} a_f(m) q^m$ and $g(z) = \sum_{m \geq m_0} a_g(m) q^m$. For $\delta = 0$ or 1 we have that*

$$\lim_{\substack{n \rightarrow \infty \\ n \equiv \delta \pmod{2}}} \frac{L_f^*(n)}{L_g^*(n)} = \frac{a_f(m_1) + (-1)^\delta a_f(-m_1)}{a_g(m_1) + (-1)^\delta a_g(-m_1)},$$

where m_1 is the smallest positive integer such that at least one of the numerator or denominator in the right hand side is nonzero.

Remark. In the limit above, the right hand side is taken to be infinity whenever the denominator is zero.

The proof of Theorem 1.1 is now almost immediate.

Proof of Theorem 1.1: Since $\mathcal{F} \in H_{2-k}^*$ is good for g , the q -expansion of $\xi_{2-k}(\mathcal{F})$ is given by

$$\xi_{2-k}(\mathcal{F})(z) = \langle g, g \rangle^{-1} \sum_{m=1}^{\infty} a_g(n) q^m$$

with $a_g(1) = 1$. Moreover, by Corollary 5.4, it follows that

$$\lim_{n \equiv \delta \pmod{2}} \lim_{n \rightarrow \infty} \frac{L_{D^{k-1}(\mathcal{F})}(n)}{L_{\xi_{2-k}(\mathcal{F})}(n)} = \lim_{n \equiv \delta \pmod{2}} \lim_{n \rightarrow \infty} \frac{L_{D^{k-1}(\mathcal{F})}^*(n)}{L_{\xi_{2-k}(\mathcal{F})}^*(n)} = (c_{\mathcal{F}}^+(1) + (-1)^\delta c_{\mathcal{F}}^+(-1)) \langle g, g \rangle.$$

The result may now be concluded since we have $\langle g, g \rangle = \omega_g^+ \omega_g^-$. □

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