Asymptotic formulas for some retricted partition functions

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Abstract

We use Rademacher's method to obtain asymptotic expressions for some restricted partition functions. For fixed positive integers $m > 1$ and l, we consider the number $p_m(n)$ of partitions of n into summands not divisible by m, and the number $p_m^l(n)$ of partitons of n with the further restriction that any integer occurs at most l times as a summand.

1 Introduction

Let n, m, l be positive integers, with $m > 1$. As usual, $p(n)$ denotes the number of distinct partitions of n with positive integral summands. We introduce the number $p_m(n)$ of partitions of n with the restriction that no summand is a multiple of m , and the number $p_m^l(n)$ of partitions of n into summands not divisible by m and with the further restriction that no integer occurs more than l times as a summand. We will investigate the asymptotic behaviour of these partition functions.

For any subset $A \subseteq \mathbb{N}$, let $p_A(n)$ be the number of partitions of n with summands restricted to A. Nathanson ([9]) proved the asymtotic formula

$$\log p_A(n) \sim \pi\sqrt{\frac{2}{3}\alpha n},$$

whenever $ggT(A) = 1$, where $ggT(A)$ denotes the greatest common divisor of A and the density $\alpha = \lim_{n\to\infty} \frac{\#\{x\in A, x\leq n\}}{n}$ exists. When we choose $A = \{n \in \mathbb{N} \mid m \nmid n\}$, we get $\alpha = 1 - \frac{1}{m}$ and

$$\log p_m(n) \sim \pi\sqrt{\frac{2(m-1)n}{3m}}. \tag{1}$$

Transferring a proof of Meinardus ([8]), one can show that

$$p_m(n) = (m-1)^{\frac{1}{4}} 2^{-\frac{5}{4}} 3^{-\frac{1}{4}} m^{-\frac{3}{4}} n^{-\frac{3}{4}} e^{\frac{\pi\sqrt{2(m-1)n}}{\sqrt{3m}}} \left(1 + O\left(n^{-d}\right)\right), \tag{2}$$

with some $d > 0$.

Hagis ([4],[5],[6]) developed an asymptotic formula for $p_m(n)$ in the case that m is prime. With this restriction, he even got exact formulas, analogous to Rademacher's exact formula for $p(n)$.

Our aim in this paper is to improve existing asymptotic formulas for $p_m(n)$. We will get rid of Hagis' restriction that m is prime, and we will develop asymptotic formulas for $p_m^l(n)$. The proofs of the main results (Theorems 4 and 5) use the circle method of Hardy and Littlewood; they are based on Rademacher's work on $p(n)$ (see [12], [13] and chapter 4 in [11]).

This paper is a condensed version of my diploma thesis at the University of Würzburg in 2002 which was supervised by Professor G. Köhler.

2 Generating functions and transformation equations

The generating functions for $p(n)$, $p_m(n)$ and $p_m^l(n)$ are

$$\begin{aligned} F(q) &:= \sum_{n=0}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} (1-q^n)^{-1}, \\ F_m(q) &:= \sum_{n=0}^{\infty} p_m(n) q^n = \prod_{\substack{n=1 \\ m \nmid n}}^{\infty} (1-q^n)^{-1}, \\ F_m^l(q) &:= \sum_{n=0}^{\infty} p_m^l(n) q^n = \prod_{\substack{n=1 \\ m \nmid n}}^{\infty} \frac{1-q^{(l+1)n}}{1-q^n}, \end{aligned}$$

respectively. They are holomorphic functions of q in the unit disc $\mathbb{D}$.
A well known result is (see for example [2] where the formula can be found in a slightly different version)

Theorem 1 *The generating function F of $p(n)$ satisfies the transformation equation*

$$F\left(\exp\left(\frac{2\pi ih}{k} - \frac{2\pi z}{k}\right)\right) = \omega(h,k)\sqrt{z}\exp\left(-\frac{\pi z}{12k} + \frac{\pi}{12kz}\right) F\left(\exp\left(\frac{2\pi ih'}{k} - \frac{2\pi}{kz}\right)\right),$$

in which $k > 0$, $(h,k) = 1$, h' *satisfies* $hh' \equiv -1 \pmod{k}$, $z \in \mathbb{C}$ *with* $\Re(z) > 0$, *and the square root has positive real part. Furthermore, let* $\omega(h,k) := \exp(\pi i \sigma(h,k))$ *with* $\sigma(h,k) := \sum_{\mu=1}^{k-1} \left(\left(\frac{\mu}{k}\right)\right) \left(\left(\frac{h\mu}{k}\right)\right)$ *a Dedekind Sum and* $((x))$ *defined for* $x \in \mathbb{R}$ *as*

$$((x)) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{for } x \in \mathbb{R} \setminus \mathbb{Z} \\ 0 & \text{for } x \in \mathbb{Z} \end{cases}.$$

From this one can conclude:

Theorem 2 *Let* $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(Z)$ *operate on* $\mathbb{H}$, *the complex upper half-plane, as usual:* $L \circ \tau = \frac{a\tau+b}{c\tau+d}$. *With* $\tau' = L\tau$ *one has*

1. $F_m(\exp(2\pi i \tau)) = F_m(\exp(2\pi i \tau'))$, *for* $c = 0$.
2. $F_m(\exp(2\pi i \tau)) = \frac{1}{\sqrt{m_1}} \omega_m(-d,c) \exp\left(\frac{\pi i}{12 m_1 c}((m_1 - d_1)a - (m-1)m_1 d) - \frac{\pi i}{12 m_1}((m-1)m_1\tau + (m_1 - d_1)\tau')\right) G_m^c\left(\exp\left(\frac{2\pi i}{m_1 c}(m_1 g' - a) + \frac{2\pi i}{m_1}\tau'\right)\right)$ *for* $c \neq 0$.

Here $(m,c) = d_1$, $m = d_1 m_1$, $c = d_1 c_1$, g' *satisfies* $m_1 h g' \equiv -1 \pmod{c_1}$. *Furthermore let* $\omega_m(h,c) := \frac{\omega(h,c)}{\omega(m_1 h, c_1)}$, $G_m^c(x) := \frac{F\left(x^{m_1} \exp\left(\frac{2\pi i r}{d_1}\right)\right)}{F\left(x^{d_1}\right)}$, *with* r *satisfying* $h' = m_1 g' + r c_1$.

Proof. Due to the identity

$$F_m(q) = \prod_{\substack{n=1 \\ m \nmid n}}^{\infty} (1 - q^n)^{-1} = \frac{\prod_{n=1}^{\infty} (1 - q^{mn})}{\prod_{n=1}^{\infty} (1 - q^n)} = \frac{F(q)}{F(q^m)}$$

one can use Theorem 1 in case $c \neq 0$ (one may choose $c > 0$ and set $k = c, h' = a$, $h = -d, z = -i(d + c\tau)$ in order to satisfy the assumptions of Theorem 1). In case $c = 0$ the theorem follows immediately.
In the same way one gets

Theorem 3 *Let* $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(Z)$; *for* $\tau \in \mathbb{H}$, $\tau' = L\tau$ *one has*

1. $F_m^l(\exp(2\pi i \tau)) = F_m^l(\exp(2\pi i \tau'))$, *if* $c = 0$.
2. $F_m^l(\exp(2\pi i \tau)) = \frac{\sqrt{m_3}}{\sqrt{m_1}} \omega_m^l(-d,c) \exp\left(\frac{\pi i}{12 m c (l+1)}(d(m-1)lm(l+1) + a((m - d_1^2)(l+1) - (m - d_3^2)d_2^2)) + \frac{\pi i}{12 m (l+1)}((m-1)lm(l+1)\tau - ((m - d_1^2)(l+1) - (m - d_3^2)d_2^2)\tau')\right) H_m^l\left(\exp\left(\frac{2\pi i}{m_1 m_3 l_2 c}(m_3 l_2 g^* - a) + \frac{2\pi i}{m_1 m_3 l_2}\tau'\right)\right)$,
   if $c \neq 0$.

Here $(h,c)=1,\ c>0,\ (m,c)=d_1,\ m=m_1d_1,\ c=c_1d_1,\ (l+1,c)=d_2,$ $l+1=d_2l_2,\ c=c_2d_2,\ (c_2,m)=d_3,\ c_2=d_3c_3,\ m=d_3m_3,\ g^*$ *is defined by* $m_3l_2hg^*\equiv -1 \pmod{c_3},\ \omega_m^l(h,c)=\frac{\omega_m(h,c)}{\omega_m(l_2h,c_2)}\ H_m^l(x):=\frac{G_m^c\left(x^{m_3l_2}\exp\left(\frac{2\pi i\lambda}{m_1d_3}\right)\right)}{G_m^{c_2}\left(x^{m_1d_2}\right)}$, *with* λ *defined by* $m_1g'=m_3l_2g^*+\lambda c_3$.

Proof. Due to the identity

$$F_m^l(q)=\prod_{\substack{\nu=1\\ m\nmid\nu}}^{\infty}\frac{1-q^{(l+1)\nu}}{1-q^\nu}=\frac{F_m(q)}{F_m(q^{l+1})}$$

one obtains Theorem 3 from Theorem 2 by making the same substitutions as before (using $\exp\left(\frac{2\pi ih}{k}-\frac{2\pi z}{k}\right)^{l+1}=\exp\left(\frac{2\pi i(l_2h)}{k_2}-\frac{2\pi(l_2z)}{k_2}\right)$).

3 Asymptotic formulas

Having transformation formulas, one is able to prove the asymptotic formulas for $p_m(n)$ and $p_m^l(n)$

Theorem 4 *Let* $m\in\mathbb{N},\ m\geq 2$*; for* $p_m(n)$ *one has the asymptotic formula*

$$p_m(n) \;=\; \frac{2(m-1)^{\frac14}\sqrt{3}}{m^{\frac34}(m-1+24n)^{\frac34}}\exp\left(\frac{\pi(m-1+24n)^{\frac12}(m-1)^{\frac12}}{6\sqrt{m}}\right)\left(1+O\left(n^{-\frac12}\right)\right).$$

Proof. Using Cauchy's formula for the generating function $F_m(q)$ and substituting $q=e^{2\pi i\tau}$ leads to

$$p_m(n)=\int_l F_m(\exp(2\pi i\tau))\exp(-2n\pi i\tau)d\tau,$$

where l is the horizontal line of length 1 in the upper half space with parametrization $l(t)=t+i\epsilon, -\frac12\leq t\leq\frac12$, with $0<\epsilon<\frac18$. We subdivide l into three parts: l_1 (with t running from $-\frac12$ to $-\sqrt{2\epsilon}$), l_2 (whith t running from $-\sqrt{2\epsilon}$ to $\sqrt{2\epsilon}$) and l_3 (whith t running from $\sqrt{2\epsilon}$ to $\frac12$). In order to estimate the integrals over l_1 and l_3 one can use Theorem 2 by choosing for $L\in SL_2(\mathbb{Z})$ the matrix that transforms a fixed $\tau:=x+iy\in l_1\cup l_3$ to the standard fundamental domain $F:=\{z\in\mathbb{H}:|z|\geq 1,|\Re z|\leq\frac12\}$. Using $y'=\frac{y}{|c\tau+d|^2}$ with $y'=\Im(\tau'),\tau'=L\tau$ one gets $|c\tau+d|<1$, which makes the case $c=0$ impossible. Therefore by applying Theorem 2 and using $G_m^c(0)=1$ one obtains

$$|F_m(\exp(2\pi i\tau))|\leq\exp\left(\frac{\pi}{12m_1}((m-1)m_1y+(m_1-d_1)y')\right)\sum_{\nu=0}^{\infty}|a_m^c(\nu)|e^{-\frac{2\pi\nu y'}{m_1}},$$

with $G_m^c(q)=\sum_{\nu=0}^\infty a_m^c(\nu)q^\nu$ for $|q|<1$. Now we distinguish two cases.
First case: $m_1<d_1$.
Due to $y=\epsilon<\frac18$ und $y'\geq\frac{\sqrt3}{2}>0$. one gets

$$|F_m(\exp(2\pi i\tau))|=O(1).$$

Second case: $m_1 \geq d_1$.
We first want to show that $|c| \geq 2$ i.g $c \notin \{0, \pm 1\}$.
If $c = 0$ we get the contradiction $d = 0$ due to $|c\tau + d|^2 = \frac{y}{y'} < 1$.
So let us assume $|c| = 1$. Then we have with $y = \epsilon$:

$$\frac{1}{y'} = \frac{|c\tau + d|^2}{y} = \frac{(x \pm d)^2}{\epsilon} + \epsilon,$$

which leads to

$$\frac{1}{y'} \geq \frac{x^2}{\epsilon} + \epsilon > 2$$

if $d = 0$ using $|x| \geq \sqrt{2\epsilon}$ and

$$\frac{1}{y'} \geq \frac{1}{4\epsilon} + \epsilon > 2$$

if $d \neq 0$, a contradiction to $y' \geq \frac{\sqrt{3}}{2}$. This leads to the estimation

$$|F_m(\exp(2\pi i\tau))| = O\left(\exp\left(\frac{\pi(m-1)}{48m\epsilon}\right)\right).$$

Therefore in both cases the integral over $l_1 \cup l_3$ is $O\left(\exp\left(2n\pi\epsilon + \frac{\pi(m-1)}{48m\epsilon}\right)\right)$. Now set $\epsilon := \frac{\sqrt{m-1}}{4\sqrt{6mn}}$ in order to minimize the error term.

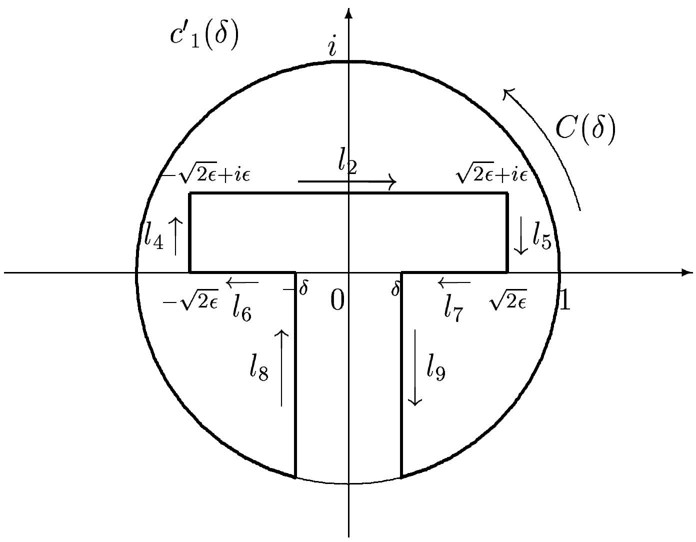
Using Theorem 2 again with $L = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in Sl_2(\mathbb{Z})$ (choosing $g' = 0$) one gets (using $\omega_m(0,1) = 1$)

$$F_m(\exp(2\pi i\tau)) = \frac{1}{\sqrt{m}} \exp\left(-\frac{\pi i(m-1)\tau}{12}\right) \sum_{\nu=0}^{\infty} a_m^1(\nu) e^{-\frac{\pi i}{m\tau}\left(2\nu - \frac{m-1}{12}\right)},$$

with $G_m^1(q) = \sum_{\nu=0}^{\infty} a_m^1(\nu) q^\nu$ for $|q| < 1$. Setting $N := \lfloor \frac{m-1}{24} \rfloor$ and using $y < \frac{1}{8}$, $|x| \leq \sqrt{2\epsilon}$ for $\tau = x + iy \in l_2$ one gets

$$\begin{aligned} p_m(n) = {} & \frac{1}{\sqrt{m}} \sum_{\nu=0}^{N} a_m^1(\nu) \int_{l_2} \exp\left(-\pi i\tau\left(2n + \frac{m-1}{12}\right) - \frac{\pi i}{\tau m}\left(2\nu - \frac{m-1}{12}\right)\right) d\tau \\ & + O\left(\exp\left(\frac{\pi\sqrt{n(m-1)}}{\sqrt{6m}}\right)\right). \end{aligned}$$

In order to calculate the integral one changes the path of integration (as Rademacher did in the case of $p(n)$) as shown in the following figure, where
$0 < \delta < \sqrt{2\epsilon}$.
The estimations of the integrand on l_4, l_5, l_6, l_7 are easy and therefore left to the reader.



We substitute $z = i\tau$ and let δ tend to 0. Then we get

$$p_m(n) = \frac{i}{\sqrt{m}} \sum_{\nu=0}^{N} a_m^1(\nu) \int_{\partial\mathbb{D}} \exp\left(-\pi z\left(\frac{m-1}{12} + 2n\right) - \frac{\pi}{zm}\left(\frac{m-1}{12} - 2\nu\right)\right) dz$$
$$+O\left(\exp\left(\frac{\pi\sqrt{n(m-1)}}{\sqrt{6m}}\right)\right)$$
$$= \frac{i}{\sqrt{m}} \sum_{\nu=0}^{N} a_m^1(\nu) 2\pi i \ \mathrm{Res}|_{z=0} \exp\left(-\pi z\left(\frac{m-1}{12} + 2n\right) - \frac{\pi}{zm}\left(\frac{m-1}{12} - 2\nu\right)\right)$$
$$+O\left(\exp\left(\frac{\pi\sqrt{n(m-1)}}{\sqrt{6m}}\right)\right).$$

One may assume that $\frac{m-1}{24} \notin \mathbb{N}$ (otherwise the associated summand is holomorphic and can be omitted). Developing the exponential function in a power series gives the searched for residue as

$$\mathrm{Res}|_{z=0} \exp\left(-\pi z\left(\frac{m-1}{12} + 2n\right) - \frac{\pi}{zm}\left(\frac{m-1}{12} - 2\nu\right)\right)$$
$$= -\frac{(m-1-24\nu)^{\frac{1}{2}}}{\sqrt{m}(m-1+24n)^{\frac{1}{2}}} I\left(\frac{\pi(m-1+24n)^{\frac{1}{2}}(m-1-24\nu)^{\frac{1}{2}}}{6\sqrt{m}}\right),$$

with $I(z) = \sum_{l=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2l+1}}{l!(l+1)!}$ a modified Bessel function of first kind. Using the well known asymptotic formula for the Bessel function
$I(x) = \frac{e^x}{(2\pi x)^{\frac{1}{2}}}\left(1 + O\left(\frac{1}{x}\right)\right)$ for $x \in \mathbb{R}$, $|x| \to \infty$ one gets the desired asymptotic formula for $p_m(n)$:

$$p_m(n) = \sum_{\nu=0}^{N} a_m^1(\nu) \frac{2\pi(m-1-24\nu)^{\frac{1}{2}}}{m(m-1+24n)^{\frac{1}{2}}} \left(\frac{2\pi^2\sqrt{(m-1+24n)(m-1-24\nu)}}{6\sqrt{m}} \right)^{-\frac{1}{2}}$$

$$\exp\left(\frac{\pi\sqrt{(m-1+24n)(m-1-24\nu)}}{6\sqrt{m}} \right)$$

$$\left(1 + O\left(\frac{6\sqrt{m}}{\pi\sqrt{(m-1+24n)(m-1-24\nu)}} \right)\right) + O\left(\exp\left(\frac{\pi\sqrt{(m-1)n}}{\sqrt{6m}} \right)\right)$$

$$= \sum_{\nu=0}^{N} a_m^1(\nu) \frac{2\sqrt{3}(m-1-24\nu)^{\frac{1}{4}}}{m^{\frac{3}{4}}(m-1+24n)^{\frac{3}{4}}} \exp\left(\frac{\pi\sqrt{(m-1+24n)(m-1-24\nu)}}{6\sqrt{m}} \right)$$

$$\left(1 + O\left(n^{-\frac{1}{2}}\right)\right) + O\left(\exp\left(\frac{\pi\sqrt{n(m-1)}}{\sqrt{6m}} \right)\right)$$

$$= \frac{2\sqrt{3}(m-1)^{\frac{1}{4}}}{m^{\frac{3}{4}}(m-1+24n)^{\frac{3}{4}}} \exp\left(\frac{\pi\sqrt{(m-1+24n)(m-1)}}{6\sqrt{m}} \right) \left(1 + O\left(n^{-\frac{1}{2}}\right)\right),$$

due to $a_m^1(0) = 1$.
One can also develop an asymptotic formula for $p_m^l(n)$

Theorem 5 *Let $m \geq 2$, $l \geq 1$; for $p_m^l(n)$ one has the asymptotic formula*

$$p_m^l(n) = \frac{2(m-1)^{\frac{1}{4}}\sqrt{3}l^{\frac{1}{4}}}{(l+1)^{\frac{1}{4}}m^{\frac{1}{4}}(24n-(m-1)l)^{\frac{3}{4}}} \exp\left(\frac{\pi(24n-(m-1)l)^{\frac{1}{2}}(m-1)^{\frac{1}{2}}\sqrt{l}}{6\sqrt{m(l+1)}} \right) \left(1 + O\left(n^{-\frac{1}{2}}\right)\right).$$

Proof. Since the proof of Theorem 5 is very similar to that of Theorem 4 (though a little bit more complicated), some of the details are left to the reader; the notations and symbols are kept the same. One has

$$p_m^l(n) = \int_l F_m^l(\exp(2\pi i\tau)) \exp(-2n\pi i\tau) d\tau.$$

For $\tau \in l_1 \cup l_3$ one obtains, after mapping τ to the fundamental domain

$$|F_m^l(\exp(2\pi i\tau))| \leq \frac{\sqrt{m_3}}{\sqrt{m_1}} \exp\left(\frac{\pi y'}{12m(l+1)}((m-d_1^2)(l+1) - (m-d_3^2)d_2^2) \right) \sum_{\nu=0}^{\infty} |b_m^l(\nu)| e^{-\frac{\pi\sqrt{3}\nu}{m_1 m_3 l_2}},$$

with $H_m^l(x) = \sum_{\nu=0}^{\infty} b_m^l(\nu)x^\nu$ for $|x| < 1$.
As in Theorem 4 we have $c \neq 0$. Therefore

$$\frac{1}{y'} = \frac{|c\tau + d|^2}{y} \geq c^2 y = c^2\epsilon \geq d_2^2 d_3^2 \epsilon$$

e.g $y' \leq \frac{1}{c^2\epsilon}$, due to $d_2 d_3 | c$.
In order to obtain an optimal error term, one has to distinguish two cases.
First case: $l \neq 1$ or $m \notin \{2,3\}$. We then get

$$|F_m^l(\exp(2\pi i\tau))| = O\left(\exp\left(\frac{\pi(m-1)}{48m\epsilon} + \frac{\pi}{12m(l+1)\epsilon}\right)\right) O\left(\exp\left(\frac{\pi((m-1)(l+1)+4)}{48m(l+1)\epsilon}\right)\right).$$

Second case: $l = 1$ and $m \in \{2,3\}$:
We now have to consider several cases:
If c is odd we have $d_2 = 1$. Therefore we get

$$|F_m^l(\exp(2\pi i\tau))| = O\left(\exp\left(\frac{\pi(m-d_1^2)}{96m\epsilon}\right)\right) = O\left(\exp\left(\frac{\pi(m-1)}{96m\epsilon}\right)\right).$$

Now let c be even. If $m \nmid \frac{c}{2}$ we have $d_3 = 1$ which leads to an error term $O(1)$. (We have an negative exponent in this case).
Now let $m|\frac{c}{2}$. We now consider the cases $m = 2$ and $m = 3$.
If $m = 2$ we have $d_1 = d_3 = 2$ and therefore $y' \leq \frac{1}{16\epsilon}$. So we get

$$|F_2^1(\exp(2\pi i\tau))| = O\left(\exp\left(\frac{\pi y'}{12}\right)\right) = O\left(\exp\left(\frac{\pi}{192\epsilon}\right)\right).$$

. If $m = 3$ we have $d_1 = d_3 = 3$. Therefore

$$|F_3^1(\exp(2\pi i\tau))| = O\left(\exp\left(\frac{\pi y'}{6}\right)\right) \leq O\left(\exp\left(\frac{\pi}{216\epsilon}\right)\right) \leq O\left(\exp\left(\frac{\pi(p-1)}{144p\epsilon}\right)\right)$$

Choosing ϵ in the first case as $\epsilon := \frac{\sqrt{(m-1)(l+1)+4}}{4\sqrt{6m(l+1)n}}$ and in the second case as $\epsilon := \frac{\sqrt{(m-1)}}{8\sqrt{3mn}}$, if we are not in the case $m = 2, c \equiv 0 \bmod 4$ and $\epsilon = \frac{1}{8\sqrt{6n}}$ in the left case. One gets that the integral along $l_1 \cup l_3$ is $O(\exp(4n\pi\epsilon))$. Applying Theorem 3 with $L = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\in Sl_2(\mathbb{Z})$ leads to

$$F_m^l(\exp(2\pi i\tau)) = \exp\left(\frac{\pi i(m-1)l\tau}{12}\right) \sum_{\nu=0}^{\infty} b_m^l(\nu) e^{-\frac{\pi i}{m^2(l+1)\tau}\left(2\nu - \frac{(m-1)lm}{12}\right)},$$

with $H_m^l(x) = \sum_{\nu=0}^{\infty} b_m^l(\nu)x^\nu$ for $|x| < 1$. with $H_m^l(x) = \sum_{\nu=0}^{\infty} b_m^l(\nu)x^\nu$ for $|x| < 1$.
Setting $N = \lfloor \frac{(m-1)lm}{24} \rfloor$ one gets

$$p_m^l(n) = \sum_{\nu=0}^{N} b_m^l(\nu) \int_{l_2} \exp\left(-\pi i\tau\left(2n - \frac{(m-1)l}{12}\right)\right.$$
$$\left. - \frac{\pi i}{\tau m^2(l+1)}\left(2\nu - \frac{(m-1)lm}{12}\right)\right) d\tau + O(\exp(4n\pi\epsilon)).$$

Changing the path of integration the same way as in Theorem 4 and substituting $z = i\tau$, we obtain

$$p_m^l(n) = i\sum_{\nu=0}^{N} b_m^l(\nu)2\pi i\mathrm{Res}|_{z=0}\exp\left(-\pi z\left(2n - \frac{(m-1)l}{12}\right)\right.$$
$$\left.-\frac{\pi}{zm^2(l+1)}\left(\frac{(m-1)lm}{12} - 2\nu\right) + O\left(\exp\left(4n\pi\epsilon\right)\right)\right).$$

As in Theorem 4 we get

$$p_m^l(n) = \sum_{\nu=0}^{N} b_m^l(\nu)\frac{2\pi((m-1)lm - 24\nu)^{\frac{1}{2}}}{\sqrt{l+1}m(24n-(m-1)l)^{\frac{1}{2}}}$$
$$I\left(\frac{\pi\sqrt{(24n-(m-1)l)((m-1)lm-24\nu)}}{6m\sqrt{l+1}}\right) + O\left(\exp\left(4n\pi\epsilon\right)\right)$$
$$= \frac{2\sqrt{3}(m-1)^{\frac{1}{4}}l^{\frac{1}{4}}}{(l+1)^{\frac{1}{4}}m^{\frac{1}{4}}(24n-(m-1)l)^{\frac{3}{4}}}\exp\left(\frac{\pi\sqrt{(24n-(m-1)l)(m-1)l}}{6\sqrt{m(l+1)}}\right)$$
$$\left(1 + O\left(n^{-\frac{1}{2}}\right)\right),$$

due to $b_m^l(0) = 1$.

As in the proof of Theorem 4 one thus gets the desired asymptotic formula. □

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