LIFTING MAPS FROM A VECTOR SPACE OF JACOBI CUSP FORMS TO A SUBSPACE OF ELLIPTIC MODULAR FORMS

KATHRIN BRINGMANN

Abstract. In this paper we construct a lifting map from a vector space of generalized Jacobi cusp forms to a certain subspace of elliptic cusp forms and vice versa such that both mappings are adjoint with respect to the Petersson scalar products.

1. Introduction and statement of results

In their paper "Heegner points and derivatives of L-series II " [GKZ], Gross, Kohnen, and Zagier constructed certain lifting maps in the dimension 1 case of Jacobi forms, to obtain deep formulas relating height pairings of Heegner points to coefficients of Jacobi forms. This work raises many natural questions. In particular: To what extent do coefficients of generalized Jacobi forms interpolate arithmetic quantities such as height pairings and values of L-functions of modular forms? The present paper is a first step in this direction. We consider Jacobi forms of several toric variables, and we relate them to spaces of elliptic modular forms. Armed with this result, following the approach of [EZ], one should then be able to develop a theory of newforms and hopefully use the Eichler-Shimura trace formula for elliptic cusp forms to compare the Hecke actions on these spaces in a nice compatible way. As an application one expects explicit formulas that express the central critical values of Hecke L-functions of elliptic Hecke eigenforms as squares of Fourier coefficients of generalized Jacobi forms, as in the classical case.

Here we construct lifting maps from the vector space of Jacobi cusp forms with respect to the generalized Jacobi group $\Gamma_g^J := SL_2(\mathbb{Z}) \ltimes \left(\mathbb{Z}^{(g,1)} \times \mathbb{Z}^{(g,1)}\right)$ to a certain subspace of elliptic modular forms.

In the following let $n, k, g \in \mathbb{N}$, where $k \geq \frac{g+3}{2}$, and $g \equiv 1 \pmod 8$; moreover let m be a positive definite symmetric half-integral $g \times g$ matrix (the last two conditions imply in particular that $\frac{1}{2}\det(2m)$ is an integer), $r \in \mathbb{Z}^{(1,g)}$, $D_0 := -\det \begin{pmatrix} 2n_0 & r_0 \\ r_0^t & 2m \end{pmatrix} < 0$ such that D_0 is a a square $\pmod{\frac{1}{2}\det(2m)}$ and a fundamental discriminant. Moreover, if p divides both $\det(2m)$ and D_0, p^2 must not divide $\det(2m)$ if $p \neq 2$, p^3 must not divide $\det(2m)$ if $p = 2$ and $\frac{D_0}{4}$ is odd, and p^4 must not divide $\det(2m)$ is $p = 2$ and $\frac{D_0}{4}$ is even. Moreover if $p \neq 2$, $\prod_{\substack{i=1 \\ i \neq j}} m_i$ has to be assumed to be a square $\pmod p$, where the m_i

Date: September 20, 2005.
2000 *Mathematics Subject Classification.* 11F50.

are choosen such that $\exists\, U \in GL_g(\mathbb{Z}/p\mathbb{Z})$ with $(2m)[U] \equiv \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_g \end{pmatrix} \pmod p$, $p|m_j$ (for the existence of such an U see [Ca]). Quadratic forms with the above conditions indeed exist (for an example see [Br1]). The last three conditions can easily be shown to be satisfied if $\det(2m) \cdot D_0$ is squarefree. Moreover it can be shown that for $g = 1$ the conditions are equivalent to the conditions given in [GKZ]. Let us denote by $J_{k,m}^{cusp}$ the vector space of Jacobi cusp forms with respect to Γ_g^J, and by $S_k(\frac{1}{2}\det(2m))^-$ the subspace of elliptic cusp forms with respect to $\Gamma_0(\frac{1}{2}\det(2m))$ that have eigenvalue -1 under the Fricke involution.

Definition 1.1. *For $\phi \in J_{k+\frac{g+1}{2},m}^{cusp}$ we define*

$$\mathcal{S}_{D_0,r_0}(\phi)(w) := 2^{1-g} \sum_{n=1}^{\infty} \left(\sum_{d|n} \left(\frac{D_0}{d} \right) d^{k-1} c_\phi \left(\frac{n^2}{d^2} n_0, \frac{n}{d} r_0 \right) \right) e^{2\pi i n w} \quad (w \in \mathbb{H}),$$

where $c_\phi(n,r)$ is the $(n,r)-$th Fourier coefficient of ϕ, and where $\left(\frac{D_0}{d}\right)$ denotes the usual Kronecker symbol.

Definition 1.2. *For $f \in S_{2k}(\frac{1}{2}\det(2m))^-$ we define for $(\tau, z) \in \mathbb{H} \times \mathbb{C}^{(g,1)}$*

$$\mathcal{S}^*_{D_0,r_0}(f)(\tau,z) := \left(\frac{i}{\det(2m)} \right)^{k-1} \sum_{\substack{n \in \mathbb{Z} \\ r \in \mathbb{Z}^{(1,g)} \\ 4n > m^{-1}[r^t]}} r_{k,\frac{1}{2}\det(2m),D_0 D, r_0 (2m)^* r^t, D_0}(f) e^{2\pi i (n\tau + rz)},$$

where $D := -\begin{pmatrix} 2n & r \\ r^t & 2m \end{pmatrix}$, and where $r_{k,\frac{1}{2}\det(2m),D_0 D,r_0(2m)^* r^t,D_0}(f)$ is a certain cycle integral, defined in Section 2.

We prove the following theorem.

Theorem 1.3. *If ϕ is an element of $J_{k+\frac{g+1}{2},m}^{cusp}$, then the function $\mathcal{S}_{D_0,r_0}(\phi)(w)$ is an element of $S_{2k}(\frac{1}{2}\det(2m))^-$.*
If $f \in S_{2k}(\frac{1}{2}\det(2m))^-$, then the function $\mathcal{S}^*_{D_0,r_0}(f)(\tau,z)$ is an element of $J_{k+\frac{g+1}{2},m}^{cusp}$.
The maps

$$\mathcal{S}_{D_0,r_0} : J_{k+\frac{g+1}{2},m}^{cusp} \to S_{2k}(\frac{1}{2}\det(2m))^-$$

and

$$\mathcal{S}^*_{D_0,r_0} : S_{2k}(\frac{1}{2}\det(2m))^- \to J_{k+\frac{g+1}{2},m}^{cusp}$$

are adjoint maps with respect to the Petersson scalar products.

For the proof we follow the same method as in [GKZ] and define a function $\Omega_{k,m,D_0,r_0}(w;\tau,z)$ that can easily be shown to be a holomorphic kernel function for the map $\mathcal{S}^*_{D_0,r_0}$. To prove the theorem we have to show that $\Omega_{k,m,D_0,r_0}(w;\tau,z)$ is also a holomorphic kernel function for the map $\mathcal{S}_{D_0,r_0}$. Using the Petersson coefficient formula for Jacobi cusp forms, we have to show that $\Omega_{k,m,D_0,r_0}(w;\tau,z)$ has a Fourier expansion, where the Fourier coefficients are certain linear combinations of Jacobi-Poincaré series. Therefore we have to manipulate certain higher dimensional congruences and compute sums of multi-variable Kloosterman sums.

Remark. This paper is a condensed version of a part of the authors Ph.D. thesis supervised by Prof. Dr. W. Kohnen.

Acknowledgements

The author thanks Winfried Kohnen and Ken Ono for their helpful comments.

2. General facts about quadratic forms, the generalized genus character, and Jacobi cusp forms

First we recall some facts about quadratic forms, the generalized genus character, and Jacobi cusp forms. For details, we refer the reader to [BK], [Br1],[Br2], [EZ], and [GKZ]. For $a,b,c\in\mathbb{Z}$ let us define the integral binary quadratic form

$$[a,b,c](x,y) := ax^2+bxy+cy^2.$$

The group $SL_2(\mathbb{Z})$ acts on these forms in the usual way by

$$[a,b,c]\circ\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}(x,y) := [a,b,c](\alpha x+\beta y,\gamma x+\delta y)\quad (x,\, y\in\mathbb{Z}).$$

Let $\Delta\in\mathbb{Z}$ be a discriminant and let us denote by $\mathcal{D}_\Delta$ the set of integral binary quadratic forms with discriminant Δ. Furthermore, let us abbreviate for $l\in\mathbb{N}$ by $\mathcal{D}_{l,\Delta}\subset\mathcal{D}_\Delta$ the set of all quadratic forms with the additional condition that $a\equiv 0 \pmod{l}$. Moreover let us define for integers $\rho \pmod{2l}$ with $\Delta\equiv\rho^2 \pmod{4l}$ the set

$$\mathcal{D}_{l,\Delta,\rho} := \left\{[a,b,c]\in\mathcal{D}_\Delta \,|\, a\equiv 0 \pmod{l}, b\equiv\rho \pmod{2l}\right\}.$$

Then the sets $\mathcal{D}_{l,\Delta}$ and $\mathcal{D}_{l,\Delta,\rho}$ are $\Gamma_0(l)$ invariant.
Now let D_0 be a fundamental discriminant that divides Δ such that both D_0 and Δ/D_0 are squares $\pmod{4l}$. Then we define for $Q=[al,b,c]\in\mathcal{D}_{l,\Delta}$ the following generalized genus character:

$$\chi_{D_0}(Q) := \begin{cases}\left(\frac{D_0}{n}\right) & \text{if } (a,b,c,D)=1\\ 0 & \text{otherwise.}\end{cases}$$

Here n is an integer coprime to D_0 represented by the form $[al_1,b,cl_2]$ for some decomposition $l=l_1l_2$, $l_i>0$ $(i=1,2)$. It is easy to show that such an n always exists and that the value of $\left(\frac{D_0}{n}\right)$ is independent of the choice of l_1,l_2, and n.

Lemma 2.1. *The function χ_{D_0} is $\Gamma_0(l)-$invariant and has the following properties:*

P1 (Multiplicativity):

$$\chi_{D_0}([al,b,c]) = \chi_{D_0}([a_1 l, b, ca_2])\chi_{D_0}([a_2 l, b, ca_1]) \qquad \text{if } a = a_1 a_2, (a_1,a_2)=1.$$

P2 (Invariance under the Fricke involution):

$$\chi_{D_0}([al,b,c]) = \chi_{D_0}([cl,-b,a]).$$

P3 (Explicit formula):

$$\chi_{D_0}([al,b,c]) = \left(\frac{D_1}{l_1 a}\right)\left(\frac{D_2}{l_2 c}\right)$$

for any splitting $D_0 = D_1 D_2$ of D_0 into coprime fundamental discriminants and $l = l_1 l_2$ of l into positive factors such that $(D_1, l_1 a) = (D_1, l_2 c) = 1$, $\chi_{D_0}([al,b,c]) = 0$ if no such splitting exists.

Next we define certain kernel functions for geodesic cycle integrals

$$f_{k,l,\Delta,\rho,D_0}(z) := \sum_{Q \in \mathcal{D}_{l,\Delta,\rho}} \frac{\chi_{D_0}(Q)}{Q(z,1)^k} \qquad (z \in \mathbb{H}).$$

Then it is known from [GKZ] that the series $f_{k,l,\Delta,\rho,D_0}(z)$ is absolutely and locally uniformly convergent for $k>1$ and is an element of $S_{2k}(l)^-$.

Lemma 2.2. *The Fourier expansion of $f_{k,l,\Delta,\rho,D_0}(z)$ $(k \geq 1)$ is given by*

$$f_{k,l,\Delta,\rho,D_0}(z) = \sum_{m=1}^{\infty} c^{\pm}_{k,l}(m,\Delta,\rho,D_0) e^{2\pi i m z},$$

where

$$c^{\pm}_{k,l}(m,\Delta,\rho,D_0) := c_{k,l}(m,\Delta,\rho,D_0) + (-1)^{k+1} c_{k,l}(m,\Delta,-\rho,D_0),$$

where $\pm = (\pm 1)^{k+1}$, and where

$$c_{k,l}(m,\Delta,\rho,D_0) = i^k \cdot (-1)^{-\frac{1}{2}} \cdot \frac{(2\pi)^k}{(k-1)!} \cdot (m^2/\Delta)^{\frac{k-1}{2}} \cdot \Big[|D_0|^{-\frac{1}{2}} \cdot \epsilon_l(m,\Delta,\rho,D_0)$$
$$+ i^{k+1} \cdot \pi \cdot \sqrt{2} \cdot (m^2/\Delta)^{\frac{1}{4}} \cdot \sum_{a \geq 1} (la)^{-\frac{1}{2}} \cdot S_{la}(m,\Delta,\rho,D_0) \cdot J_{k-1/2}\left(\frac{\pi m \sqrt{\Delta}}{la}\right)\Big].$$

Here

$$\epsilon_l(m,\Delta,\rho,D_0) := \begin{cases} \left(\frac{D_0}{m/f}\right) & \text{if } \Delta = D_0^2 \cdot f^2 \ (f>0),\ f|m,\ D_0 f \equiv \rho \pmod{2l} \\ 0 & \text{otherwise} \end{cases},$$

$$S_{la}(m,\Delta,\rho,D_0) = \sum_{\substack{b(2la) \\ b \equiv \rho(2l) \\ b^2 \equiv \Delta(4la)}} \chi_{D_0}\left(\left[al, b, \frac{b^2-\Delta}{4la}\right]\right) \cdot e_{2la}(mb),$$

and $J_{k-1/2}(t)$ is the Bessel function of order $k-1/2$.

Moreover we define for $f \in S_{2k}(l)$ and $Q=[a,b,c] \in \mathcal{D}_{l,\Delta,\rho}$

$$r_{k,l,Q}(f) := \int_{\gamma_Q} f(z)\cdot Q(z,1)^{k-1}\,dz,$$

where γ_Q is the image in $\Gamma_0(l)\backslash\mathbb{H}$ of the semicircle $a|z|^2+bx+c=0$ ($x=\operatorname{Re}(z)$), orientated from $\frac{-b-\sqrt{\Delta}}{2a}$ to $\frac{-b+\sqrt{\Delta}}{2a}$ if $a\neq 0$ or if $a=0$ of the vertical line $bx+c=0$, orientated from $-\frac{c}{b}$ to $i\infty$ if $b>0$ and from $i\infty$ to $-\frac{c}{b}$ if $b<0$. Then we can easily show that the above definition makes sense (i.e., the integral is invariant with respect to the subgroup of $\Gamma_0(l)$ perserving Q) and depends only on the $\Gamma_0(l)$ equivalence class of Q. Furthermore, we define

$$r_{k,l,\Delta,\rho,D_0}(f) := \sum_{Q\in\mathcal{D}_{l,\Delta,\rho}/\Gamma_0(l)} \chi_{D_0}(Q)\cdot r_{k,l,Q}(f).$$

Theorem 2.3. *For $f\in S_{2k}(l)^-$ we have*

$$\langle f, f_{k,l,\Delta,\rho,D_0}\rangle = \pi\cdot\binom{2k-2}{k-1}\cdot 2^{-2k+2}\cdot\Delta^{-k+1/2}\cdot r_{k,l,\Delta,\rho,D_0}(f),$$

where $<\cdot,\cdot>$ denotes the usual Petersson scalar product for elliptic cusp forms with respect to $\Gamma_0(l)$.

To finish, we want to repeat some facts about Jacobi cusp forms. The Jacobi group Γ_g^J acts on $\mathbb{H}\times\mathbb{C}^{(g,1)}$ in the usual way by

$$\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix},(\lambda,\mu)\right)\circ(\tau,z) := \left(\frac{a\tau+b}{c\tau+d},\frac{z+\lambda\tau+\mu}{c\tau+d}\right).$$

Let k be an integer, m be a positive definite symmetric half-integral $g\times g$ matrix, $\gamma=\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix},(\lambda,\mu)\right)\in\Gamma_g^J$, and $\phi:\mathbb{H}\times\mathbb{C}^{(g,1)}\to\mathbb{C}$. Then we define the following action

$$\phi|_{k,m}\gamma(\tau,z) := (c\tau+d)^{-k}\cdot e(-c(c\tau+d)^{-1}m[z+\lambda\tau+\mu]+m[\lambda]\tau+2\lambda^t mz)) \cdot\phi(\gamma\circ(\tau,z)),$$

where $e(z):=e^{2\pi iz}\quad(\forall\, z\in\mathbb{C})$. A holomorphic function $\phi:\mathbb{H}\times\mathbb{C}^{(g,1)}\to\mathbb{C}$ is called a Jacobi cusp form of weight k and index m with respect to Γ_g^J, if $\phi|_{k,m}\gamma(\tau,z)=\phi(\tau,z)\,\forall\,\gamma\in\Gamma_g^J$, and if it has a Fourier expansion of the form

$$\phi(\tau,z) = \sum_{\substack{n\in\mathbb{Z}\\ r\in\mathbb{Z}^{(1,g)}\\ 4n>m^{-1}[r^t]}} c(n,r)e\left(n\tau+rz\right).$$

Let us denote by $J_{k,m}^{cusp}$ the vector space of these Jacobi cusp forms.
For ϕ and $\psi \in J_{k,m}^{cusp}$ we define

$$< \phi, \psi >:= \int_{\Gamma_g^J \backslash \mathbb{H} \times \mathbb{C}^{(g,1)}} \phi(\tau, z) \cdot \overline{\psi(\tau, z)} \cdot v^k \cdot \exp\left(-4\pi m[y] \cdot v^{-1}\right) dV_g^J,$$

where $dV_g^J = v^{-g-2}\, dudvdxdy$, and where we have written $\tau = u + iv$, $z = x + iy$.
For this vector space of Jacobi cusp forms one can define certain Poincaré series $P_{k,m;(n,r)}(\tau, z)$ as in [BK] for $k > g + 2$ and in [Br1] or [Br2] for $k = g + 2$. These series satisfies the following Petersson coefficient formula

Lemma 2.4. *Let $\phi \in J_{k,m}^{cusp}$. Then we have*

$$\left\langle \phi, P_{k,m;(n,r)} \right\rangle = \lambda_{k,m,D} \cdot c_\phi(n, r), \tag{2.1}$$

where $c_\phi(n, r)$ denotes the (n, r)-th Fourier coefficient of ϕ and

$$\lambda_{k,m,D} := 2^{(g-1)(k-g/2-1)-g} \cdot \Gamma\left(k - g/2 - 1\right) \cdot \pi^{-k+g/2+1} \cdot (\det m)^{k-(g+3)/2} \cdot |D|^{-k+g/2+1}.$$

Lemma 2.5. *The function $P_{k,m;(n,r)}$ has a Fourier expansion of the form*

$$P_{k,m;(n,r)}(\tau, z) = \sum_{\substack{n' \in \mathbb{Z} \\ r' \in \mathbb{Z}^{(1,g)} \\ 4n' > m^{-1}[r'^t]}} g_{k,m;(n,r)}^{\pm}(n', r') e(n'\tau + r'z),$$

where $\pm = (\pm 1)^k$, where

$$g_{k,m;(n,r)}^{\pm}(n', r') := g_{k,m;(n,r)}(n', r') + (-1)^k g_{k,m;(n,r)}(n', -r'),$$

where

$$g_{k,m;(n,r)}(n', r') := \delta_m(n, r, n', r') + 2\pi i^k \cdot (\det(2m))^{-1/2} \cdot (D'/D)^{k/2-g/4-1/2}$$
$$\times \sum_{c \geq 1} e_{2c}(r' m^{-1} r^t) \cdot H_{m,c}(n, r, n', r') \cdot J_{k-g/2-1}\left(\frac{2\pi\sqrt{D'D}}{\det(2m) \cdot c}\right) \cdot c^{-g/2-1},$$

and where

$$\delta_m(n, r, n', r') := \begin{cases} 1 & \text{if} \quad D' = D,\, r' \equiv r \pmod{\mathbb{Z}^{(1,g)} \cdot 2m} \\ 0 & \text{else} \end{cases}$$

and

$$H_{m,c}(n, r, n', r') := \sum_{\substack{x(c) \\ y(c)^*}} e_c((m[x] + rx + n)\bar{y} + n'y + r'x),$$

where x and y run over a complete set of representatives for $\mathbb{Z}^{(g,1)}/c\mathbb{Z}^{(g,1)}$ and $(\mathbb{Z}/c\mathbb{Z})^*$, respectively, where $\bar{y}$ denotes an inverse of $y \pmod{c}$.

3. Construction of the lifting maps

Our aim of this section is the proof of Theorem 1.3. For details we refer the reader to [Br1]. Before we can prove the theorem we have to show that Definition 1.2 is allowed. For this it is due to the assumptions given on m and D_0 in the introduction enough to show that

$$DD_0 \equiv \big(r_0(2m)^* r^t\big)^2 \pmod{p^\nu} \tag{3.1}$$

$\forall\, p^\nu$ with $p^\nu | 2\det 2m$. For the proof of (3.1) we change $\begin{pmatrix} 2n_0 & r_0 \\ r_0^t & 2m \end{pmatrix}$ into $\begin{pmatrix} 2n_0 & r_0 \\ r_0^t & 2m \end{pmatrix}\left[\begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}\right]$, with $U \in GL_g(\mathbb{Z}/p^\nu\mathbb{Z})$ for a suitabel choice of U. It can be shown that by doing so, congruence (3.1) is changed in an equivalent congruence. Moreover none of the assumptions made for D_0 and m is needed for the proof of the new congruence. We want to regard the cases $p \neq 2$ and $p = 2$ separately.
Since for $p \neq 2$ a quadratic form over the ring of $p-$adic numbers $\mathbb{Z}_p$ is equivalent to a sum of forms lx^2 with $l \in \mathbb{Z}_p$ (cf [Ca]) we may assume that in case $p \neq 2$ the matrix $2m$ is a diagonal matrix with diagonal elements $m_1, \dots, m_g$. Moreover we can easily show that p divides exactly one m_i $(1 \leq i \leq g)$. Inserting this in both sides of congruence (3.1) we can show that this congruence holds for $p \neq 2$.
Since for $p = 2$ a quadratic form over the ring of $p-$adic numbers $\mathbb{Z}_p$ is equivalent to a sum of forms $l \cdot x^2$, $2^\nu \cdot xy$ and $2^\nu \cdot (x^2 + xy + y^2)$, where $l \in \mathbb{Z}_2$ and $\nu \in \mathbb{N}_0$ (cf. [Ca]), we may assume that $2m$ is a block-diagonal matrix with blocks from the set $\left\{2l, 2^\nu \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, 2^\nu \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}$. Morover we can easily show that the type $2l$ occurs exactly once and that $\nu = 0$. Inserting this in both sides of (3.1) we can show that this congruence holds for $p = 2$.

Now we can come to the proof of Theorem 1.3. For this let us define

$$\Omega_{k,m,D_0,r_0}(w;\tau,z) := c_{k,m,D_0} \cdot \sum_{\substack{n\in\mathbb{Z} \\ r\in\mathbb{Z}^{(1,g)} \\ 4n > m^{-1}[r^t]}} |D|^{k-1/2} \cdot f_{k,\frac{1}{2}\det(2m),D_0D,r(2m)^*r_0^t,D_0}(w) \cdot e^{2\pi i(n\tau + rz)},$$

where

$$c_{k,m,D_0} := \frac{(-2i)^{k-1} \cdot |D_0|^{k-1/2}}{\left(\frac{1}{2}\det(2m)\right)^{k-1} \cdot \pi \cdot \binom{2k-2}{k-1}}.$$

One can see easily, using the Fourier expansion of $f_{k,\frac{1}{2}\det(2m),D_0D,r(2m)^*r_0^t,D_0}(w)$, that the series $\Omega_{k,m,D_0,r_0}(w;\tau,z)$ is absolutely convergent. As a function of w it is an element of

$S_{2k}\left(\frac{1}{2}\det(2m)\right)^{-}$. Moreover we can show the following identity by using Theorem 2.3

$$\mathcal{S}^*_{D_0,r_0}(f)(\tau,z)=\langle f,\Omega_{k,m,D_0,r_0}(\cdot;-\bar{\tau},-\bar{z})\rangle \qquad \left(\forall\, f\in S_k\left(\frac{1}{2}\det(2m)\right)^{-}\right).$$

Next we want to show that

$$\mathcal{S}_{D_0,r_0}(\phi)(\omega)=\langle \phi,\Omega_{k,m,D_0,r_0}(-\bar{\omega},\cdot,\cdot)\rangle \quad \left(\forall\,\phi\in J^{cusp}_{k+\frac{g+1}{2},m}\right).$$

Since $\Omega_{k,m,D_0,r_0}(-\bar{\tau},-\bar{z};w)=\overline{\Omega_{k,m,D_0,r_0}(\tau,z;-\bar{w})}$ we have due to the Petersson coefficient formula to prove that

$$\Omega_{k,m,D_0,r_0}(w;\tau,z)=c_{k,m,D_0}\cdot\frac{i^{k-1}\cdot(2\pi)^k}{(k-1)!} \times\sum_{l\geq 1} l^{k-1}\left(\sum_{dd'=l}\left(\frac{D_0}{d}\right)\cdot d'^k\cdot P_{k+\frac{g+1}{2},m,(n_0d'^2,r_0d')}(\tau,z)\right)e^{2\pi i l w}. \tag{3.2}$$

For this we expand both sides of (3.2) in double Fourier series and compare Fourier coefficients. Using the Fourier expansion of $f_{k,\frac{1}{2}\det(2m),D_0D,r(2m)^*r_0^t,D_0}(w)$ and of $P_{k+\frac{g+1}{2},m,(n_0d'^2,r_0d')}$, it can be seen directly that we have to show

(3.3)

$$\begin{aligned}
&l^{k-1}\cdot(D/D_0)^{k/2}\cdot\epsilon_{\frac{1}{2}\det(2m)}(l,D_0D,r(2m)^*r_0^t,D_0)+i^{k+1}\cdot(D/D_0)^{k/2-1/4}\cdot l^{k-1/2}\cdot\pi\\
&\quad\times 2(\det(2m))^{-1/2}\cdot\sum_{a\geq 1}a^{-1/2}\cdot S_{\frac{1}{2}\det(2m)a}(l,D_0D,r(2m)^*r_0^t,D_0)\\
&\quad\cdot J_{k-1/2}\left(\frac{2\pi\cdot l}{\det(2m)\cdot a}\cdot\sqrt{D_0D}\right)\\
&=l^{k-1}\cdot\sum_{d|l}\left(\frac{D_0}{d}\right)\cdot(l/d)^k\cdot\delta_m\left(\left(\frac{l}{d}\right)^2n_0,\frac{l}{d}r_0,n,r\right)+2\cdot i^{k+(g+1)/2}\\
&\quad\times(D/D_0)^{k/2-1/4}\cdot l^{k-1/2}\cdot\pi\cdot(\det 2m)^{-1/2}\cdot\sum_{d|l}\left(\frac{D_0}{d}\right)\cdot d^{-1/2}\sum_{c\geq 1}e_{2c}(rm^{-1}r_0^tl/d)\\
&\quad\times H_{m,c}\left(\frac{l^2}{d^2}n_0,\left(\frac{l}{d}\right)r_0,n,r\right)\cdot J_{k-1/2}\left(\frac{2\pi\cdot l}{\det(2m)\cdot c\cdot d}\cdot\sqrt{D_0D}\right)\cdot c^{-g/2-1}.
\end{aligned}$$

We first want to show that the first terms of (3.3) agree with each other. For this we have to show the following

Lemma 3.1. *We have*

$$(D/D_0)^{k/2}\cdot \epsilon_{\frac{1}{2}\det(2m)}(l, D_0D, r(2m)^* r_0^t, D_0) = \sum_{d|l}\left(\frac{D_0}{d}\right)\cdot (l/d)^k\cdot \delta_m\left(\left(\frac{l}{d}\right)^2 n_0, \frac{l}{d} r_0, n, r\right). \tag{3.4}$$

Proof. The left-hand side of (3.4) is zero unless $D = D_0 f^2$ for some $f\in\mathbb{N}$ with $f|l$ and $D_0 f \equiv r(2m)^* r_0^t \pmod{\det(2m)}$. Using $D_0 = r_0(2m)^* r_0^t - 2n_0\det(2m)$ (which follows directly from the Jacobi decomposition of D_0) we see that this congruence is equivalent to $r_0(2m)^* r_0^t f \equiv r(2m)^* r_0^t \pmod{\det(2m)}$. In this case the left-hand side of (3.4) is equal to $\left(\frac{D_0}{l/f}\right)\cdot f^k$.

The right-hand side of (3.4) is zero unless $D = D_0(l/d)^2$ and $r \equiv r_0 l/d \pmod{\mathbb{Z}^{(1,g)}\cdot 2m}$. Setting $f = l/d$, we see that in this case it has the value $\left(\frac{D_0}{l/f}\right)\cdot f^k$.

Thus we have to show that under the assumptions for m and D_0 the following congruences are equivalent:

$$(r - r_0 f)(2m)^* r_0^t \equiv 0 \pmod{\det(2m)}, \tag{3.5}$$

$$r - r_0 f \equiv 0 \pmod{\mathbb{Z}^{(1,g)}\cdot 2m}. \tag{3.6}$$

One directly sees that if r is a solution of (3.6), then r is also a solution of (3.5). Moreover it can be shown easily that (3.6) is equivalent to the following condition

(3.7) The congruence $\lambda\cdot 2m \equiv r - r_0 f \pmod{\mathbb{Z}^{(1,g)}\cdot \det(2m)}$ is solvable.

Now we let p be a prime, $\nu\in\mathbb{N}$ such that p^ν divides $\det(2m)$ and consider the congruences:

$$(r - r_0 f)(2m)^* r_0^t \equiv 0 \pmod{p^\nu}, \tag{3.8}$$

$$\lambda\cdot 2m \equiv r - r_0 f \pmod{\mathbb{Z}^{(1,g)}\cdot p^\nu}, \tag{3.9}$$

and show that every solution r of (3.8) gives a solution λ of (3.9). For this we again change $\begin{pmatrix} 2n_0 & r_0 \\ r_0^t & 2m\end{pmatrix}$ into $\begin{pmatrix} 2n_0 & r_0 \\ r_0^t & 2m\end{pmatrix}\left[\begin{pmatrix} 1 & 0 \\ 0 & U\end{pmatrix}\right]$ with U as before and show the claim in the case that $2m$ a diagonal matrix if $p\neq 2$ and a block diagonal matrix if $p = 2$. We just have to be careful about using the restrictions given for m and D_0 since these are changed by this substitution (For example the assumption about the common divisors of D_0 and $\det(2m)$ is changed). Let us abbreviate $(s_1,\dots,s_g)^t := r - r_0 U f$, $(r_1',\dots,r_g')^t := r_0$, and $(\lambda_1,\dots,\lambda_g)^t := \lambda$. For the proof we treat the cases $p\neq 2$ and $p = 2$ separately.

In the case $p \neq 2$ congruence (3.8) is equivalent to

$$\sum_{i=1}^{g} \left(\prod_{j \neq i} m_j \right) s_i r_i' \equiv 0 \pmod{p^\nu} \tag{3.10}$$

and congruence (3.9) is equivalent to the solvability of the congruences

$$\lambda_i \cdot m_i \equiv s_i \pmod{p^\nu} \qquad (1 \leq i \leq g). \tag{3.11}$$

Moreover it can be shown easily that p divides exactly one m_i $(1 \leq i \leq g)$ and that the other m_j are coprime to p. We may without loss of generality assume that p divides m_1. Using $\left(\prod_{j=2}^{g} m_j, p\right) = 1$ and $(r_1', p) = 1$ (which can be seen from the Jacobi decomposition of $\begin{pmatrix} 2n_0 & r_0 \\ r_0^t & 2m \end{pmatrix}$) congruence (3.10) is equivalent to $s_1 \equiv 0 \pmod{p^\nu}$. Thus the claim follows since the congruences $\lambda_i \cdot m_i \equiv s_i \pmod{p^\nu}$ $(2 \leq i \leq n)$ are solvable due to $\left(\prod_{j=2}^{g} m_j, p\right) = 1$.

In the case $p = 2$ we get with the same abbreviations as before (and without loss of generality assuming that the block $2l$ occurs at the first position) that (3.8) has the form

$$s_1 r_1' \equiv 0 \pmod{2}, \tag{3.12}$$

and (3.9) is equivalent to the solvability of the system of congruences

$$\begin{aligned} s_1 &\equiv 0 \pmod{2}, \\ \lambda_3 &\equiv s_2 \pmod{2}, \\ \lambda_2 &\equiv s_3 \pmod{2}, \\ &\vdots \\ \lambda_{g-1} &\equiv s_g \pmod{2}. \end{aligned}$$

Clearly the last $g-1$ congruences are solvable. Moreover we obtain as in the case $p \neq 2$ that $2 \nmid r_1'$. Thus (3.12) is equivalent to $s_1 \equiv 0 \pmod{2}$, i.e., we have proved Lemma 3.1. □

Thus the first terms in (3.3) agree. Next we have to show that the second terms in (3.3) agree. In the second term on the right-hand side of (3.3) we substitute $cd = a$ to get, using $g \equiv 1 \pmod{8}$,

$$i^{k+1} \cdot (D/D_0)^{k/2-1/4} \cdot l^{k-1/2} \cdot 2\pi \cdot (\det(2m))^{-1/2} \cdot \sum_{a \geq 1} \sum_{d \mid (a,l)} \left(\frac{D_0}{d} \right) \cdot d^{-1/2} \cdot e_{2a}(r m^{-1} r_0^t)$$
$$\times H^{\pm}_{m,a/d} \left(\frac{l^2}{d^2} n_0, \frac{l}{d} r_0, n, r \right) \cdot J_{k-1/2} \left(\frac{2\pi \cdot l}{\det(2m) \cdot a} \cdot \sqrt{D_0 D} \right) \cdot (a/d)^{-g/2-1}.$$

Thus it is sufficient to show

Lemma 3.2. *For $l \geq 1, n \geq 0,$ and $r \in \mathbb{Z}^{(1,g)}$ we have*

$$S_{\frac{1}{2}\det(2m)a}(l, DD_0, r_0(2m)^* r^t, D_0) = \sum_{d|(a,l)} \left(\frac{D_0}{d}\right) \cdot (a/d)^{g/2} \cdot e_{2a/d}(rm^{-1}r_0^t) \cdot H_{m,a/d}\left(\frac{l^2}{d^2}n_0, \frac{l}{d}r_0, n, r\right).$$

Proof. If we insert the definitions of $S_{\frac{1}{2}\det(2m)a}$ and $H_{m,a/d}$ and multiply both sides with $e_{2a}(-r_0 m^{-1} r^t)$, then we see that we have to show

$$\sum_{\substack{b(a\det(2m)) \\ b\equiv r_0(2m)^* r^t(\det(2m)) \\ b^2\equiv DD_0(2\det(2m)a)}} \chi_{D_0}\left(\left[\frac{a}{2}\det(2m), b, \frac{b^2-DD_0}{2\det(2m)a}\right]\right) \cdot e_a\left(\frac{b-r_0(2m)^{-1}r}{\det(2m)}l\right)$$
$$= \sum_{d|(a,l)} \left(\frac{D_0}{d}\right) \cdot (a/d)^{-(g+1)/2} \cdot \sum_{\substack{\rho(a/d)^* \\ \lambda(a/d)}} e_{a/d}\left(\left(m[\lambda] + \frac{l}{d}r_0\lambda + \frac{l^2}{d^2}n_0\right)\bar{\rho} + n\rho + r\lambda\right).$$

Since both sides are periodic in l with period a, it is sufficient to show that their Fourier transforms are equal, i.e., we have to show that for every $h' \in \mathbb{Z}/a\mathbb{Z}$ we have

$$\frac{1}{a} \cdot \sum_{\substack{b(a\det(2m)) \\ b\equiv r_0(2m)^* r^t(\det 2m) \\ b^2\equiv DD_0(2\det(2m)a)}} \sum_{l(a)} \chi_{D_0}\left(\left[\frac{a}{2}\det(2m), b, \frac{b^2-DD_0}{2\det(2m)a}\right]\right) \times e_a\left(\left(\frac{b-r_0(2m)^{-1}r^t}{\det(2m)} - h'\right)l\right)$$

$$(3.13) \quad = \frac{1}{a} \cdot \sum_{l(a)} \sum_{d|(a,l)} \left(\frac{D_0}{d}\right) \cdot (a/d)^{-(g+1)/2} \cdot \sum_{\substack{\rho(a/d)^* \\ \lambda(a/d)}} \times e_{a/d}\left(\left(m[\lambda] + \frac{l}{d}r_0\lambda + \frac{l^2}{d^2}n_0\right)\bar{\rho} + n\rho + r\lambda - h'\frac{l}{d}\right).$$

Setting $h = \det(2m)h' + r_0(2m)^* r^t$ we see the left-hand side of (3.13) is equal to

$$\frac{1}{a} \cdot \sum_{\substack{b(a\det(2m)) \\ b\equiv r_0(2m)^* r^t(\det(2m)) \\ b^2\equiv DD_0(2\det(2m)a)}} \chi_{D_0}\left(\left[\frac{a}{2}\det(2m), b, \frac{b^2-DD_0}{2\det(2m)a}\right]\right) \cdot \sum_{l(a)} e_a\left(\frac{l}{\det(2m)}(b-h)\right)$$

$$= \begin{cases} \chi_{D_0}\left(\left[a\frac{1}{2}\det(2m), h, \frac{h^2-DD_0}{2\det(2m)a}\right]\right) & \text{if } h^2 \equiv DD_0 \pmod{2a\det(2m)} \\ 0 & \text{otherwise} \end{cases}.$$

For the right-hand side of (3.13) we obtain, after replacing l by ld and then (λ, l) by $(\rho\lambda, \rho l)$,

$$\frac{1}{a}\cdot\sum_{d|a}\left(\frac{D_0}{d}\right)\cdot(d/a)^{(g+1)/2}\cdot\sum_{\substack{\rho(a/d)^*\\ \lambda,l(a/d)}} e_{a/d}\left(\rho\left(m[\lambda]+r_0l\lambda+n_0l^2+r\lambda-h'l+n\right)\right).$$

Thus it is left to prove the following

Lemma 3.3. *Suppose that* $b \equiv r(2m)^* r_0^t \pmod{\det(2m)}$. *Let*

$$F(x,y) := m[x] + r_0xy + n_0y^2 + rx + sy + n \qquad (x \in \mathbb{Z}^{(g,1)}, y \in \mathbb{Z}),$$

where

$$s = r(2m)^{-1}r_0^t - \frac{b}{\det(2m)},$$

and where

$$F_c(m, r_0, n_0, r, s, n) := F_c := c^{-(g+1)/2}\cdot\sum_{\lambda(c)^*}\sum_{x,y(c)} e_c(\lambda F(x,y)).$$

Then we have for any $a \geq 1$

$$(3.14)\quad \frac{1}{a}\cdot\sum_{d|a}\left(\frac{D_0}{d}\right)\cdot F_{a/d} = \begin{cases} \chi_{D_0}\left(\left[\frac{a}{2}\det(2m), b, \frac{b^2-DD_0}{2\det(2m)a}\right]\right) & \text{if } a|\frac{b^2-DD_0}{2\det(2m)} \\ 0 & \text{otherwise}\end{cases}.$$

Proof. For the proof we need the following formulas for Gauss sums

Lemma 3.4. *Let* $a, b \in \mathbb{Z}$, $\nu \in \mathbb{N}_0$, *and let* p *be a prime number.* *Define* $G(a,b,p^\nu) := \sum_{x(p^\nu)} e_{p^\nu}(ax^2+bx)$. *Let* $\alpha := \nu_p(a)$, *where* $a = p^\alpha a'$ *with* $(a',p)=1$.

(1) *For* $\alpha \geq \nu$ *we have* $G(a,b,p^\nu) = \begin{cases} p^\nu & \text{if } b \equiv 0 \pmod{p^\nu} \\ 0 & \text{otherwise}\end{cases}$.

(2) *For* $0 \leq \alpha < \nu$ *and* $b \not\equiv 0 \pmod{p^\alpha}$ *we have* $G(a,b,p^\nu) = 0$.

(3) *If* $p \neq 2$ *and* $b \equiv 0 \pmod{p^\alpha}$, $0 \leq \alpha < \nu$, *we have*

$$G(a,b,p^\nu) = p^{\frac{\alpha+\nu}{2}}\cdot\epsilon(p^{\nu-\alpha})\cdot\left(\frac{a/p^\nu}{p^{\nu-\alpha}}\right)\cdot e_{p^{\nu+\alpha}}\left(-b^2\overline{\frac{4a}{p^\alpha}}\right),$$

where $\overline{\frac{4a}{p^\alpha}}$ *is an inverse of* $\frac{4a}{p^\alpha} \pmod{p^{\nu+\alpha}}$, *and where* $\epsilon(x) = 1$ *or* i *according as* $x \equiv 1$ *or* $3 \pmod 4$.

(4) *The sum* $G(a,b,2^\nu)$ *is equal to* 2^ν *if* $\nu-\alpha = 1$ *and* $b \not\equiv 0 \pmod 2$, *has the value*

$$2^{\frac{\nu+\alpha}{2}}\cdot(i+1)\cdot\left(\frac{-2^{\nu+\alpha}}{a/2^\alpha}\right)\cdot\epsilon(a/2^\alpha)\cdot e_{2^{\nu+\alpha}}\left(-\overline{\frac{a}{2^\alpha}}\frac{b^2}{4}\right)$$

if $\nu-\alpha > 1$ *and* $b \equiv 0 \pmod{2^{\alpha+1}}$, *and is* 0 *otherwise. Here* $\overline{a/2^\alpha}$ *is an inverse of* $a/2^\alpha \pmod{p^{\nu+\alpha+2}}$.

Lemma 3.5. *Let $p \neq 2$ be a prime, c be a p-power, $A \in \mathbb{Z}$ with $p|A$. Then we have*

$$\sum_{\lambda(c)} \left(\frac{\lambda}{p}\right) \cdot e_{pc}(\lambda A) = \begin{cases} \epsilon(p) \cdot \frac{c}{\sqrt{p}} \cdot \left(\frac{A/c}{p}\right) & \text{if } c|A \\ 0 & \text{otherwise} \end{cases}.$$

Proof of Lemma 3.3. For the proof we set $C := \frac{b^2 - DD_0}{2\det(2m)}$. Since both sides of (3.14) can be shown to be multiplicative functions in a we may assume that a is a $p-$power, where p is a prime number. Moreover, using Lemma 2.1, it can be shown that for the proof of Lemma 3.3 it is sufficient to show that

$$\frac{1}{a} \cdot \sum_{d|a} \left(\frac{D_0}{d}\right) \cdot F_{a/d} = \begin{cases} 0 & \text{if } a \nmid C \\ \left(\frac{D_0}{a}\right) & \text{if } a|C \text{ and } p \nmid D_0 \\ \left(\frac{D_0/p^*}{a}\right)\left(\frac{\frac{1}{2}\det(2m)C/a}{p}\right) & \text{if } a|C \text{ and } p|D_0, \end{cases} \tag{3.15}$$

where p^* is the prime discriminant divisor of D divisible by p. For the proof of (3.15) we again want to change $\begin{pmatrix} 2n_0 & r_0 \\ r_0^t & 2m \end{pmatrix}$ into $\begin{pmatrix} 2n_0 & r_0 \\ r_0^t & 2m \end{pmatrix}\left[\begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}\right]$ with U as before. We can show that by doing so we neither change the left- nor the right-hand side of (3.15). Nevertheless we again have to be careful since we again change the assumptions made on D_0 and m. Let us abbreviate $(r_1, \ldots, r_g) := r$, and$(r_1', \ldots, r_g') := r_0$. We now want to treat the cases $p \neq 2$ and $p = 2$ separately.
In case $p \neq 2$ we may assume that m is a diagonal matrix with entries $m_1, \ldots, m_g$. We can show with the same arguments as before that $(m_i, p) = 1$ $(1 \leq i \leq g)$ if $p \nmid \det(2m)$ and p divides m_i for exactly one m_i $(1 \leq i \leq g)$ if $p|\det(2m)$. Morover we have

$$\begin{aligned} D &= \det(2m)/2 \cdot \left(\sum_{i=1}^g r_i^2/m_i - 4n\right), \\ D_0 &= \det(2m)/2 \cdot \left(\sum_{i=1}^g r_i'^2/m_i - 4n_0\right), \\ b &= \det(2m)/2 \cdot \left(\sum_{i=1}^g r_i r_i'/m_i - 2s\right). \end{aligned} \tag{3.16}$$

Furthermore in this case the sum F_c has the form

$$F_c = c^{-(g+1)/2} \cdot \sum_{\lambda(c)^*} e_c(n\lambda) \sum_{y(c)} e_c\left(\left(n_0 y^2 + sy\right)\lambda\right)$$
$$\times \prod_{i=1}^{g} \sum_{x_i(c)} e_c\left(\left(m_i x_i^2 + (y r_i' + r_i)\, x_i\right)\lambda\right).$$

(3.17)

Let us first assume $p \nmid \det(2m)$. Applying Lemma 3.4 and using $g \equiv 1 \pmod 4$ leads to

$$F_c = c^{-1/2} \cdot \epsilon(c) \cdot \left(\frac{\frac{1}{2}\det(2m)}{c} \right) \cdot \sum_{\lambda(c)^*} \left(\frac{\lambda}{c} \right) \cdot e_c\left(\lambda \left(n - \sum_{i=1}^{g} r_i^2/(4m_i) \right) \right)$$
$$\sum_{y(c)} e_c\left(\left(\lambda \left(n_0 - \sum_{i=1}^{g} r_i'^2/(4m_i) \right) y^2 + \left(s - \sum_{i=1}^{g} r_i' r_i/(2m_i) \right) y \right) \right)$$
$$= c^{-1/2} \cdot \epsilon(c) \cdot \sum_{\lambda(c)^*} \left(\frac{\lambda}{c} \right) \cdot e_c(-\lambda D) \sum_{y(c)} e_c(-D_0 \lambda y^2 - 2b\lambda y),$$

where for the last identity we have replaced λ by $2\det(2m) \cdot \lambda$ and have used (3.16) We now proceed, treating the cases $p \nmid D_0$ and $p | D_0$, separately. In the case $p \nmid D_0$ we have, again using Lemma 3.4 and (3.16)

$$F_c = \left(\frac{D_0}{c} \right) \cdot \sum_{\lambda(c)^*} e_c\left(\lambda(b^2 - DD_0)/D_0 \right) = \left(\frac{D_0}{c} \right) \cdot \sum_{\lambda(c)^*} e_c(\lambda C),$$

where for the last identity we have replaced λ by $\lambda \cdot D_0 \cdot \overline{2\det(2m)}$, where $\overline{2\det(2m)}$ is an inverse of $2\det(2m) \pmod c$. Thus we obtain

$$\sum_{d|a} \left(\frac{D_0}{d} \right) \cdot F_{a/d} = \left(\frac{D_0}{a} \right) \sum_{\lambda(a)} e_a(\lambda C) = \begin{cases} \left(\frac{D_0}{a} \right) \cdot a & \text{if } a|C \\ 0 & \text{otherwise} \end{cases}.$$

In the case $p|D_0$ we obtain, using Lemma 3.4, that F_c vanishes if $p \nmid b$ (which is equivalent to $p \nmid C$) and otherwise has the value

$$\begin{aligned} F_c &= p^{1/2} \cdot \epsilon(p) \cdot \left(\frac{D_0/p^*}{c/p} \right) \cdot \sum_{\lambda(c)^*} \left(\frac{\lambda}{p} \right) \cdot e_c\left(\lambda \left(b^2 - DD_0 \right) / D_0 \right) \\ &= p^{1/2} \cdot \epsilon(p) \cdot \left(\frac{D_0/p^*}{c/p} \right) \cdot \left(\frac{\frac{1}{2}\det(2m)}{p} \right) \cdot \left(\frac{D_0/p}{p} \right) \cdot \sum_{\lambda(c)^*} \left(\frac{\lambda}{p} \right) \cdot e_c(\lambda C/p), \end{aligned}$$

where for the last identity we have replaced λ by $\overline{2\det(2m)} \cdot D_0/p \cdot \lambda$, where $\overline{2\det(2m)}$ is an inverse of $2\det(2m) \pmod c$. Thus we obtain in case $c|C$, using Lemma 3.5,

$$F_c = c \cdot \left(\frac{D_0/p^*}{c} \right) \cdot \left(\frac{\frac{1}{2}\det(2m)C/c}{p} \right). \tag{3.18}$$

Otherwise the sum vanishes. Clearly expression (3.18) is zero if $p|(C/c)$. Therefore the sum on the left-hand side of (3.15) is reduced to a single term F_a.
Now let us assume that $p|\det(2m)$. Then we can show that p divides exactly one m_i $(1 \le i \le g)$ of order $\nu = \nu_p(\det 2m)$. Without loss of generality we may assume $p|m_g$.

In the case $p^\nu | c$ we have, again using Lemma 3.4 and $g \equiv 1 \pmod 4$

$$F_c = c^{-1/2} \cdot p^{\nu/2} \cdot \epsilon(c/p^\nu) \cdot \left(\frac{\prod_{i=1}^{g-1} m_i}{c}\right) \cdot \left(\frac{m_g/p^\nu}{c/p^\nu}\right) \cdot \sum_{\lambda(c)^*} e_c\left(\left(n - \sum_{i=1}^{g} r_i^2/(4m_i)\right)\lambda\right)$$

$$\left(\frac{\lambda}{c/p^\nu}\right) \cdot \sum_{\substack{y(c) \\ yr'_g + r_g \equiv 0 (p^\nu)}} e_c\left(\lambda\left(y^2\left(n_0 - \sum_{i=1}^{g} r_i'^2/(4m_i)\right) + y\left(s - \sum_{i=1}^{g} r_i r'_i/(2m_i)\right)\right)\right).$$

$$= c^{-1/2} \cdot p^{\nu/2} \cdot \epsilon(c/p^\nu) \cdot \left(\frac{\prod_{i=1}^{g-1} m_i}{c}\right) \cdot \left(\frac{m_g/p^\nu}{c/p^\nu}\right) \cdot \left(\frac{2\det(2m)/p^\nu}{c/p^\nu}\right)$$

$$\times \sum_{\lambda(c)^*} \left(\frac{\lambda}{c/p^\nu}\right) \cdot e_c(-\lambda D/p^\nu) \sum_{\substack{y(c) \\ yr'_g + r_g \equiv 0 (p^\nu)}} e_c\left(\lambda\left(-y^2 D_0/p^\nu - 2b/p^\nu y\right)\right),$$

where for the last identity we have replaced λ by $\frac{2\det(2m)}{p} \cdot \lambda$. Using $p \nmid D_0$ we can show with the same arguments as before that $(r'_g, p) = 1$. Thus we can replace y by $-r_g\overline{r'_g} + p^\nu y$, where $\overline{r'_g}$ is an inverse of $r'_g \pmod c$, and where the new y runs $\pmod{c/p^\nu}$. Moreover we can easily show that $\left(\frac{\prod_{i=1}^{g-1} m_i}{p}\right) = 1$. Thus we obtain

$$F_c = c^{-1/2} \cdot p^\nu \cdot \epsilon(c/p) \cdot \sum_{\lambda(c)^*} \left(\frac{\lambda}{c/p^\nu}\right) \cdot e_c\left(\lambda\left(-D/p^\nu - D_0/p^\nu \cdot \left(r_g\overline{r'_g}\right)^2 + 2b/p^\nu\left(r_g\overline{r'_g}\right)\right)\right)$$

$$\times \sum_{y(c/p^\nu)} e_c\left(\lambda\left(-p^\nu y^2 D_0 + 2y\left(D_0 r_g \overline{r'_g} - b\right)\right)\right).$$

Due to (3.16) we have $D_0 r_g \overline{r'_g} - b \equiv 0 \pmod{p^\nu}$, i.e., we get, using Lemma 3.4, $p \nmid D_0$ and $\left(\frac{D_0}{p}\right) = 1$

$$F_c = \sum_{\lambda(c)^*} e_c\left(\frac{\lambda(b^2 - DD_0)}{p^\nu D_0}\right) = \sum_{\lambda(c)^*} e_c(C\lambda), \tag{3.19}$$

where for the last identity we have changed λ into $D_0 \cdot \overline{2\det(2m)/p^\nu} \cdot \lambda$, where $\overline{2\det(2m)/p^\nu}$ denotes an inverse of $2\det(2m)/p^\nu \pmod c$.
The case $p^\nu \nmid c$ is treated similarly and gives the same value for F_c.
Thus we infer that

$$\sum_{d|a} \left(\frac{D_0}{d}\right) \cdot F_{a/d} = \sum_{\lambda(c)} e_a(\lambda C) = \begin{cases} a & \text{if } a|C \\ 0 & \text{otherwise} \end{cases}.$$

Therefore formula (3.15) is proved in the case $p \neq 2$.
In the case $p = 2$ we only prove (3.15) in case that D_0 is odd since the case that D_0 is

even can be treated with the same arguments. In the case that D_0 is odd we may assume with the same arguments as before that m is a block diagonal matrix with blocks from the set $\left\{l, \left(\begin{smallmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{smallmatrix}\right)\right\}$, where the type l occurs exactly once. Let the type $\left(\begin{smallmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{smallmatrix}\right)$ occur g_1 times and the type $\left(\begin{smallmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{smallmatrix}\right)$ occur g_2 times, i.e., $g = 1 + 2g_1 + 2g_2$. Let us set $I := \{2, 4, \ldots, 2g_1\}$, and $J := \{2g_1 + 2, 2g_1 + 4, \ldots, g - 1\} \subset \mathbb{N}$. Then we have

$$\begin{aligned}
D &= \frac{1}{2}\det(2m) \cdot \left(-4n + \frac{r_1^2}{l} + \frac{4}{3} \cdot \sum_{i \in I} \left(r_i^2 - r_i r_{i+1} + r_{i+1}^2 \right) + 4 \cdot \sum_{i \in J} r_i r_{i+1} \right), \\
D_0 &= \frac{1}{2}\det(2m) \cdot \left(-4n_0 + \frac{r_1'^2}{l} + \frac{4}{3} \cdot \sum_{i \in I} \left(r_i'^2 - r_i' r_{i+1}' + r_{i+1}'^2 \right) + 4 \cdot \sum_{i \in J} r_i' r_{i+1}' \right), \\
b &= \frac{1}{2}\det(2m) \cdot \left(-2s + \frac{r_1 r_1'}{l} + \frac{2}{3} \cdot \sum_{i \in I} \left(2r_i r_i' + 2r_{i+1} r_{i+1}' - r_i' r_{i+1} - r_{i+1}' r_i \right) \right. \\
&\qquad \left. + 2 \cdot \sum_{i \in J} \left(r_i' r_{i+1} + r_i r_{i+1}' \right) \right), \qquad (3.20)
\end{aligned}$$

and

$$\begin{aligned}
C = \frac{1}{2}\det(2m) \cdot \Bigg(\Bigg(& -s + \frac{r_1 r_1'}{2l} + \frac{1}{3} \cdot \sum_{i \in I} \left(2r_i' r_i - r_i' r_{i+1} - r_{i+1}' r_i + 2r_{i+1}' r_{i+1} \right) \\
& + \sum_{i \in J} \left(r_i' r_{i+1} + r_{i+1}' r_i \right) \Bigg)^2 \\
& - \left(-2n + \frac{r_1^2}{2l} + \frac{2}{3} \cdot \sum_{i \in I} (r_i^2 - r_i r_{i+1} + r_{i+1}^2) + 2 \cdot \sum_{i \in J} r_i r_{i+1} \right) \\
& \times \left(-2n_0 + \frac{r_1^2}{2l} + \frac{2}{3} \cdot \sum_{i \in I} \left(r_i'^2 - r_i' r_{i+1}' + r_{i+1}'^2 \right) + 2 \cdot \sum_{i \in J} r_i' r_{i+1}' \right) \Bigg). \qquad (3.21)
\end{aligned}$$

Moreover we have

$$\begin{aligned}
F_c = c^{-(g+1)/2} \cdot & \sum_{\lambda(c)^*} e_c(n\lambda) \sum_{y(c)} e_c\left(\left(n_0 y^2 + sy \right) \lambda \right) \sum_{x_1(c)} e_c\left(\left(l x_1^2 + \left(y r_1' + r_1 \right) x_1 \right) \lambda \right) \qquad (3.22) \\
& \times \prod_{i \in I} \sum_{\substack{x_i(c) \\ x_{i+1}(c)}} e_c\left(\lambda \left(x_i^2 + x_i x_{i+1} + x_{i+1}^2 + \left(r_i' y + r_i \right) x_i + \left(r_{i+1}' y + r_{i+1} \right) x_{i+1} \right) \right) \\
& \times \prod_{i \in J} \sum_{\substack{x_i(c) \\ x_{i+1}(c)}} e_c\left(\lambda \left(x_i x_{i+1} + \left(r_i' y + r_i \right) x_i + \left(r_{i+1}' y + r_{i+1} \right) x_{i+1} \right) \right).
\end{aligned}$$

We now want to determine the different types of sums over the x_i that can appear. For this we need Lemma 3.4. We only compute the sums in the case $c \neq 2$ since the case $c = 2$ is treated quite similarly. If $r_1' y + r_1 \equiv 0 \pmod 2$ we obtain, using Lemma 3.4 and $(l, 2) = 1$, that the first type of sum equals

$$c^{1/2} \cdot \left(\frac{-c}{\lambda l}\right) \cdot \epsilon(l\lambda) \cdot (1+i) \cdot e_c\left(-\lambda \left(r_1' y + r_1\right)^2 / (4l)\right).$$

Otherwise the sum has the value 0. Moreover we have, again using Lemma 3.4, that the second type of sum is equal to

$$\begin{aligned}
&\sum_{\substack{x_{i+1}(c) \\ x_{i+1}+r_i'y+r_i \equiv 0(2)}} e_c\left(\lambda\left(x_{i+1}^2 + \left(r_{i+1}' y + r_{i+1}\right) x_{i+1}\right)\right) \cdot c^{1/2} \left(\frac{-c}{\lambda}\right) \cdot \epsilon(\lambda) \cdot (1+i) \\
&\qquad\qquad \times e_c\left(-\lambda \frac{(x_{i+1} + r_i' y + r_i)^2}{4}\right) \\
&= c \cdot \left(\frac{-c}{3}\right) \cdot (-1) \cdot e_c\left(\bar{3}\lambda\left(r_i r_{i+1} - r_i^2 - r_{i+1}^2\right) + y^2\left(r_i' r_{i+1}' - r_i'^2 - r_{i+1}'^2\right)\right. \\
&\qquad\qquad \left. + y\left(r_i' r_{i+1} + r_{i+1}' r_i - 2r_i' r_i - 2r_{i+1}' r_{i+1}\right)\right),
\end{aligned}$$

where for the last identity we have replaced x_{i+1} by $-r_i' y - r_i + 2x_{i+1}$, with the new x_{i+1} running $\pmod{c/2}$ and have used that $\epsilon(\lambda) \cdot \epsilon(3\lambda) = i$. The third type of sum is equal to

$$c \cdot (-1) \cdot \left(\frac{-c}{-1}\right) \cdot e_c\left(-\lambda\left(y^2 r_i' r_{i+1}' + y\left(r_i' r_{i+1} + r_{i+1}' r_i\right) + r_i r_{i+1}\right)\right).$$

Thus we obtain by changing λ into $\frac{1}{2}\det(2m) \cdot \lambda$, by using that $g \equiv 1 \pmod 4$, equation (3.20), and $\epsilon(l \cdot \det(2m) \cdot \lambda) = \epsilon(\lambda)$

$$\begin{aligned}
F_c &= c^{-1/2} \cdot (1+i) \cdot \sum_{\lambda(c)^*} \left(\frac{-c}{\lambda}\right) \cdot \epsilon\left(l \cdot \frac{1}{2}\det(2m)\lambda\right) \cdot e_c\left(-\lambda D/4\right) \\
&\qquad \times \sum_{\substack{y(c) \\ r_1' y + r_1 \equiv 0(2)}} e_c(\lambda(-D_0/4y^2 - b/2y)) \\
&= c^{-1/2} \cdot (1+i) \cdot \sum_{\lambda(c)^*} \left(\frac{-c}{\lambda}\right) \cdot \epsilon(\lambda) \cdot e_c\left(\lambda\left(-D/4 - D_0/4(r_1\bar{r}_1)^2 + b/2r_1\overline{r_1'}\right)\right) \\
&\qquad \times \sum_{y(c/2)} e_c\left(\lambda\left(-D_0 y^2 + \left(D_0 r_1 \overline{r_1'} - b\right) y\right)\right),
\end{aligned}$$

where for the last identity we have replaced y by $2y - r_1\overline{r_1'}$, where $\overline{r_1'}$ is an inverse of $r_1 \pmod c$, and where the new y runs $\pmod{c/2}$. Due to $D_0 \cdot r_1 \cdot \overline{r_1'} - b \equiv 0 \pmod 2$ we obtain, using Lemma 3.4 and $\epsilon(\lambda)\cdot\epsilon(-D_0\lambda)=i$,

$$F_c=\left(\frac{c}{D_0}\right)\cdot\sum_{\lambda(c)^*}e_c\left(\lambda\left(b^2/(4D_0)-D/4\right)\right)=\left(\frac{c}{D_0}\right)\cdot\sum_{\lambda(c)^*}e_c(\lambda C),$$

where for the last identity we have changed λ into $\overline{\det(2m)/2}\cdot D_0\cdot\lambda$, where $\overline{\det 2m/2}$ is an inverse of $\det 2m/2 \pmod c$. We now can proceed similarly as in the case $p\neq 2$. □

References

[BK] Böcherer, S., Kohnen, W.: Estimates for Fourier coefficients of Siegel cusp forms. Math. Ann. **297**, 499-517 (1993).

[Br1] Bringmann, K.: Applications of Poincaré series on Jacobi groups. Ph.D. Thesis 2004. University of Heidelberg (http://www.math.wisc.edu/ bringman)

[Br2] Bringmann, K.: Estimates of Fourier coefficients of Siegel cusp forms for subgroups and in the case of small weight . Journal of the London Mathematical Society, accepted for publication.

[Ca] Cassels, J.W.S.: Rational quadratic forms. L.M.S. Monographs, No.13. London, New York, San Francisco: Academic Press (1978).

[EZ] Eichler, M., Zagier, D.: The theory of Jacobi forms. Progress in Mathematics, **Vol. 55**. Boston-Basel-Stuttgart: Birkhuser (1985).

[GZ] B. Gross, D. Zagier: Heegner points and derivatives of L-series. Invent. Math **84**, 225-320 (1986).

[GKZ] Gross, B.; Kohnen, W.; Zagier, D.: Heegner points and derivatives of L-series. II. Math. Ann. **278**, 497-562 (1987)

[SZ] N. Skoruppa, D. Zagier: Jacobi forms and a certain space of modular forms. Invent. Math. **94**, 111-146 (1988).

[Zi] Ziegler, C.: Jacobi forms of higher degree. Abh. Math. Semin. Univ. Hamb. **59**, 191-224 (1989).

Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706
E-mail address: `bringman@math.wisc.edu`