ESTIMATES OF FOURIER COEFFICIENTS OF SIEGEL CUSP FORMS FOR SUBGROUPS AND IN THE CASE OF SMALL WEIGHT

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Abstract. Here we develop estimates for Fourier coefficients of Siegel cusp forms. First we consider the case of Siegel modular forms for the full modular group Γ_g having small weight (i.e., weight $k = g+1$). Afterwards the case of a certain subgroup $\Gamma_{g,0}(N)$ of Γ_g is considered.

1. Introduction and statement of results

Let F be a cusp form of integral weight k with respect to the Siegel modular group $\Gamma_g = Sp_g(\mathbb{Z}) \subset GL_{2g}(\mathbb{Z})$ with Fourier coefficients $a(T)$, where T is a positive definite symmetric half-integral $g \times g$ matrix. Then a conjecture of Resnikoff and Saldaña (cf. [RS]) says that

$$a(T) \ll_{\epsilon,F} (\det T)^{k/2-(g+1)/4+\epsilon} \qquad (\epsilon > 0).$$

For $g = 1$ this conjecture is true (cf. [De] and [DS]), but for arbitrary g there are known counter examples (cf. [K2]). For $g \geq 2$ the estimate is at least true on average (cf. [K1]). For $k > g+1$, the best known estimate is

$$a(T) \ll_{\epsilon,F} (\det T)^{k/2-c_g+\epsilon} \qquad (\epsilon > 0), \tag{1.1}$$

where

$$c_g := \begin{cases} \frac{13}{36} & \text{if } g = 2 \quad [K2] \\ \frac{1}{4} & \text{if } g = 3 \quad [Bre] \\ \frac{1}{2g} + (1-\frac{1}{g})\alpha_g & \text{if } g > 3 \quad [BK] \end{cases}$$

and

$$\alpha_g^{-1} := 4(g-1) + 4\left[\frac{g-1}{2}\right] + \frac{2}{g+2}.$$

One directly sees that $\alpha_g \to 0$ for $g \to \infty$ (i.e., far from the conjecture of Resnikoff and Saldaña).

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The method in [BK] is the following: If we write $Z \in \mathbb{H}_g$ as $Z = \begin{pmatrix} \tau & z \\ z^t & \tau' \end{pmatrix}$, where $\tau \in \mathbb{H}$, $z \in \mathbb{C}^{(1,g-1)}$, and $\tau' \in \mathbb{H}_{g-1}$, we see that the function $F(Z) \in S_k(\Gamma_g)$ has a so-called Fourier-Jacobi expansion of the kind

$$F(Z) = \sum_{m>0} \phi_m(\tau, z)\, e^{2\pi i \mathrm{tr}\,(m\tau')} \qquad (\tau' \in \mathbb{H}_{g-1}),$$

where m runs through all positive definite symmetric half-integral $(g-1) \times (g-1)$ matrices, and where the coefficients $\phi_m(\tau, z)$ are Jacobi cusp forms with respect to the full Jacobi group $\Gamma^J_{1,g} := \Gamma_1 \ltimes \left(\mathbb{Z}^{(g,1)} \times \mathbb{Z}^{(g,1)}\right)$. In [BK] the Fourier coefficients of Jacobi cusp forms are estimated by developing a Petersson coefficient formula for Jacobi cusp forms and by estimating the Fourier coefficients of certain Jacobi-Poincaré series $P_{k,m;(n,r)}$ uniformly in $\det m$. The restriction $k > g+1$ is needed for the absolute convergence of $P_{k,m;(n,r)}$. Then the Petersson norm of a Jacobi cusp form is estimated for the particular case where the Jacobi cusp form arises from a Fourier-Jacobi expansion. This can be done by using certain Dirichlet series of Rankin-Selberg type which have a meromorphic continuation to the whole complex plane with finitely many poles and satisfy a certain functional equation (cf. [Ya]). The method in [Bre] is similar, only he uses a different splitting of $Z \in \mathbb{H}_3$, with $\tau \in \mathbb{H}_2$, $z \in \mathbb{C}^{(2,1)}$, and $\tau' \in \mathbb{H}$.

Here we prove generalizations of the estimates of [BK] in various directions. More precisely we regard the limiting case $k = g+1$, and the case where Γ_g is replaced by the subgroup

$$\Gamma_{g,0}(N) := \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \,\middle|\, C \equiv 0 \pmod N \right\}$$

of Γ_g. In the case $k = g+1$ the Poincaré series for Jacobi forms cannot be defined as before, because one can show that these series are not absolutely convergent. Therefore we use the so-called "Hecke trick" and multiply every summand of the Poincaré series by a factor depending on a complex variable s such that the new series $P_{k,m;(n,r),s}$ is again absolutely convergent for $\sigma = \mathrm{Re}(s)$ sufficiently large. Moreover this factor is chosen such that the new series is again invariant under the slash operation of the Jacobi group. Now the method is the following one: we compute the Fourier expansion of $P_{k,m;(n,r),s}$ and show that it is absolutely and locally uniformly convergent in an even larger domain of $\mathbb{C}$ that contains the point $s = 0$ if $k = g+1$. For this we have to estimate certain integrals and generalized Kloosterman sums to show that the series defined through the above Fourier expansion is absolutely and locally uniformly convergent in s in the larger domain. Therefore we have a definition for the Poincaré series for $k = g+1$. What is left to show is that these series are Jacobi cusp forms and that the Petersson coefficient formula is still valid. In this way we obtain the following theorem.

Theorem 1.1. *If $g > 3$, $k \geq g+1$, then estimate (1.1) holds.*

As a second generalization we estimate Fourier coefficients of Siegel cusp forms with respect to $\Gamma_{g,0}(N)$. Thus in what follows we let F be a cusp form of integral weight k with respect to $\Gamma_{g,0}(N)$ with Fourier coefficients $a(T)$, where T is given as before. For $g = 2, 3$ we obtain estimates of the same quality as in the case of the full modular group; for $g > 3$ we obtain a slightly weaker estimate. For the proof we define a vector space of certain Jacobi cusp forms for subgroups. We estimate the Fourier coefficients of these Jacobi cusp forms (again using for $k = g + 1$ the Hecke-trick). The difficulty lies in the estimates of the Petersson norm of the Fourier-Jacobi coefficients since it is not obvious how to define similar Dirichlet series of Rankin-Selberg type with a simple functional equation. Thus we instead use the classical Hecke argument to obtain a slightly weaker estimate for the Petersson norm which leads to the following theorem.

Theorem 1.2. *If $g \geq 2$, $k \geq g + 1$, then*

$$a(T) \ll_{\epsilon,F} (m_{g-1}(T))^{1/2} \cdot (\det T)^{(k-1)/2+\epsilon} \qquad (\epsilon > 0),$$

where $m_{g-1}(T) := min\{T[U]|_{g-1} | \, U \in GL_g(\mathbb{Z})\}$, and where $T[U] := U^t T U$.

Corollary 1.3. *If $g \geq 2$, $k \geq g + 1$, then*

$$a(T) \ll_{\epsilon,F} (\det T)^{k/2-1/(2g)+\epsilon} \qquad (\epsilon > 0).$$

We now proceed using a different splitting in the Fourier-Jacobi expansion. Doing so, we again can define certain Dirichlet series of Rankin-Selberg type which have a meromorphic continuation to the whole complex plane and satisfy a certain complicated functional equation that connects this Dirichlet series to other Dirichlet series. Thus we obtain, again using a modified version of a theorem of Sato and Shintani, an estimate of the same quality as in (1.1).

Theorem 1.4. *If $g = 2$ and $k \geq 3$, then*

$$a(T) \ll_{\epsilon,F} (min(T))^{5/18+\epsilon} \cdot (\det T)^{(k-1)/2+\epsilon} \qquad (\epsilon > 0),$$

where $min(A) := min_{g \in \mathbb{Z}^g} A[g]$ for an $n \times n$ matrix A.

Corollary 1.5. *If $g = 2$ and $k \geq 3$, then*

$$a(T) \ll_{\epsilon,F} (\det T)^{k/2-13/36+\epsilon} \qquad (\epsilon > 0).$$

Theorem 1.6. *If $g = 3$ and $k \geq 8$ an even integer, then*

$$a(T) \ll_{\epsilon,F} (min(T))^{-3/13+\epsilon} \cdot (\det T)^{k/2-1/4+\epsilon} \qquad (\epsilon > 0).$$

Corollary 1.7. *If $g = 3$ and $k \geq 8$ an even integer, then*

$$a(T) \ll_{\epsilon,F} (\det T)^{k/2-1/4+\epsilon} \qquad (\epsilon > 0).$$

Moreover, for $g = 2$ we obtain in the same way as in the case of the full Siegel modular group (cf. [K1]) the following estimate on average:

Corollary 1.8. *If $g = 2$ and $k \geq 3$ an integer, then*

$$\sum_{\{T>0, tr\ (T)=N\}} |a(T)|^2 \ll_{\epsilon,F} N^{k-1/2+\epsilon} \quad (\epsilon > 0).$$

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2. General facts about Jacobi cusp forms

First we briefly want to recall some basic facts about Jacobi cusp forms. For details we refer the reader to [EZ] and [Zi]. For later purposes we give a more general definition than needed in this section. The Jacobi group $\Gamma^J_{l,n} := \Gamma_l \ltimes \left(\mathbb{Z}^{(n,l)} \times \mathbb{Z}^{(n,l)}\right)$ acts on $\mathbb{H}_l \times \mathbb{C}^{(n,l)}$ in the usual way by

$$\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda, \mu)\right) \circ (\tau, z) := \left((A\tau + B)(C\tau + D)^{-1}, (z + \lambda\tau + \mu)(C\tau + D)^{-1}\right).$$

Let k be an integer, m a positive definite symmetric half-integral $n \times n$ matrix, $\gamma = \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda, \mu)\right) \in \Gamma^J_{l,n}$, and $\phi : \mathbb{H}_l \times \mathbb{C}^{(n,l)} \to \mathbb{C}$. Then we define the following action

$$\begin{aligned} \phi|_{k,m}\gamma(\tau, z) \;:=\; & \det(C\tau + D)^{-k} \cdot e\left(-\mathrm{tr}\left(m(C\tau + D)^{-1}Cm[(z + \lambda\tau + \mu)^t]\right.\right. \\ & \left.\left. + m\tau[\lambda^t] + 2m\lambda z^t\right)\right) \cdot \phi(\gamma \circ (\tau, z)), \end{aligned}$$

where $e(x) := e^{2\pi i x} \quad (\forall\, x \in \mathbb{R})$, and where $\mathrm{tr}(A)$ and $\det(A)$ denote as usual the trace and determinant of a matrix A, respectively.

In the following we let Γ be a subgroup of $\Gamma^J_{l,n}$ of finite index. In particular we denote by $\Gamma^J_{l,n,0}(N) := \Gamma_{l,0}(N) \ltimes \left(\mathbb{Z}^{(n,l)} \times \mathbb{Z}^{(n,l)}\right)$. A holomorphic function $\phi : \mathbb{H}_l \times \mathbb{C}^{(n,l)} \to \mathbb{C}$ is called a Jacobi cusp form of weight k and index m with respect to Γ, if $\phi|_{k,m}\gamma(\tau, z) = \phi(\tau, z)\, \forall\, \gamma \in \Gamma$, and if for all $\gamma \in \Gamma^J_{l,n}$ there exists a positive integer M such that the function $\phi|_{k,m}\gamma$ has a Fourier expansion of the form

$$\phi|_{k,m}\gamma(\tau, z) = \sum_{\substack{n \in \mathbb{Z}^{(l,l)} \\ r \in \mathbb{Z}^{(l,n)} \\ \frac{4n}{N} > m^{-1}[r^t]}} c(n, r) e\left(\frac{1}{M}\mathrm{tr}\ (n\tau) + \mathrm{tr}\ (rz)\right).$$

If ϕ comes from a Fourier-Jacobi expansion of a function $F \in S_k(\Gamma_g)$, we can choose $M = 1$. Moreover, if ϕ comes from a Fourier-Jacobi expansion of a function $F \in S_k(\Gamma_{g,0})$, we can choose $M = 1$ if $\gamma \in \Gamma^J_{g,0}$ and $M = N$ else. For Jacobi cusp forms ϕ and ψ with respect to Γ we define

$$< \phi, \psi > := \frac{1}{[\Gamma^J_{l,n} : \Gamma]} \int_{\Gamma \setminus \mathbb{H}_l \times \mathbb{C}^{(n,l)}} \phi(\tau, z) \cdot \overline{\psi(\tau, z)} \cdot v^k \cdot \exp\left(-4\pi m[y] \cdot v^{-1}\right) dV^J_n,$$

where $dV^J_n = (\det v)^{-n-2}\, dudvdxdy$, and where we have written $\tau = u + iv$, and $z = x + iy$. For this and the next chapter we only need the case $l = 1$. Let us denote by $J^{cusp}_{k,m}$ and $J^{cusp}_{k,m}(N)$ the vector spaces of Jacobi cusp forms with respect to $\Gamma^J_{1,g}$ and $\Gamma^J_{1,g,0}(N)$, respectively. Next we shortly want to repeat the construction of Poincaré series for Jacobi cusp forms given in [BK]. Therefore we let $n \in \mathbb{Z}, r \in \mathbb{Z}^{(1,g)}$, and m be a positive definite symmetric half-integral $g \times g$ matrix such that $4n > m^{-1}[r^t]$. Then we define a Poincaré series of exponential type by

$$P_{k,m;(n,r)}(\tau, z) := \sum_{\gamma \in \left(\Gamma^J_{1,g}\right)_\infty \setminus \Gamma^J_{1,g}} e^{n,r}|_{k,m}\gamma(\tau, z) \qquad \left(\tau \in \mathbb{H},\ z \in \mathbb{C}^{(g,1)}\right),$$

where $e^{n,r}(\tau, z) := e(n\tau + rz) := e^{2\pi i(n\tau+rz)} \qquad \left(\tau \in \mathbb{H}, z \in \mathbb{C}^{(g,1)}\right)$, and where $(\Gamma^J_{1,g})_\infty := \left\{ \left(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, (0, \mu) \right) \middle|\ n \in \mathbb{Z},\ \mu \in \mathbb{Z}^{(g,1)} \right\}$ is the stabilizer group of the function $e^{n,r}$. Then it is stated in [BK] and proved in [Bri] that the series $P_{k,m;(n,r)}(\tau, z)$ is absolutely and locally uniformly convergent on $\mathbb{H} \times \mathbb{C}^{(g,1)}$ if $k > g + 2$. For $k \le g + 2$ the series is not absolutely convergent at the point $(i, 0)$ (cf. [Bri]). Moreover, for $k > g+2$, the function $P_{k,m;(n,r)}$ is an element of $J^{cusp}_{k,m}$ and the Petersson coefficient formula

$$\left\langle \phi, P_{k,m;(n,r)} \right\rangle = \lambda_{k,m,D} \cdot c_\phi(n, r) \qquad \left(\forall\, \phi \in J^{cusp}_{k,m}\right) \tag{2.1}$$

holds. Here $c_\phi(n, r)$ denotes the (n, r)-th Fourier coefficient of ϕ and

$$\lambda_{k,m,D} := 2^{(g-1)(k-g/2-1)-g} \cdot \Gamma\left(k - g/2 - 1\right) \cdot \pi^{-k+g/2+1} \cdot (\det m)^{k-(g+3)/2} \cdot |D|^{-k+g/2+1},$$

where $D := -\det \begin{pmatrix} 2n & r \\ r^t & 2m \end{pmatrix}$.

3. Estimates for the full modular group (the case $k = g + 1$)

In this section we want to consider the case $k = g+2$, using the Hecke-trick as already told in the introduction. For details we refer the reader to [Bri].
Let us define

$$P_{k,m;(n,r),s}(\tau, z) := \sum_{\gamma \in \left(\Gamma^J_{1,g}\right)_\infty \setminus \Gamma^J_{1,g}} \left(\frac{v}{|c\tau + d|^2}\right)^s \cdot e^{n,r}|_{k,m}\gamma(\tau, z) \quad \left((\tau, z) \in \mathbb{H} \times \mathbb{C}^{(g,1)}\right),$$

where $v = \mathrm{Im}(\tau)$. We can show similarly, as before, that the series $P_{k,m;(n,r),s}(\tau, z)$ is absolutely and locally uniformly convergent on $\mathbb{H}\times\mathbb{C}^{(g,1)}$ if $\sigma := \mathrm{Re}(s) > \frac{1}{2}(g - k + 2)$.

In this case it satisfies the transformation law

$$P_{k,m;(n,r),s}|_{k,m}\gamma(\tau,z) = P_{k,m;(n,r),s}(\tau,z) \qquad (\forall\, \gamma \in \Gamma^J_{1,g}\,, (\tau,z)\in \mathbb{H}\times\mathbb{C}^{(g,1)}).$$

Moreover we can easily show

Lemma 3.1. *Suppose that $\sigma > \frac{1}{2}(g-k+2)$. Then the Poincaré series has the Fourier expansion*

$$P_{k,m;(n,r),s}(\tau,z) = \sum_{\substack{n'\in\mathbb{Z}\\ r'\in\mathbb{Z}^{(1,g)}}} g^{\pm}_{k,m;(n,r);s,v}(n',r')e(n'\tau+r'z),$$

where $\pm = (\pm 1)^k$, and where

$$g^{\pm}_{k,m;(n,r);s,v}(n',r') := g_{k,m;(n,r);s,v}(n',r') + (-1)^k g_{k,m;(n,r);s,v}(n',-r').$$

Here

$$g_{k,m;(n,r);s,v}(n',r') := v^s\cdot\delta_m(n,r,n',r') + \sum_{c\geq 1} H_{m,c}(n,r,n',r')\cdot\Phi_{k,m,c,v}(n',r',s)\cdot c^{-k-2s},$$

with $D' := -\det\begin{pmatrix} 2n' & r' \\ r'^t & 2m\end{pmatrix}$.
Furthermore,

$$\delta_m(n,r,n',r') := \begin{cases} 1 & \text{if} \qquad D'=D,\ r'\equiv r \pmod{\mathbb{Z}^{(1,g)}\cdot 2m} \\ 0 & \text{otherwise}\end{cases}$$

and

$$H_{m,c}(n,r,n',r') := \sum_{\substack{x(c)\\ y(c)^*}} e_c((m[x]+rx+n)\bar{y}+n'y+r'x),$$

where x and y run over a complete set of representatives for $\mathbb{Z}^{(g,1)}/c\mathbb{Z}^{(g,1)}$ and $(\mathbb{Z}/c\mathbb{Z})^*$, respectively, and where $\bar{y}$ denotes an inverse of $y \pmod c$. Moreover

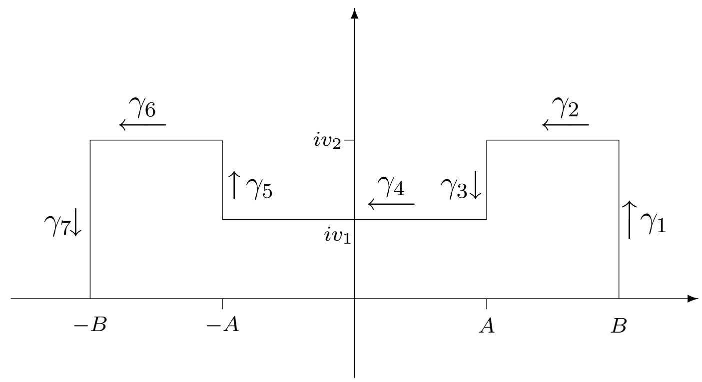
$$\text{(3.1)}\quad \Phi_{k,m,c,v}(n',r',s) := (\det(2m))^{-1/2}\cdot i^{-g/2}\cdot v^{g/2-k-s+1}\cdot e_{2c}(r'm^{-1}r^t)$$
$$\times\int_{-\infty}^{\infty}(u+i)^{g/2-k-s}\cdot(u-i)^{-s}\cdot e\left((2\det(2m))^{-1}\left(D'v(u+i)+\frac{D}{vc^2(u+i)}\right)\right)du.$$

Lemma 3.2. *The coefficients $\Phi_{k,m,c,v}(n',r',s)$ are holomorphic functions in s with $\sigma > \frac{1}{2}(1+g/2-k)$. In particular they are holomorphic at $s=0$ if $k > g/2+1$. If K is a compact subset of the domain $\sigma > \frac{1}{2}(1+\frac{g}{2}-k)$ with $s\in K$, then they satisfy the following estimate*

$$\Phi_{k,m,c,v}(n',r',s) \ll_K v^{g/2-k-\sigma+1}\cdot e^{\frac{-D}{Av}}\cdot e^{\frac{-\pi D'v}{\det(2m)}(1+sign(D')v_1)},$$

where v_1 and A are positive constants.

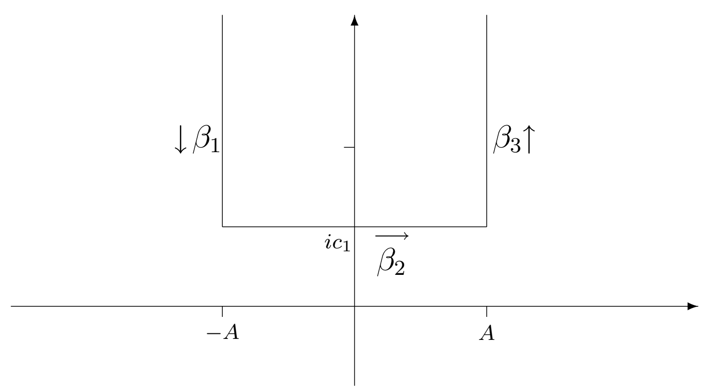
Proof. For the proof let us abbreviate the integrand in (3.1) with $g_s(u)$. Let us treat the three cases $D' = 0$, $D' > 0$, and $D' < 0$ separately. The simplest case $c_2 = 0$ is a straight-forward estimation and is therefore left to the reader. In the case $c_2 < 0$ we consider the following path of integration:



where A, B, v_1, and v_2 are positive constants with $v_1 < 1 < v_2, A < B$. Applying the residue theorem and estimating the integrals along γ_j for $j \in \{1, 2, 6, 7\}$ in a suitable way, we can show, letting B and v_2 both tend to infinity,

$$\int_{-\infty}^{\infty} g_s(u)\, du = \sum_{j=1}^{3} \int_{\beta_j} g_s(u)\, du,$$

where the paths β_j are given as



Now the claim follows if we estimate the integrals along the paths β_1, β_2, and β_3 in a suitable way. In the case $D' < 0$ we reflect the paths from the case $D' > 0$ along the real line and also obtain the claim after making some minor modifications. $\square$

Lemma 3.3. *We have the estimate*

$$|H_{m,c}(n,r,n',r')| \ll_{D,m,\epsilon} c^{g/2+1+\epsilon}. \tag{3.2}$$

Proof. Since both sides of inequality (3.2) can be shown to be multiplicative in c, it is sufficient to show that for all primes p and all $\nu \in \mathbb{N}$ we have

$$|H_{m,p^\nu}(n,r,n',r')| \leq \begin{cases} p^{\frac{g}{2}\cdot v_p(2\det(2m)D)} \cdot (p^\nu)^{g/2+1} & \text{if } p \neq 2 \\ 2^{\frac{g}{2}} \cdot 2^{\frac{g}{2}\cdot v_2(2\det(2m)D)} \cdot (2^\nu)^{g/2+1} & \text{if } p = 2. \end{cases} \tag{3.3}$$

For the proof of (3.3) we may replace $\begin{pmatrix} n & \frac{r}{2} \\ \frac{r^t}{2} & m \end{pmatrix}$ by $\begin{pmatrix} n & \frac{r}{2} \\ \frac{r^t}{2} & m \end{pmatrix} \left[\begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix} \right]$, where $U \in GL_g(\mathbb{Z}/p^\nu\mathbb{Z})$ since both sides of (3.3) are invariant under this substitution. For the proof we distinguish the cases $p \neq 2$ and $p = 2$ and use that a non-degenerate integral quadratic form over the ring of $p-$adic numbers $\mathbb{Z}_p$ is diagonisable over $\mathbb{Z}_p$ if $p \neq 2$ and is equivalent to a sum of forms $2^l \cdot \epsilon \cdot x^2$, $2^l \cdot xy$, $2^l \cdot (x^2 + xy + y^2)$, where $l \in \mathbb{Z}$, and $\epsilon \in \{1,3,5,7\}$ if $p = 2$ (cf. [Ca]). Then the claim can be shown using well-known formulas for Gauß sums (cf. [Bri]). $\square$

Theorem 3.4. *The Fourier series*

$$P_{k,m;(n,r),s}(\tau,z) := \sum_{\substack{n' \in \mathbb{Z} \\ r' \in \mathbb{Z}^{(1,g)}}} g^{\pm}_{k,m;(n,r);s,v}(n',r') e(n'\tau + r'z) \qquad \left((\tau,z) \in \mathbb{H} \times \mathbb{C}^{(g,1)}\right)$$

is absolutely and locally uniformly convergent in s and defines a holomorphic function in s for $\sigma > \frac{1}{2}(g/2+2-k)$. In particular the series $P_{k,m;(n,r),0}(\tau,z)$ is absolutely convergent if $k > g/2+2$. Let us define

$$\begin{aligned} g^{(1)}(n',r') &:= \textstyle\sum_{c \geq 1} H_{m,c}(n,r,n',r') \cdot \Phi_{k,m,c,v}(n',r',s) \cdot c^{-k-2s}, \\ g^{(1)}(n',r')^{\pm} &:= g^{(1)}(n',r') + (-1)^k g^{(1)}(n',-r'). \end{aligned}$$

Let $k = g+2$, $(\tau,z) \in \mathbb{F}$, where $\mathbb{F}$ is the standard fundamental domain for the action of the Jacobi group on $\mathbb{H} \times \mathbb{C}^{(g,1)}$ given by

$$\begin{aligned} \Big\{ (\tau,z) \in \mathbb{H} \times \mathbb{C}^{(g,1)} \,\Big|\, \tau \in \mathcal{F},\ 0 \leq x_\nu \leq 1\, (1 \leq \nu \leq g),\ 0 \leq (yv^{-1})_{\nu\mu} \leq 1 \\ (1 \leq \nu,\mu \leq g) \Big\} / \{(\tau,z) \mapsto (\tau,-z)\}, \end{aligned}$$

and suppose that s varies in a compact set K such that $0 < \sigma < 1$. Then we have the estimate

$$\left| \sum_{\substack{n' \in \mathbb{Z} \\ r' \in \mathbb{Z}^{(1,g)}}} g^{(1)}(n',r')^{\pm} e(n'\tau + r'z) \right| \ll_K v^{-g/2-1}.$$

Proof. For the proof we need the estimates from Lemma 3.2 and 3.3 and that $D' = \frac{1}{2}\det(2m)\cdot(-4n'+m^{-1}[r'^t])$ (which follows directly from the Jacobi decomposition of D'). Then we obtain the absolute and local uniform convergence in s by estimating the sum $g^{(1)}(n',r')^{\pm}$ against the product of a special value of a Jacobi theta series and a geometric sum. This establishes the holomorphicity in the variable s and the desired estimate. □

In the remainder we restrict ourselves to the case $k = g+2$.

Lemma 3.5. *The function $P_{g+2,m;(n,r)}(\tau,z) := P_{g+2,m;(n,r),0}(\tau,z)$ is an element of $J^{cusp}_{g+2,m}$.*

Proof. Since the transformation law is clear it is left to show that the Fourier coefficients of $P_{g+2,m;(n,r)}(\tau,z)$ are constant functions of $v = \mathrm{Im}(\tau)$. Checking the definitions, it is therefore enough to show that the integral

$$\int_{iv-\infty}^{iv+\infty} \tau^{-g/2-2}\cdot e\left((2\det(2m))^{-1}\left(D'\tau+\frac{D}{c^2\tau}\right)\right)d\tau \tag{3.4}$$

is independent of v. This is done by using Laplace transforms which lead to certain Bessel functions independent of v (cf. [Bri]). The vanishing of the Fourier coefficients for $D' \geq 0$ can be established by deforming the path of integration up to infinity. □

Theorem 3.6. *For $\phi \in J^{cusp}_{g+2,m}$ we have*

$$\left\langle \phi, P_{g+2,m;(n,r)}\right\rangle = \lambda_{g+2,m,D}\cdot c_\phi(n,r),$$

where $c_\phi(n,r)$ and $\lambda_{g+2,m,D}$ are defined as before.

Proof. First we have to show that the scalar products $\left\langle \phi, P_{g+2,m;(n,r),\sigma}\right\rangle$ are absolutely convergent. This can be done by using the usual unfolding argument and the boundness condition

$$|\phi(\tau,z)| \ll_\phi (\det v)^{-k/2}\cdot e^{2\pi \mathrm{tr}\,(mv^{-1}[y])} \tag{3.5}$$

(cf. [Zi]).
Next we can prove that

$$\left\langle \phi, P_{g+2,m;(n,r),\sigma}\right\rangle = \lambda_{g+2,m,D,\sigma}\cdot c_\phi(n,r), \tag{3.6}$$

where

$$\begin{aligned}\lambda_{g+2,m,D,\sigma} \;&:=\; 2^{(g-1)(g/2+\sigma+1)-g}\cdot\Gamma\left(g/2+\sigma+1\right)\cdot\pi^{-g/2-\sigma-1}\cdot(\det m)^{g/2+\sigma+1/2}\\ &\quad\cdot|D|^{-g/2-\sigma-1}.\end{aligned}$$

What is left to show is that we may interchange limit and integration, i.e.,

$$\lim_{\sigma\to 0}\left\langle \phi, P_{g+2,m;(n,r),\sigma}\right\rangle = \left\langle \phi, P_{g+2,m;(n,r)}\right\rangle. \tag{3.7}$$

For this we let $(\tau, z) \in \mathbb{F}$ and use Lebesgue's Theorem of bounded convergence with majorant $g(\tau, z) > 0$ on $\mathbb{H} \times \mathbb{C}^{(g,1)}$ chosen as

$$\sum_{\gamma \in \left(\Gamma^J_{1,g}\right)_\infty \backslash \Gamma^J_{1,g}} \frac{v}{|c\tau + d|^2} \cdot \left|e^{n,r}|_{g+2,m}\gamma(\tau, z)\right| + v^{-g/2-1} + \chi_{\{(\tau,z)\in\mathbb{F} \mid \frac{\sqrt{3}}{2} < v < 1\}} \cdot \sum_{\substack{\gamma \in \left(\Gamma^J_{1,g}\right)_\infty \backslash \Gamma^J_{1,g} \\ c=0}} \left|e^{n,r}|_{g+2,m}\gamma(\tau, z)\right|,$$

where χ_M denotes the characteristic function of a set $M \subset \mathbb{H} \times \mathbb{C}^{(g,1)}$, and $c = 0$ means that if $\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda a, \lambda b)\right) \in \Gamma^J_{1,g}$, then $c = 0$. It is easy to show that these three series converge. That the above function is indeed a majorant follows if we separate the series $P_{g+2,m;(n,r),\sigma}(\tau, z)$ into two parts according to $c = 0$ or $c \neq 0$ and use Theorem 3.4. In the following we abbreviate the terms of $g(\tau, z)$ with $g_1(\tau, z)$, $g_2(\tau, z)$ and $g_3(\tau, z)$ in an obvious sense. The convergence of

$$\int_{\mathbb{F}} g_j(\tau, z) \cdot |\phi(\tau, z)| \cdot e^{-4\pi m[y]v^{-1}}\, dvdudxdy \qquad (j \in \{1, 2\})$$

is already shown for $j = 1$ and follows easily for $j = 2$. To show the convergence of

$$\int_{\mathbb{F}} g_3(\tau, z) \cdot |\phi(\tau, z)| \cdot \exp(-4\pi m[y]v^{-1})\, dudvdxdy,$$

it is sufficient to regard one of the 3^g subseries of g_2 with $sign(r'_i)\,(1 \leq i \leq g)$ fixed. We define for the components $r'_i\,(1 \leq i \leq g)$ of r': $\epsilon(r'_i) := \begin{cases} 1 & r'_i < 0 \\ 0 & r'_i \geq 0 \end{cases}$, using a fixed but arbitrarily chosen subseries from among the 3^g subseries, which we denote by $\sum^{**}_{n',r'}$. Then we have, using (3.5),

$$\int_{\mathbb{F}} |\phi(\tau, z)| \cdot \sum_{n',r'}^{**} e^{-2\pi n'v - 2\pi r'y} \cdot e^{-4\pi m[y]v^{-1}}\, dudvdxdy$$

$$\ll \sum_{n',r'}^{**} e^{-2\pi n'\frac{\sqrt{3}}{2} - 2\pi\epsilon(r'_i)r'_i} \int_{\frac{\sqrt{3}}{2}}^{1} v^{-g/2-1}\, dv,$$

which is clearly finite since the integral is finite and $\sum^{**}_{n',r'} e^{-2\pi n'\frac{\sqrt{3}}{2} - 2\pi\epsilon(r'_i)r'_i}$ is a special value of a subseries of the convergent series $\sum^{**}_{n',r'} e^{2\pi i(n'\tau + r'z)}$. □

Theorem 3.7. *Suppose that $k \geq g + 2$. Let $\phi \in J^{cusp}_{k,m}$ with Fourier coefficients $c(n, r)$. Then we have*

$$c(n, r) \ll_{\epsilon,k} \left(1 + \frac{|D|^{g/2+\epsilon}}{(\det m)^{(g+1)/2}}\right)^{1/2} \cdot \frac{|D|^{k/2-g/4-1/2}}{(\det m)^{k/2-(g+3)/4}} \cdot \parallel \phi \parallel \qquad (\epsilon > 0).$$

Proof. Using the Cauchy-Schwarz inequality and the Petersson coefficient formula two times leads to

$$|c(n,r)|^2 \leq \lambda_{k,m,D}^{-1} \cdot b_{n,r}(P_{k,m;(n,r)}) \cdot \parallel \phi \parallel^2,$$

where $b_{n,r}(P_{k,m;(n,r)})$ denotes the (n,r)-th Fourier coefficient of the Poincaré series $P_{k,m;(n,r)}$. In order to prove Theorem 3.7, we therefore only need to estimate the Fourier coefficients of the Poincaré series. Since these are of the same type as in the case $k > g+2$, we can proceed as in [BK] . □

From Theorem 3.7, Theorem 1.1 now follows very similarly as in the case of the case $k > g+1$ (cf. [BK]) □

4. The subgroup $\Gamma_{g,0}(N)$

Since the estimates of the Fourier coefficients of Jacobi cusp forms (uniformly in $\det m$) are very similar as in the case of the full modular group we only give the results here and refer the reader to [Bri].

Lemma 4.1. *Suppose that $k \geq g+2$. Let ψ be a Jacobi cusp form with respect to $\Gamma_{1,g,0}(N)$ with Fourier coefficients $c(n,r)$. Then we have*

$$c(n,r) \ll_{\epsilon,k} \left(1 + \frac{|D|^{g/2+\epsilon}}{(\det m)^{(g+1)/2}}\right)^{1/2} \cdot \frac{|D|^{k/2-g/4-1/2}}{(\det m)^{k/2-(g+3)/4}} \cdot \parallel \phi \parallel \qquad (\epsilon > 0).$$

Lemma 4.2. *Let $k \geq 8$ be an even integer and let ϕ be a Jacobi cusp form with respect to $\Gamma_{2,1,0}(N)$ with Fourier coefficients $c(n,r)$. Then for a Minkowski-reduced matrix (for a definition cf. [Fr]) $\begin{pmatrix} m & \frac{r^t}{2} \\ \frac{r}{2} & n \end{pmatrix}$ we have*

$$c(n,r) \ll_{\epsilon,k} \left(1 + m^{-1/2+\epsilon} \cdot (\det \eta)^{1+\epsilon} + m^{-1/2+\epsilon} \cdot (min(\eta))^{-1} \cdot (\det \eta)^{3/2+\epsilon}\right) \cdot \parallel \phi \parallel,$$

where $\eta := n - \frac{rr^t}{4m}$.

We now want to restrict to the case where the Jacobi cusp forms come from a Fourier-Jacobi expansion. For this, we let $F \in S_k(\Gamma_{g,0}(N))$ and regard the following two Fourier-Jacobi expansions

$$F(Z) = \sum_{\tilde{m}>0} \psi_{\tilde{m}}(\tilde{\tau}, \tilde{z}) e^{2\pi i \mathrm{tr}\,(\tilde{m}\tilde{\tau}')} = \sum_{m \geq 1} \phi_m(\tau, z) e^{2\pi i m \tau'}. \tag{4.1}$$

Here the first sum extends over all positive definite symmetric half-integral $(g-1)\times(g-1)$ matrices, and $Z \in \mathbb{H}_g$ is written as $Z = \begin{pmatrix} \tilde{\tau} & \tilde{z} \\ \tilde{z}^t & \tilde{\tau}' \end{pmatrix}$, with $\tilde{\tau} \in \mathbb{H}$, $\tilde{z} \in \mathbb{C}^{(1,g-1)}$, and $\tilde{\tau}' \in \mathbb{H}_{g-1}$. The second sum extends over all positive integers, and $Z \in \mathbb{H}_g$ is written as $Z = \begin{pmatrix} \tau & z^t \\ z & \tau' \end{pmatrix}$, with $\tau \in \mathbb{H}_{g-1}$, $z \in \mathbb{C}^{(1,g-1)}$, and $\tau' \in \mathbb{H}$. Moreover $\psi_{\tilde{m}}$ and ϕ_m are Jacobi cusp forms with respect to $\Gamma^J_{1,g-1,0}(N)$ and $\Gamma^J_{g-1,1,0}(N)$, respectively. From the classical Hecke argument we obtain quite trivially:

Lemma 4.3. *We have the estimates*

$$\begin{aligned} \| \psi_{\tilde{m}} \| &\ll_F (\det \tilde{m})^{k/2} \\ \| \phi_m \| &\ll_F m^{k/2}. \end{aligned}$$

The aim of this section is to give the following improvement.

Lemma 4.4. *We have*

$$\| \phi_m \| \ll_{\epsilon,F} m^{k/2-g/(4g+1)+\epsilon} \qquad (\epsilon > 0),$$

where ϕ_m denotes the m-th Fourier-Jacobi coefficients of F.

For the proof we need some properties about certain non-holomorphic Eisenstein series.

Definition 4.5. *We formally define the following non-holomorphic Eisenstein series*

$$\begin{aligned} E_{s,N}(Z) &:= \sum_{M \in \mathcal{C}_N \backslash \Gamma_{g,0}(N)} \left(\frac{\det(\mathit{Im}(M<Z>))}{\det(\mathit{Im}(M<Z>)_1)} \right)^s, \\ E_s(Z) &:= E_{s,1}(Z), \end{aligned}$$

where $\mathcal{C}_N$ denotes the subgroup of $\Gamma_{g,0}(N)$ consisting of all those matrices with $(0, \dots 0, 1)$ as last row, and $Im(M < Z >)_1$ means the upper $(g-1) \times (g-1)$ block of the matrix $Im(M < Z >)$. Moreover let

$$E^*_{s,N}(Z) := \pi^{-s} \cdot \Gamma(s) \cdot \zeta_N(2s) \cdot E_{s,N}(Z),$$

where

$$\zeta_N(s) := \sum_{\substack{n \in \mathbb{N} \\ (n,N)=1}} n^{-s}.$$

It can be easily shown that the series $E_{s,N}(Z)$ is well-defined, converges absolutely and locally uniformly for $\mathrm{Re}(s) > g$, and is invariant under $\Gamma_{g,0}(N)$. Since for a primitive vector $(c, d) \in \mathbb{Z}^{(1,2g)}$ (i.e., $gcd(c, d) = 1$) with $c \equiv 0 \pmod N$ there exists a matrix $M \in \Gamma_{g,0}(N)$ with (c, d) as last row (cf. [Bri]), we have the identity

$$E_{s,N}(Z) = \sum_{\substack{(c,d) \in \mathbb{Z}^{(2g,1)} \\ (c,d)=1 \\ c \equiv 0 (N)}} \left(P_Z \left[\begin{pmatrix} c \\ d \end{pmatrix} \right] \right)^{-s} \qquad (\forall\, Z \in \mathbb{H}_g),$$

where $P_Z := \begin{pmatrix} Y & 0 \\ 0 & Y^{-1} \end{pmatrix} \left[\begin{pmatrix} I & 0 \\ X & I \end{pmatrix} \right] > 0.$

Lemma 4.6. *The function $E^*_{s,N}(Z)$ has a meromorphic continuation to the whole complex s-plane. If $N = 1$ it is holomorphic except for two simple poles at $s = 0$ and at $s = g$ with residues -1 and 1, respectively. In this case it satisfies the functional equation*

$$E^*_s(Z) = E^*_{g-s}(Z).$$

If $N \neq 1$ the only singularity is a simple pole at $s = g$ with residue

$$N^{-g} \cdot \sum_{l|N} \mu(l) l^{-g},$$

where $\mu(l)$ denotes the Möbius function. In this case it satisfies the functional equation

$$E^*_{g-s,N}(Z) = N^{2s-2g} \cdot \sum_{l_1|N} \mu\left(\frac{N}{l_1}\right) \cdot l_1^{g-2s} \cdot \sum_{l_2|l_1} l_2^{2s} \cdot E^*_{s,l_2}(Z).$$

Moreover we have the following identity between the Eisenstein series

$$E^*_{s,N}(Z) = \sum_{l|N} \mu(l) \cdot (Nl)^{-s} \cdot E^*_s\left(\frac{N}{l} Z\right).$$

Proof. Since the case $N = 1$ is treated in [Kr] we may assume $N > 1$.
Writing $\lambda = \begin{pmatrix} c \\ d \end{pmatrix}$, using the identity $P_Z[\lambda] = Y[c] + Y^{-1}[d + Xc] \quad \left(\forall\, c, d \in \mathbb{Z}^{(g,1)}\right)$, and the well-known property of the Möbius function $\sum_{l|n} \mu(l) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$, we deduce

$$\begin{aligned}
\zeta_N(2s) E_{s,N}(Z) &= \sum_{\substack{(c,d) \in \mathbb{Z}^{(g,1)} \times \mathbb{Z}^{(g,1)} \\ (N,d)=1}} \left(Y[Nc] + Y^{-1}[NXc + d]\right)^{-s} \\
&= \sideset{}{'}\sum_{(c,d) \in \mathbb{Z}^{(g,1)} \times \mathbb{Z}^{(g,1)}} \left(Y[Nc] + Y^{-1}[NXc + d]\right)^{-s} \sum_{l|(N,d)} \mu(l) \\
&= \sum_{l|N} \mu(l) \sideset{}{'}\sum_{(c,d) \in \mathbb{Z}^{(g,1)} \times \mathbb{Z}^{(g,1)}} \left(Y[Nc] + Y^{-1}[NXc + ld]\right)^{-s} \\
&= \sum_{l|N} (Nl)^{-s} \cdot \mu(l) \sideset{}{'}\sum_{(c,d) \in \mathbb{Z}^{(g,1)} \times \mathbb{Z}^{(g,1)}} \left(\left(\frac{N}{l} Y\right)[c] + \left(\frac{N}{l} Y\right)^{-1} \left[\left(\frac{N}{l} X\right) c + d\right]\right)^{-s} \\
&= \zeta(2s) \cdot \sum_{l|N} \mu(l) \cdot (Nl)^{-s} \cdot E_s\left(\frac{N}{l} Z\right).
\end{aligned}$$

Here $\sum'$ means as usual that the vector 0 is omitted in the summation.
Hence

$$E^*_{s,N}(Z) = \sum_{l|N} \mu(l) \cdot (Nl)^{-s} \cdot E^*_s\left(\frac{N}{l} Z\right) = N^{-2s} \cdot \sum_{l|N} \mu\left(\frac{N}{l}\right) \cdot l^s \cdot E^*_s(lZ). \tag{4.2}$$

Applying the Möbius inversion formula to (4.2), we get

$$E_s^*(NZ) = N^{-s} \cdot \sum_{l|N} l^{2s} \cdot E_{s,l}^*(Z). \tag{4.3}$$

Thus the claim about the poles, residues and functional equation of $E_{s,N}^*(Z)$ follows easily from the poles and residues of the function $E_s^*(Z)$. □

In order to prove a later claim on the meromorphicity of $\langle FE_{s,N}, G\rangle$ as a function of s, where $F, G \in S_k(\Gamma_{g,0}(N))$, we need some knowledge about the growth behaviour of the Eisenstein series $E_{s,N}$ for $\mathrm{tr}(Y) \to \infty$.

Lemma 4.7. *Fix $s \in \mathbb{C}$. Then for every $Z \in \mathbb{F}_g$, $\gamma \in \Gamma_g$, there exists a real constant α such that*

$$E_{s,N}^*(\gamma \circ Z) - \sum_{l|N} \mu(l) \cdot (Nl)^{-s} \cdot \left(\frac{1}{s} + \frac{1}{g-s}\right) \ll_{g,N,\gamma} (tr\, Y)^{\alpha},$$

where $\mathbb{F}_g$ denotes the standard fundamental domain for the action of Γ_g on $\mathbb{H}_g$, and where the constant implied in $\ll_{g,N,\gamma}$ only depends on g, N, and γ.

Proof. For the proof we first show an integral representation

$$E_s^*(Z) - \left(\frac{1}{s} + \frac{1}{g-s}\right) = \int_1^{\infty} \sum_{\lambda \in \mathbb{Z}^{(2g,1)}}{}' e^{-\pi t P_Z[\lambda]} \cdot (t^s + t^{g-s}) \frac{dt}{t} \qquad (\forall\, Z \in \mathbb{H}_g).$$

Then we write γ as a product of the generators $\pm \begin{pmatrix} E & S \\ 0 & E \end{pmatrix}$ $(S = S^t)$ and $\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$ of Γ_g and make an induction of the minimal number of these matrices that are needed. For details we refer the reader to [Bri]. □

Definition 4.8. *Let us now formally define the Dirchlet series*

$$\begin{aligned} D_{F,G,N}(s) &:= \tfrac{k_N}{i_N} \cdot \zeta_N(2s-2k+2g) \cdot \textstyle\sum_{m \geq 1} \langle \phi_m, \psi_m \rangle\, m^{-s}, \\ D_{F,G,N}^*(s) &:= (4\pi)^{-s} \cdot \Gamma(s) \cdot \Gamma(s-k+g) \cdot D_{F,G,N}(s), \end{aligned}$$

where $F, G \in S_k(\Gamma_{g,0}(N))$, ϕ_m and ψ_m are the m-th Fourier-Jacobi coefficients of F and G, respectively, $i_N := [\Gamma_g : \Gamma_{g,0}(N)]$, and $k_N := [\Gamma_1 : \Gamma_0(N)]$. Moreover we define for a positive divisor l of N the Dirichlet series

$$D_{F,G,N,l}(s) := \zeta_l(2s-2k+2g) \cdot \sum_{m \geq 1} b_l(m) m^{-s},$$

where

$$b_l(m) := \frac{1}{i_l} \cdot \sum_j \int_{\mathbb{F}_l} \phi_{m,\gamma_j}(\tau, z) \cdot \overline{\psi_{m,\gamma_j}(\tau, z)} \cdot (\det v)^{k-(g+1)} \cdot e^{-4\pi m v^{-1}[y]}\, dudvdxdy,$$

and where

$$D_{F,G,N,l}^*(s) := (4\pi)^{-s} \cdot \Gamma(s) \cdot \Gamma(s-k+g) \cdot D_{F,G,N,l}(s).$$

Here $\mathbb{F}_l$ *is a fixed fundamental domain of the action of* $\Gamma^J_{g-1,1}(l)$ *on* $\mathbb{H}_{g-1}\times\ \mathbb{C}^{(1,g-1)}$. *Moreover* $\phi_{m,\gamma_j}(\tau,z)$ *and* $\psi_{m,\gamma_j}(\tau,z)$ *denote the Fourier-Jacobi coefficient of* $F|\gamma_j$ *and* $G|\gamma_j$*, respectively, where* γ_j *runs through a set of representatives of* $\Gamma_{g,0}(N)\backslash\Gamma_{g,0}(l)$.

We can easily show that

$$\left.\begin{array}{l}\langle\phi_m,\psi_m\rangle\\ b_l(m)\end{array}\right\}\ll_{F,G} m^k.$$

Thus the series $D_{F,G,N}(s)$ and $D_{F,G,N,l}(s)$ are absolutely and locally uniformly convergent for $\mathrm{Re}(s)>k+1$ and therefore holomorphic.

Theorem 4.9. *The functions* $D_{F,G,N}(s)$ *and* $D_{F,G,N,l}(s)$ *have meromorphic continuations to the whole complex plane with only finitely many poles. The function* $D_{F,G,N}(s)$ *is entire if* $\langle F,G\rangle=0$*, and otherwise at* $s=k$ *has a simple pole of residue*

$$\frac{(4\pi)^k}{(k-1)!(g-1!)}\cdot\pi^{g-k}\cdot\langle F,G\rangle\cdot N^{-g}\cdot\sum_{l|N}\mu(l)\cdot l^{-g}.$$

It satisfies the functional equation

$$D^*_{F,G,N}(2k-g-s)=N^{2s-2k}\cdot\sum_{l_1|N}\mu\left(\frac{N}{l_1}\right)\cdot l_1^{-2s+2k-g}\sum_{l_2|l_1}l_2^{2(s-k+g)}\cdot D^*_{F,G,N,l_2}(s).$$

Proof. Since the proof for $N=1$ is given in [Kr] we may assume $N>1$. Hence we can show quite similary as in the case $N=1$ that

$$\pi^{g-k}\cdot\left\langle E^*_{s-k+g,N}F,G\right\rangle=D^*_{F,G,N}(s). \tag{4.4}$$

Since $F,G\in S_k(\Gamma_{g,0}(N))$ we have $F(Z)\,,G(Z)\ll_{F,G} e^{-a\sum_{i=1}^g y_i}$, where a is a positive constant. Thus we find, by using Lemma 4.7, that $D^*_{F,G,N}(s)$ has a meromorphic continuation to $\mathbb{C}$ having at most simple poles at $s=k$ and $s=k-g$ and we have also a justification for the computatations made in (4.4). Now the claim about the poles and residues of the Dirichlet series follows easily from the poles and residues of the Eisenstein series and the Gamma function.

To prove the functional equation of $D^*_{F,G,N}(s)$ we use Lemma 4.6 (with $g-k+s$ instead of s) to find

$$\begin{aligned}D^*_{F,G,N}(2k-g-s)&=\pi^{g-k}\cdot\left\langle E^*_{g-(g-k+s),N}F,G\right\rangle\\ &=N^{2s-2k}\cdot\sum_{l_1|N}\mu\left(\frac{N}{l_1}\right)\cdot l_1^{-2s+2k-g}\sum_{l_2|l_1}l_2^{2(s-k+g)}\cdot\pi^{g-k}\cdot\left\langle\iota\left(E^*_{s-k+g,l_2}\right)F,G\right\rangle,\end{aligned}$$

where ι denotes the inclusion map. Since $\iota^*(\bar{F}G)=\sum_j(\bar{F}G)|\gamma_j$, where ι^* denotes the adjoint of the map ι, and where γ_j runs through a set of representatives of $\Gamma_{g,0}(N)\backslash\Gamma_{g,0}(l)$, we find using the usual unfolding argument

$$\langle \iota(E_{s,l})F, G\rangle = i_l^{-1} \cdot \sum_j \int_{\mathcal{C}_l\backslash\mathbb{H}_g} F|_k\gamma_j(Z) \cdot \overline{G|_k\gamma_j(Z)} \cdot (\det v)^{-s} \cdot (\det Y)^{k-(g+1)+s}\, dXdY.$$

As a fundamental domain for the action of $\mathcal{C}_l$ on $\mathbb{H}_g$, we may choose

$$\left\{ \left(\begin{array}{cc} \tau & z^t \\ z & \tau' \end{array} \right) \middle| \, (\tau, z) \in \mathcal{F}_l,\, v' > v^{-1}[y],\, 0 \le u' \le 1, \right\},$$

where we have written $\tau = u + iv$, $z = x + iy$, and $\tau' = u' + iv'$, respectively. Inserting the Fourier-Jacobi expansions of $F|_k\gamma_j(z)$ and $G|_k\gamma_j(z)$, we find for $\mathrm{Re}(s) > g+1$, after a straightforward computation,

$$\frac{1}{i_l} \cdot (4\pi)^{-(s+k-g)} \cdot \Gamma(s+k-g) \cdot \sum_{m\ge 1} \left(\sum_j \int_{\mathcal{F}_l} \phi_{m,\gamma_j}(\tau, z) \cdot \overline{\psi_{m,\gamma_j}(\tau, z)} \cdot (\det v)^{k-(g+1)} \times e^{-4\pi m v^{-1}[y^t]}\, dudvdxdy \right) m^{-(s+k-g)},$$

where $\phi_{m,\gamma_j}(\tau, z)$ and $\psi_{m,\gamma_j}(\tau, z)$ are the Fourier Jacobi coefficients of F and G, respectively. Thus we have the identity

$$D^*_{F,G,N,l}(s) = \pi^{g-k} \cdot \left\langle \iota(E^*_{s-k+g,l})F, G \right\rangle.$$

Now we can show with the same arguments as before that $D^*_{F,G,N,l}(s)$ has a meromorphic continuation to $\mathbb{C}$ with a possible simple pole at $s = k$. Moreover the functional equation follows directly. □

To prove Theorem 4.4 we only need the case $F = G$. Clearly $D_{F,F}(s)$ and $D_{F,F,N,l}(s)$ have non-negative coefficients. Thus a classical Theorem of Landau says that they must have the first real singularity at their abscissa of convergence. Thus they converge for $\mathrm{Re}(s) > k$. We now need the following modified version of Landau's Hauptsatz that follows quite easily fom [SS].

Lemma 4.10. *Suppose* $Z(s) = \sum_{n\ge1} c(n)n^{-s}$ *and* $\eta_i(s) = \sum_{n\ge1} b_i(n)n^{-s}$, $1 \le i \le l$ *(*$l \in \mathbb{N}$*) are Dirichlet series with non-negative coefficients which converge for Re*$(s) > \sigma_0$*, have a meromorphic continuation to* $\mathbb{C}$ *with finitely many poles and satisfy a functional equation*

$$Z^*(\delta - s) = \sum_{i=1}^{l} \pm \eta_i^*(s),$$

where

$$\begin{aligned} Z^*(s) &= A^{-s} \cdot \textstyle\prod_{j=1}^J \Gamma(a_j s + b_j) \cdot Z(s), \\ \eta_i^*(s) &= A_i^{-s} \cdot \textstyle\prod_{j=1}^J \Gamma(a_j s + b_j) \cdot \eta_i(s), \end{aligned}$$

where A , $A_i > 0$, $J \in \mathbb{N}$, $a_j > 0$, $b_j \in \mathbb{R}$. *Suppose that*

$$\kappa := (2\sigma_0 - \delta) \cdot \sum_{j=1}^{J} a_j - \frac{1}{2} > 0.$$

Then

$$\sum_{n \le x} c(n) = \sum_{all\ poles} Res\left(\frac{\zeta(s)}{s} x^s\right) + O_\eta(x^\eta),$$

for any $\eta > \eta_0 := (\delta + \sigma_0(\kappa - 1))/(\kappa + 1)$.

We want to use Lemma 4.10 with $Z(s) = D_{F,F,N}(s)$, $\eta_i(s) = D_{F,F,N,p(i)}(s) \cdot N^{-2k} \cdot (p(i))^{2k-g} \cdot (q(i))^{-2k+2g}$, where $(p(i), q(i))$ runs through the elements (l_1, l_2) with $l_1 | N$, $l_2 | l_1$ and $\mu\left(\frac{N}{l_1}\right) \neq 0$. Moreover $\sigma_0 = k$, $\delta = 2k - g$, $A = 4\pi$, $A_i = 4\pi (p(i))^2 (q(i))^{-2} N^{-2}$, $J = 2$, $a_1 = a_2 = 1$, $b_1 = 0$, $b_2 = g - k$. Therefore we have

$$\sum_{n \le x} c(n) = cx^k + O_\eta\left(x^{k - \frac{2g}{4g+1} + \epsilon}\right) \qquad (\forall \epsilon > 0),$$

where $c = Res_{s=k} \frac{D_{F,F}(s)}{s}$. Thus we get, taking $x = m$ and $x = m - 1$ and substracting, that

$$c(m) \ll_{\epsilon,F} m^{k - \frac{2g}{4g+1} + \epsilon}.$$

Writing $\zeta_N(s)^{-1} := \sum_{n \ge 1} \mu_N(n) n^{-s}$ and using that the coefficients of $\zeta_N(s)^{-1}$ are bounded by 1, we find

$$\begin{aligned} \| \phi_m \|^2 &= \sum_{d^2 | m} \mu_N(d) \cdot d^{2k-2g} \cdot c\left(\frac{m}{d^2}\right) \ll_{\epsilon,F} m^{k - \frac{2g}{4g+1} + \epsilon} \cdot \sum_{d \ge 1} d^{-2g(1 - \frac{2}{4g+1})} \\ &\ll_{\epsilon,F} m^{k - \frac{2g}{4g+1} + \epsilon}. \end{aligned}$$

□

To prove Theorem 1.2 and Corollary 1.3 we write T as $T = \begin{pmatrix} n & \frac{r}{2} \\ \frac{r^t}{2} & m \end{pmatrix}$ in the same way as before. Then $a(T)$ is the (n, r)-th Fourier coefficient of the m-th Fourier-Jacobi coefficient of F. Thus we can use Lemma 4.1 (with $g - 1$ instead of g) and Lemma 4.3. Thererfore the claim follows if we assume that $\det m = m_{g-1}(T)$ (since we else can replace T by $T[U]$, with $U \in GL_g(\mathbb{Z})$ such that $T[U]|_{g-1} = m_{g-1}(T)$, which changes neither the left- nor the right-hand side of the estimate in Theorem 1.2 and Lemma 1) and use the estimate $m_{g-1}(T) \ll_g (\det T)^{1-1/g}$, which follows directly from reduction theory.
The improvements for $g = 2$ and $g = 3$ are obtained with the same arguments. Moreover Theorem 1.8 is gained similarly as in the case of the full Siegel modular group. □

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