A CONVERSE THEOREM FOR HILBERT-JACOBI FORMS

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1. Introduction and Statement of Results

Doi and Naganuma (see [6]) constructed a lifting map from elliptic modular forms to Hilbert modular forms in the case of a real quadratic field with narrow class number one. A Converse Theorem for Hilbert modular forms was one of their basic tools. This gives rise to the question of constructing a lifting map in the case of Jacobi forms. Here we do the first step in this direction and prove a Converse Theorem for Hilbert-Jacobi forms.

Studying the connection between functions that satisfy certain transformation laws and the functional equation of their associated L-functions has value on its own and a long history. In a celebrated paper (see [9]), Hecke showed that the automorphy of a cusp form with respect to $\mathrm{SL}_2(\mathbb{Z})$ is equivalent to the functional equation of its associated L-functions. That only one functional equation is needed is in a way atypical and highly depends on the fact that $\mathrm{SL}_2(\mathbb{Z})$ is generated by the matrices $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. This situation already changes if one considers cusp forms with respect to a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ which have a character. In this case the functional equation of twists is required (see [18]).

Hecke's work has inspired an astonishing number of people and a lot of generalizations of his "Converse Theorem" have been made, e.g. generalizations to Hilbert modular forms as mentioned above (see [6]), Siegel modular forms (see [1], [10]) or Jacobi forms (see [14],[15]). Maass showed an analogue of Hecke's result for nonholomorphic modular forms (see [13]). He proved that these correspond to certain L-functions in quadratic fields. An outstanding generalization of a Converse Theorem for $\mathrm{GL}(n)$ was done by Jacquet and Langlands for $n = 2$ (see [11]), Jacquet, Piatetski-Shapiro, and Shalika for $n = 3$ (see [12]) and Cogdell and Piatetski-Shapiro for general n (see [5]).

In this paper, we prove a Converse Theorem for Hilbert-Jacobi cusp forms over a totally real number field K of degree $g := [K : \mathbb{Q}]$ with discriminant D_K and narrow class number 1. The case $g = 1$, i.e., Jacobi forms over $\mathbb{Q}$ as considered by Eichler and Zagier (see [7]), is treated in two interesting papers by Martin (see [14] and [15]). To describe our result, we consider functions $\phi(\tau, z)$ from $\mathbb{H}^g \times \mathbb{C}^g$ into $\mathbb{C}$ that have a Fourier expansion with certain conditions on the Fourier coefficients (see (3.4),(3.5), and (3.6)). We show that ϕ is a Hilbert-Jacobi cusp form (for the definition see Section 2) if

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and only if certain Dirichlet series $\mathcal{L}(s,\phi,r,\chi_{m,\nu})$ (see (3.9)) satisfy functional equations. More precisely, we show the following.

Theorem 1.1. *Let k be an integer and $m \in \mathfrak{d}_K^{-1}$, the inverse different of K. A function ϕ satisfying (3.3), (3.4), (3.5), and (3.6) is a Hilbert-Jacobi cusp form of weight k and index m if and only if for all ν satisfying (3.1) and (3.2) and for all $r \in \mathfrak{d}_K^{-1}/(2m\mathcal{O}_K)$ the functions $\mathcal{L}(s,\phi,r,\chi_{m,\nu})$ (see Definition 3.2) have analytic continuations to the whole complex plane, are bounded in every vertical strip and satisfy the functional equations*

$$\mathcal{L}(s,\phi,r,\chi_{m,\nu}) = \frac{1}{\sqrt{D_K}}\, i^{-kg}\, \mathbb{N}(2m)^{-1/2} \sum_{\mu \in (\mathfrak{d}_K^{-1}/2m\mathcal{O}_K)} e_{2m}(-\mu r)\mathcal{L}(k-s-1/2,\phi,\mu,\chi_{m,-\nu}),$$

see Section 2 for the definition of $\mathbb{N}$ and $e_{2m}(\cdot)$.

We proceed as follows: In Section 2 we recall basic facts about Hilbert-Jacobi cusps forms. In particular we show that these have a theta decomposition (see (2.3)), where the involved theta series satisfy some transformation law (see Lemma 2.1). Section 3 deals with certain characters of Hecke type and the Dirichlet series needed for the Converse Theorem. In Section 4, we prove Theorem 1.1.

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2. Basic Facts about Hilbert-Jacobi Cusp Forms

We let K be a totally real number field of degree $g := [K:\mathbb{Q}]$ and denote by $\mathcal{O}_K$, $\mathcal{O}_K^\times$, $\mathfrak{d}_K$, and D_K its ring of integers, units, different, and discriminant, respectively. We denote the j-th embedding $(1 \le j \le g)$ of an element $l \in K$ by $l^{(j)}$. An element $l \in K$ is said to be totally positive ($l \gg 0$) if all its embeddings into $\mathbb{R}$ are positive.

Let us now briefly recall some basic facts about Hilbert-Jacobi cusp forms (see also [16]). We put $\Gamma_K := \mathrm{SL}_2(\mathcal{O}_K)$. Let the Hilbert-Jacobi group be defined as the set $\Gamma_K^J := \Gamma_K \ltimes (\mathcal{O}_K \times \mathcal{O}_K)$, with the group multiplication

$$\gamma_1 \cdot \gamma_2 := \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, (\lambda_1,\mu_1) \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} + (\lambda_2,\mu_2) \right),$$

where we put $\gamma_i := \left(\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, (\lambda_i,\mu_i) \right) \in \Gamma_K^J$, $\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \Gamma_K$, and $(\lambda_i,\mu_i) \in \mathcal{O}_K \times \mathcal{O}_K$. The Hilbert-Jacobi group is generated by the following three types of elements

$$\left(\begin{pmatrix} \epsilon & \lambda \\ 0 & \epsilon^{-1} \end{pmatrix}, (0,0) \right), \quad \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, (0,0) \right), \text{ and } \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, (\lambda,\mu) \right), \tag{2.1}$$

where $\lambda, \mu \in \mathcal{O}_K$ and $\epsilon \in \mathcal{O}_K^\times$ (see [2], [4] and [17]).

The Hilbert-Jacobi group acts on $\mathbb{H}^g \times \mathbb{C}^g$ ($\mathbb{H}$ is the usual upper half-plane) by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu)\right) \circ (\tau, z)$$
$$:= \left(\left(\frac{a^{(1)}\tau_1 + b^{(1)}}{c^{(1)}\tau_1 + d^{(1)}}, \cdots, \frac{a^{(g)}\tau_g + b^{(g)}}{c^{(g)}\tau_g + d^{(g)}}\right), \left(\frac{z_1 + \lambda^{(1)}\tau_1 + \mu^{(1)}}{c^{(1)}\tau_1 + d^{(1)}}, \ldots, \frac{z_g + \lambda^{(g)}\tau_g + \mu^{(g)}}{c^{(g)}\tau_g + d^{(g)}}\right)\right),$$

where $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu)\right) \in \Gamma_K^J$, $\tau = (\tau_1, \cdots, \tau_g) \in \mathbb{H}^g$ and $z = (z_1, \cdots, z_g) \in \mathbb{C}^g$. Throughout this paper we write $\tau = u + iv$, $z = x + iy$, $\tau_j = u_j + iv_j$, and $z_j = x_j + iy_j$ $(1 \le j \le g)$.

Let $k \in \mathbb{N}$, $m \in \mathfrak{d}_K^{-1}$ totally positive, $\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu)\right) \in \Gamma_K^J$, and a function $\phi : \mathbb{H}^g \times \mathbb{C}^g \to \mathbb{C}$. Then we define

$$\phi|_{k,m}\gamma(\tau, z) := \mathbb{N}(c\tau + d)^{-k} \cdot e\left(-\left(\frac{cm(z + \lambda\tau + \mu)^2}{c\tau + d} + m\tau\lambda^2 + 2m\lambda z\right)\right) \cdot \phi(\gamma \circ (\tau, z)),$$

where for $\alpha \in K$ and for $z \in \mathbb{C}^g$, we define $\mathbb{N}(\alpha z) := \prod_{j=1}^g \left(\alpha^{(j)} z_j\right)$, $\mathrm{tr}(az) := \sum_{j=1}^g a^{(j)} z_j$, and $e(\alpha z) := e^{2\pi i \mathrm{tr}(\alpha z)}$.

A holomorphic function $\phi : \mathbb{H}^g \times \mathbb{C}^g \to \mathbb{C}$ is called a *Hilbert-Jacobi cusp form* of weight k and index m if $\phi|_{k,m}\gamma(\tau, z) = \phi(\tau, z)$ for all $\gamma \in \Gamma_K^J$, and if it has a Fourier expansion of the form $\sum_{\substack{n,r \in \mathfrak{d}_K^{-1} \\ 4nm - r^2 > 0}} c(n, r)\, e\,(n\tau + rz)$.

In [16] m is chosen to be in $\mathcal{O}_K$, but our choice $m \in \mathfrak{d}_K^{-1}$ seems more natural since in this way the coefficients of Hilbert-Siegel modular forms are examples for Jacobi forms as in the classical case.

If ϕ is a Hilbert-Jacobi cusp form, then the transformation $(\tau, z) \to (\tau, z + \lambda\tau + \mu)$ leads to

$$c(n, r) \;=\; c(n + \lambda r + \lambda^2 m, r + 2\lambda m) \qquad (\forall \lambda \in \mathcal{O}_K). \tag{2.2}$$

From this we can deduce that

$$\phi(\tau, z) \;=\; \sum_{r \in (\mathfrak{d}_K^{-1}/2m\mathcal{O}_K)} f_r(\tau)\, \vartheta_{m,r}(\tau, z)\ , \tag{2.3}$$

where for $r \in (\mathfrak{d}_K^{-1}/2m\mathcal{O}_K)$, we define

$$f_r(\tau) \;:=\; \sum_{\substack{n \in \mathfrak{d}_K^{-1} \\ 4nm - r^2 > 0}} c(n, r)\, e_{4m}\left(\left(4nm - r^2\right)\tau\right)\ , \tag{2.4}$$

$$\vartheta_{m,r}(\tau, z) \;:=\; \sum_{\lambda \in \mathcal{O}_K} e_{4m}\left((r + 2\lambda m)^2\,\tau + 2m\,(r + 2\lambda m)\,z\right), \tag{2.5}$$

and where for $\alpha, \beta \in K$, $\beta \neq 0$, and $z \in \mathbb{C}^g$, we define $e_\beta(\alpha z) := e(\beta^{-1}\alpha z)$.

The theta series $\vartheta_{m,r}$ satisfy the following transformation law.

Lemma 2.1. *If $m \in \mathfrak{d}_K^{-1}$ totally positive, and $\mu \in (\mathfrak{d}_K^{-1}/2m\mathcal{O}_K)$, then we have*

$$\vartheta_{m,\mu}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \frac{1}{\sqrt{D_K}} \mathbb{N}\left((\tau/i)^{1/2}\right) \cdot \mathbb{N}\left(2m\right)^{-1/2} \cdot e\left(\frac{m\cdot z^2}{\tau}\right) \sum_{r \in \left(\mathfrak{d}_K^{-1}/2m\mathcal{O}_K\right)} e_{2m}(-\mu r)\vartheta_{m,r}(\tau, z),$$

where we put $(\tau/i)^{1/2} := ((\tau_1/i)^{1/2}, ..., (\tau_g/i)^{1/2})$, and we take the principal value of the square root, namely $-\pi/2 < \arg(w) \leq \pi/2$ for $w \in \mathbb{C}$.

From Lemma 2.1 we obtain

Corollary 2.2. *A function $\phi : \mathbb{H}^g \times \mathbb{C}^g$ having a decomposition of the form (2.3) satisfies*

$$\phi\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \mathbb{N}(\tau)^k e\left(\frac{mz^2}{\tau}\right)\phi(\tau, z)$$

if and only if

$$f_r(\tau) = \frac{1}{\sqrt{D_K}} i^{-kg}\, \mathbb{N}\left((\tau/i)^{1/2-k}\right)\mathbb{N}(2m)^{-1/2} \sum_{\mu \in (\mathfrak{d}_K^{-1}/2m\mathcal{O}_K)} e_{2m}(-\mu r) f_\mu\left(-\frac{1}{\tau}\right), \tag{2.6}$$

for all $r \in (\mathfrak{d}_K^{-1}/2m\mathcal{O}_K)$. In particular, if ϕ is a Hilbert-Jacobi cusp form, then ϕ satisfies (2.6).

Exactly as in the case of elliptic modular forms, one can show;

Lemma 2.3. *Assume that ϕ is a Hilbert-Jacobi cusp form, with f_r defined as in (2.4). Let c_1 be a positive real number and let S be the subset of $\mathbb{H}^g$ such that for all $\tau \in S$ the components v_j $(1 \leq j \leq g)$ are larger than c_1. Then we have*

$$|f_r(\tau)| \ll_{\phi, c_1} e^{-c_2\left(\sum_{j=1}^g v_j\right)}, \tag{2.7}$$

where c_2 is a positive constant, and where the constant implied in $\ll_{\phi,c_1}$ depends on ϕ and on c_1.

Lemma 2.4. *If ϕ is a Hilbert-Jacobi cusp form of weight k and index m, then the function*

$$g(\tau, z) := \mathbb{N}(v)^{k/2} \exp\left(-2\pi\, tr\left(\frac{my^2}{v}\right)\right)\phi(\tau, z)$$

is bounded on $\mathbb{H}^g \times \mathbb{C}^g$.

By using Lemma 2.4, we have the following.

Lemma 2.5. *If ϕ is a Hilbert-Jacobi cusp form of weight k and index m with Fourier coefficients $c(n, r)$, then $|c(n, r)| \ll_\phi \mathbb{N}(4mn - r^2)^{k/2}$.*

3. Hecke-type characters and Dirichlet series

For the remaining we assume that k is an integer. For $m \in \mathfrak{d}_K^{-1}$, we let T_m be the subgroup of $\mathcal{O}_K^\times$ defined by

$$T_m := \left\{ \epsilon \in \mathcal{O}_K^\times \mid \epsilon - 1 \in 2m\mathfrak{d}_K \right\} .$$

We have that $\epsilon \in \mathcal{O}_K^\times$ is in T_m if and only if $\epsilon r - r \in 2m\mathcal{O}_K$ for every $r \in \mathfrak{d}_K^{-1}/(2m\mathcal{O}_K)$.

We let $u_1, \ldots, u_{g-1}$ be a basis of T_m^2, where $T_m^2 := \{\epsilon^2 \mid \epsilon \in T_m\}$. We take $\epsilon_1, \ldots, \epsilon_{g-1} \in T_m$ which satisfy $\epsilon_l^2 = u_l$ for $l = 1, ..., g-1$. If m is not a generator of the inverse different, then T_m does not contain -1, hence the ϵ_l are uniquely determined. If m is a generator of the inverse different, then T_m contains -1, and we choose $\epsilon_l > 0$ as a solution of the above equation.

For integers N_l $(1 \leq l \leq g-1)$, we choose pure imaginary solutions $\nu_1, \ldots, \nu_g$ which satisfy the following equations

$$\sum_{j=1}^{g} \nu_j = 0, \tag{3.1}$$

$$\sum_{j=1}^{g} \nu_j \log\left(u_l^{(j)}\right) = 2\pi i \left(N_l + \frac{1}{2}\delta_l\right), \tag{3.2}$$

where we put $\delta_l = 0$ or 1 if $\mathbb{N}(\epsilon_l)^k = 1$ or -1, respectively. For any integers N_l $(l = 1, .., g-1)$ we have a solution to (3.1) and (3.2), because

$$\det \begin{pmatrix} 1 & ... & 1 \\ \log(u_1^{(1)}) & ... & \log(u_1^{(g)}) \\ \vdots & ... & \vdots \\ \log(u_{g-1}^{(1)}) & ... & \log(u_{g-1}^{(g)}) \end{pmatrix} = (-1)^{g+1} g \cdot \det((\log(u_l^{(j)}))_{l,j=1,\ldots,g-1}) \neq 0,$$

where the last non-equality can be obtained from the fact that basis elements u_l are multiplicatively independent.

For $x \in K$ and $\nu := (\nu_1, \ldots, \nu_g)$ satisfying (3.1) and (3.2), we set

$$\chi_{m,\nu}(x) := \prod_{j=1}^{g} \left|x^{(j)}\right|^{\nu_j} .$$

To define the Dirichlet series needed, we consider functions $\phi(\tau, z)$ from $\mathbb{H}^g \times \mathbb{C}^g$ into $\mathbb{C}$ that have a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{\substack{n,r \in \mathfrak{d}_K^{-1} \\ 4nm - r^2 > 0}} c(n, r) e(n\tau + rz) \tag{3.3}$$

that is absolutely and locally uniformly convergent. We regard $c(n,r)=0$ unless $4nm-r^2>0$ or unless $n,r\in\mathfrak{d}_K^{-1}$. Moreover we demand that its Fourier coefficients satisfy

$$c(n,r) = c(n+\lambda r+\lambda^2 m, r+2\lambda m) \qquad (\forall\lambda\in\mathcal{O}_K), \tag{3.4}$$

$$c(\epsilon^2 n,\epsilon r) = \mathbb{N}(\epsilon)^k c(n,r) \qquad (\forall\epsilon\in\mathcal{O}_K^\times), \tag{3.5}$$

$$c(n,r) \ll_\phi \mathbb{N}(4nm-r^2)^M \tag{3.6}$$

for an integer M.

Lemma 3.1. (1) *Condition (3.4) implies that we can decompose $\phi(\tau,z)$ as in (2.3).*

(2) *Conditions (3.4) and (3.5) imply by the definition of T_m that*

$$c_r(N) := c\left(\frac{N+r^2}{4m}, r\right) \qquad \left(N\in\mathfrak{d}_K^{-2}\right)$$

is well defined on $r\in\mathfrak{d}_K^{-1}/(2m\mathcal{O}_K)$, where we put $\mathfrak{d}_K^{-2}:=\mathfrak{d}_K^{-1}\cdot\mathfrak{d}_K^{-1}$.

(3) *ϕ is a Hilbert-Jacobi cusp form if and only if (3.3), (3.4), (3.5), and (3.6) hold, and if ϕ satisfies the transformation law*

$$\phi\left(-\frac{1}{\tau},\frac{z}{\tau}\right) = \mathbb{N}(\tau)^k e\left(\frac{mz^2}{\tau}\right)\phi(\tau,z). \tag{3.7}$$

(4) *From Corollary 2.2, we see that a function ϕ satisfying (3.3), (3.4), (3.5), and (3.6) is a Hilbert-Jacobi cusp form if and only if (2.6) is satisfied for all $r\in(\mathfrak{d}_K^{-1}/2m\mathcal{O}_K)$.*

Proof. These are straightforward. We omitted the proof of this Lemma. □

Let us now define the Dirichlet series needed for Theorem 1.1.

Definition 3.2. *For a function ϕ satisfying (3.3), (3.4), (3.5), and (3.6), $r\in\mathfrak{d}_K^{-1}/(2m\mathcal{O}_K)$, and ν satisfying (3.1) and (3.2), we define*

$$L(s,\phi,r,\chi_{m,\nu}) := \sum_{\alpha\in\mathfrak{d}_K^{-2}/T_m^2} \chi_{m,\nu}(\alpha)\cdot c_r(\alpha)\cdot\mathbb{N}(\alpha)^{-s}, \tag{3.8}$$

$$\mathcal{L}(s,\phi,r,\chi_{m,\nu}) := 2^{gs}\pi^{-gs}\prod_{j=1}^{g}\Gamma(s-\nu_j)\mathbb{N}(m)^s\prod_{j=1}^{g}\left(m^{(j)}\right)^{-\nu_j} L(s,\phi,r,\chi_{m,\nu}). \tag{3.9}$$

Due to (3.6) the series $L(s,\phi,r,\chi_{m,\nu})$ is absolutely convergent for $\sigma=Re(s)>M+1$. We have the following lemma.

Lemma 3.3. *For $\sigma>M+1$, we have the identity*

$$\mathcal{L}(s,\phi,r,\chi_{m,\nu}) = \int_{T_m^2\backslash\mathbb{R}_+^g} f_r(iy)\mathbb{N}(y^{s-\nu})\frac{dy}{\mathbb{N}(y)}\ , \tag{3.10}$$

where $f_r(\tau)$ is the form defined in (2.4) with Fourier coefficients $c(n,r)$.

Proof. This can be directly calculated by using the Fourier expansion of $f_r(iy)$ and by using the relation $\mathbb{N}(u_l^\nu) = \mathbb{N}(\epsilon_l)^k$ for $l = 1, ..., g-1$, where u_l and ϵ_l are defined in the beginning of this section. We leave the details to the reader (see also [3] p.87). □

4. Proof of Theorem 1.1

Theorem 1.1 follows directly from the two lemmas proven in this section.

Lemma 4.1. *If ϕ is a Hilbert-Jacobi cusp form, $r \in \mathfrak{d}_K^{-1}/(2m\mathcal{O}_K)$, and ν satisfies (3.1) and (3.2), then the functions $\mathcal{L}(s, \phi, r, \chi_{m,\nu})$ have analytic continuations to the whole complex plane. They are of rapid decay, and satisfy the functional equations*

$$\begin{aligned}(4.1)\qquad &\mathcal{L}(s, \phi, r, \chi_{m,\nu}) \\ &= \frac{1}{\sqrt{D_K}} i^{-kg}\, \mathbb{N}(2m)^{-1/2} \sum_{\mu \in (\mathfrak{d}_K^{-1}/2m\mathcal{O}_K)} e_{2m}(-\mu r)\mathcal{L}(k - s - 1/2, \phi, \mu, \chi_{m,-\nu}).\end{aligned}$$

Proof. To prove the analytic continuation of $\mathcal{L}(s, \phi, r, \chi_{m,\nu})$, we show that the right-hand side of (3.10) is analytic for all s. For this, we separate the integral in a part with $\mathbb{N}(y) \geq 1$ and a part with $\mathbb{N}(y) \leq 1$. Using the transformation law of f_r, one can see that it is enough to consider the part with $\mathbb{N}(y) \geq 1$. To estimate this, we use the variables $y_0 \in \mathbb{R}_+$ and $t = (t_1, ..., t_{g-1}) \in \mathbb{R}^{g-1}$, where

$$y_j := y_0 \cdot e^{\sum_{l=1}^{g-1} t_l \log\left(u_l^{(j)}\right)}.$$

Then a fundamental domain of $T_m^2 \backslash \mathbb{R}_+^g$ is given by the inequalities $y_0 > 0$ and $0 \leq t_l < 1$ ($l = 1, ..., g-1$) and the part with $\mathbb{N}(y) \geq 1$ is given by $y_0 \geq 1$. The analyticity now follows if we use Lemma 2.3, since for $c > 0$ and $\sigma \in \mathbb{R}$ arbitrary the integral $\int_1^\infty e^{-cy} y^\sigma dy$ is convergent. The boundness of $\mathcal{L}(s, \phi, r, \chi_{m,\nu})$ in every vertical strip follows also from this convergence.

Moreover, by using the transformation law of f_r and Lemma 3.3, equation (4.1) follows since $1/y$ runs through $T_m^2 \backslash \mathbb{R}_+^g$ if y does. □

Lemma 4.2. *Assume that ϕ is a function satisfying (3.3), (3.4), (3.5), and (3.6), and that for all $r \in \mathfrak{d}_K^{-1}/(2m\mathcal{O}_K)$ and for all ν satisfying (3.1) and (3.2) the series $\mathcal{L}(s, \phi, r, \chi_m)$ have analytic continuations, satisfy (4.1) and are of rapid decay. Then ϕ is a Hilbert-Jacobi cusp form of weight k and of index m.*

Proof. By analytic continuation it is enough to show (2.2) for $\tau = iy$. We parametrize the integrals as before and use the Mellin inversion formula to get for σ sufficiently large

$$(4.2)\qquad \int_{[0,1]^{g-1}} f_r(iy_0 \cdot e^{tR}) e^{-\nu tR} dt = \frac{1}{2gR_m\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \mathcal{L}(s/g, \phi, r, \chi_{m,\nu}) y_0^{-s} ds,$$

where

$$R_m := \det((\log(u_l^{(j)}))_{l,j=1,\dots,g-1})$$

$$f_r\left(iy_0\cdot e^{tR}\right) := f_r\left(iy_0 e^{\sum_{l=1}^{g-1} t_l\, log\left(u_l^{(1)}\right)},\cdots,iy_0 e^{\sum_{l=1}^{g-1} t_l\, log\left(u_l^{(g)}\right)}\right),$$

$$e^{-\nu tR} := \prod_{j=1}^{g}\prod_{l=1}^{g-1} e^{-\nu_j t_l \log\left(u_l^{(j)}\right)} = \prod_{l=1}^{g-1} e^{-2\pi i\left(N_l+\frac{1}{2}\delta_l\right)t_l}, \tag{4.3}$$

where N_l and δ_l appeared in (3.2). Applying (4.1) and making the substitution $s\to g(k-1/2-s)$ gives that the right-hand side of (4.2) equals

$$\frac{1}{\sqrt{D_K}} i^{-kg}\,\mathbb{N}(2m)^{-1/2} \sum_{\mu\in(\mathfrak{d}_K^{-1}/2m\mathcal{O}_K)} e_{2m}(-\mu r) y_0^{-g(k-1/2)} \times \frac{1}{2gR_m\pi i}\int_{g(k-1/2)-\sigma-i\infty}^{g(k-1/2)-\sigma+i\infty} \mathcal{L}(s/g,\phi,\mu,\chi_{m,\nu}) y_0^s ds.$$

If $\mathrm{Re}(s) > M+1$, the series $L(s,\phi,r,\chi_{m,\nu})$ is absolutely convergent, and the series $\mathcal{L}(s,\phi,r,\chi_{m,\nu})$ is of rapid decay for $|\mathrm{Im}(s)|\to\infty$. Also $\mathcal{L}(s,\phi,r,\chi_{m,-\nu})$ is bounded in every vertical strip and has a functional equation. By using the Phragmén-Lindelöf principle, we can conclude that $\mathcal{L}(s,\phi,r,\chi_{m,-\nu})$ is of uniformly rapid decay for $|\mathrm{Im}(s)|\to\infty$ in every vertical strip. Hence, we use Cauchy's Theorem and shift the path of integration to the line $\mathrm{Re}(s)=\sigma$. Thus the left-hand side of (4.2) equals

$$\frac{1}{\sqrt{D_K}} i^{-kg}\,\mathbb{N}(2m)^{-1/2} \sum_{\mu\in(\mathfrak{d}_K^{-1}/2m\mathcal{O}_K)} e_{2m}(-\mu r) y_0^{-g(k-1/2)} \times \frac{1}{2gR_m\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{L}(s/g,\phi,\mu,\chi_{m,-\nu}) y_0^s ds.$$

But the latter integral equals $2gR_m\pi i\int_{[0,1]^{g-1}} f_\mu(iy_0^{-1}\cdot e^{-tR})e^{-\nu tR}dt$.

Thus

$$\int_{[0,1]^{g-1}} f_r(iy_0\cdot e^{tR})e^{-\nu tR}dt = \frac{1}{\sqrt{D_K}} i^{-kg}\,\mathbb{N}(2m)^{-1/2} \sum_{\mu\in(\mathfrak{d}_K^{-1}/2m\mathcal{O}_K)} e_{2m}(-\mu r) y_0^{-g(k-1/2)} \int_{[0,1]^{g-1}} f_\mu(iy_0^{-1}\cdot e^{-tR})e^{-\nu tR}dt. \tag{4.4}$$

We now let

$$g_r(t) := f_r(iy_0\cdot e^{tR}) - \frac{1}{\sqrt{D_K}} i^{-kg}\,\mathbb{N}(2m)^{-1/2} \sum_{\mu\in(\mathfrak{d}_K^{-1}/2m\mathcal{O}_K)} e_{2m}(-\mu r) y_0^{-g(k-1/2)} f_\mu(iy_0^{-1}\cdot e^{-tR}).$$

To prove the lemma is suffices to show that $g_r(t)$ is identically zero. But this follows since the function $\hat{g}_r(t) := g_r(t)\prod_{l=1}^{g-1} e^{-\pi i\,\delta_l\,t_l}$ has period 1 in every component of t and all $(N_1, ..., N_{g-1})$-th Fourier coefficients of $\hat{g}_r(t)$ are 0 due to (4.3) and (4.4). □

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