FROM SHEAVES ON $\mathbb{P}^2$ TO A GENERALIZATION OF THE RADEMACHER EXPANSION

KATHRIN BRINGMANN AND JAN MANSCHOT

Abstract. Moduli spaces of stable coherent sheaves on a surface are of much interest for both mathematics and physics. Yoshioka computed generating functions of Poincaré polynomials of such moduli spaces if the surface is $\mathbb{P}^2$ and the rank of the sheaves is 2. Motivated by physical arguments, this paper investigates the modular properties of these generating functions. It is shown that these functions can be written in terms of the Lerch sum and theta function. Based on this, we prove a conjecture by Vafa and Witten, which expresses the generating functions of Euler numbers as a mixed mock modular form. Moreover, we derive an exact formula of Rademacher-type for the Fourier coefficients of this function. This formula requires a generalization of the classical Circle Method. This is the first example of an exact formula for the Fourier coefficients of mixed mock modular forms, which is of independent mathematical interest.

1. Introduction and Statement of Results

In the past interactions between physics and mathematics have led to many interesting results. Motivated by strong-weak coupling duality (or S-duality) in physics, this article considers various generating functions which appear in the study of moduli spaces of stable coherent sheaves on the projective plane $\mathbb{P}^2$. We express the generating functions of Poincaré polynomials of moduli spaces of rank 2 sheaves in terms of the Lerch sum and theta function which we will recall later. Using these expressions, we prove a conjecture by Vafa and Witten [45] for the generating functions of Euler numbers. These functions appear to be related to Ramanujan's mock theta functions and therefore transform almost as weakly holomorphic modular forms, i.e., meromorphic modular forms whose poles (if there are any) may only lie in cusps. Our second main result is an exact formula for the Fourier coefficients of these generating functions that formally resembles the Rademacher expansion for the coefficients of weakly holomorphic modular forms.

Moduli spaces of coherent sheaves on a complex surface S receive much attention (see for example [28] for an extensive work on such moduli spaces). More specifically, one is interested in the moduli space $\mathcal{M}(r, c_1, c_2)$ of semi-stable sheaves of rank r with first Chern class c_1 and second Chern class c_2. We will consider topological invariants of $\mathcal{M}$, in particular the Poincaré polynomial $p(\mathcal{M}, s) := \sum_{i=0}^{2\dim_{\mathbb{C}}\mathcal{M}} b_i(\mathcal{M})\, s^i$ and the Euler number $\chi(\mathcal{M}) := p(\mathcal{M}, -1)$, where $b_i(\mathcal{M})$ is the ith Betti number: $b_i(\mathcal{M}) := \dim H_i(\mathcal{M}, \mathbb{Z})$. Ellingsrud and Strømme [21] computed the Betti numbers of the moduli space of sheaves with rank 1 on $\mathbb{P}^2$ and other ruled surfaces. Göttsche [23] derived the generating function for $p(\mathcal{M}(1, 0, n), s) :=$

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$\sum_{i=0}^{4n} b_i(\mathcal{M})s^i$ for rank 1 sheaves on a smooth projective surface, and wrote it as an elegant product formula (2.2).

Subsequent work by Yoshioka [49, 50] derived the generating functions of Poincaré polynomials for sheaves of rank 2 on the projective plane $\mathbb{P}^2$ (Eqs. (2.11) and (2.12)). Recently, the generating functions of the Euler and Betti numbers for rank 3 have also been computed [30, 35]. Closely related are the computations of the Euler numbers of the moduli spaces of vector bundles for rank 2 [29] and for rank 3 [48].

These generating functions have also enjoyed much interest in physics, in particular in the context of strong-weak coupling duality and instanton moduli spaces. The duality was first conjectured by Montonen and Olive [37] as a duality of gauge theory. Their conjecture claims that gauge theory with gauge group G and coupling constant g has a dual description in terms of the gauge theory with gauge group ${}^LG$ (the Langlands dual group) and coupling constant $4\pi/g$. If the theta angle θ is included in the analysis, then this $\mathbb{Z}/2$ group is enlarged to the modular group $\mathrm{SL}_2(\mathbb{Z})$ which acts on the complex parameter $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \in \mathcal{H}$ by linear fractional transformations. The symmetry of gauge theory under this larger group is known as S-duality.

Vafa and Witten [45] have tested S-duality for topologically twisted gauge theory with $\mathcal{N} = 4$ supersymmetry. They showed that for fixed instanton number the path integral of this theory equals the Euler number of a suitable compactification of the instanton moduli space, which turns out to be the Gieseker-Maruyama compactification of the moduli space of semi-stable sheaves whose Chern classes are determined by the instanton data. S-duality led Vafa and Witten [45] (see in particular Section 3 of [45]) to the conjecture that the generating function of the Euler numbers transforms as a (weakly holomorphic) modular form with a specific weight and multiplier. These properties were later also understood from the point of view of M5-branes, see for example [36].

The generating functions (2.11) and (2.12) allow a precise test of the conjectured modular properties for rank 2 and $\mathbb{P}^2$. In fact, we derive similar modular properties for these generating functions of Poincaré polynomials as for those of Euler numbers. This is quite remarkable since present discussions in the literature are limited to the Euler numbers. To make the modular properties manifest, we express in Proposition 2.2 these functions in terms of automorphic functions, in particular the Lerch sum (2.15). Since the modular properties of the Lerch sum are well established, thanks to Zwegers' thesis [54], it is straightforward to derive the modular properties of these generating functions.

Specialization of the Poincaré polynomials to the Euler numbers requires one to take the derivative of the Lerch sum. Using this relation we prove that the generating function of the Euler numbers contains the generating function of the Hurwitz class numbers.A connection between the Euler numbers of the moduli space of vector bundles and class numbers was earlier proposed by Klyachko [29]. To state our result, let $H(n)$ be the Hurwitz class number, i.e., the number of equivalence classes of quadratic forms of discriminant $-n$, where each class C is counted with multiplicity $1/\mathrm{Aut}(C)$. We note that $H(0) = -\frac{1}{12}$ and $H(3) = \frac{1}{3}$. Moreover, we let (throughout $q := e^{2\pi i\tau}$)

$$h_j(\tau) := \sum_{n=0}^{\infty} H(4n+3j)q^{n+\frac{3j}{4}}, \qquad j \in \{0,1\}. \tag{1.1}$$

In Section 2, we prove:

Proposition 1.1. *The generating functions of the Euler numbers $\chi(\mathcal{M}(2, c_1, c_2))$ take the form:*

$$\begin{aligned} q^{-\frac{1}{2}} \sum_{n=1}^{\infty} \chi\left(\mathcal{M}(2,-1,n)\right) q^n &= \frac{3h_1(\tau)}{\eta^6(\tau)}, \\ q^{-\frac{1}{4}} \sum_{n=2}^{\infty} \chi\left(\mathcal{M}(2,0,n)\right) q^n &= \frac{3h_0(\tau)}{\eta^6(\tau)} + \frac{1}{4\eta^3(2\tau)}, \end{aligned}$$

where $\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$ is Dedekind's eta-function.

This proposition is the conjecture [45] mentioned in the abstract. Results of Refs. [29, 49] led Ref. [45] to this conjecture, and it was verified by a comparison of the first coefficients of $3h_1(\tau)/\eta^6(\tau)$ with Eq. (2.11). The good modular properties of $h_j(\tau)$ after addition of a suitable non-holomorphic term (see Eq. (2.10)) was a strong confirmation of the S-duality conjecture. We refer to Sec. 4.2 of Ref. [45] for more details.

In the following we recall in more detail what is known about modularity of generating functions of class numbers of imaginary quadratic fields. Recall that the Fourier coefficients $r(n)$ of Θ_0^3, with $\Theta_0(\tau) := \sum_{n\in\mathbb{Z}} q^{n^2}$, themselves encode class numbers. To be more precise, by a result of Gauss, we have that

$$r(n) = 12\left(H(4n) - 2H(n)\right). \tag{1.2}$$

If one wants to study the full generating function for the Hurwitz class numbers, then one has to move to the world of harmonic weak Maass forms [14] (see also Section 2). These are generalizations of modular forms in that they satisfy the same modular transformation laws but instead of being meromorphic they are annihilated by the weight k hyperbolic Laplacian. To be more precise, Zagier [51] showed that the generating function

$$h(\tau) := \sum_{\substack{n\geq 0 \\ n\equiv 0,3 \pmod 4}} H(n) q^n,$$

is a mock modular form with shadow $\Theta_0(\tau)$, notions which we will recall shortly. We note that the full generating function for class numbers of real and imaginary quadratic fields requires one to consider even more generalized automorphic objects [20].

Mock modular forms are related to Ramanujan's so-called mock theta functions, which he introduced in his last letter to Hardy (see [43], pp. 127-131) in 17 examples. Ramanujan stated that these forms have properties which resemble those of theta functions, but are not modular forms. The mock theta functions occur on the one hand in a vast variety of papers (see for example [2, 4, 15, 27, 47] just to mention a few), but were on the other hand not well understood for a long time since they lack real modularity properties. The mystery surrounding these functions was finally solved by Zwegers in his famous PhD thesis [54] in which he related the mock theta functions to harmonic weak Maass forms. Placing the mock theta functions into the world of harmonic weak Maass forms has many applications: for example the first author and Ono proved an exact formula for the coefficients of one of the mock theta functions [11] and explained how to construct an infinite family of mock theta functions related to Dyson's rank statistic on partitions [12]. Further applications are for example a relation between Hurwitz class numbers and overpartition rank differences [9], and a duality relating the coefficients of mock theta functions to coefficients of weakly holomorphic modular forms [22, 55]. Part of the difficulty of really understanding the mock theta functions was grounded in the fact that these functions have a certain hidden companion, which Zagier calls the shadow of the mock theta function, and without which the mock theta functions are not fully understood. These shadows may be obtained from the associated harmonic Maass form by applying the differential operator $\xi_{2-k} := 2iy^{2-k}\overline{\frac{\partial}{\partial\overline{\tau}}}$ (with $k = 1/2$ and $y := \mathrm{Im}\ (\tau)$) and turn out to be unary theta functions. Mock modular forms are then generalizations of mock theta functions in that the associated shadow does not necessarily have to be a unary theta function but may be a general (weakly holomorphic) modular form. Mixed mock modular forms are functions which lie in the tensor space of mock modular forms and modular forms.

The functions

$$f_j(\tau) := \frac{h_j(\tau)}{\eta^6(\tau)} = \sum_{n=0}^{\infty} \alpha_j(n) q^{n-\frac{j+1}{4}}, \qquad j \in \{0,1\}, \tag{1.3}$$

of Proposition 1.1 are examples of such forms. In this paper we prove an exact formula for $\alpha_j(n)$, which is the first exact formula for coefficients of mixed mock modular forms. This result provides an exact formula for $\chi(\mathcal{M}(2,c_1,c_2))$, which is clearly of interest for both mathematics and physics. The proof requires a generalization of the Hardy-Ramanujan Circle Method due to the first author and Mahlburg [10] which may be applied to mixed mock modular forms. Let us next place this result in its mathematical context. As usual we denote by $p(n)$ the number of partitions of an integer n. Recall that Hardy and Ramanujan [25, 26], in work which gave birth to the Circle Method, derived their famous asymptotic formula for the partition function $p(n)$,

$$p(n) \sim \frac{1}{4n\sqrt{3}} \cdot e^{\pi\sqrt{2n/3}} \qquad (n \to \infty).$$

Rademacher [41] then subsequently proved the following exact formula

$$p(n) = 2\pi(24n-1)^{-\frac{3}{4}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} \cdot I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right).$$

Here $I_\ell(x)$ is the I-Bessel function of order ℓ, and $A_k(n)$ is the Kloosterman sum

$$A_k(n) := \frac{1}{2}\sqrt{\frac{k}{12}} \sum_{\substack{x \pmod{24k} \\ x^2\equiv -24n+1 \pmod{24k}}} \chi_{12}(x)\cdot e\left(\frac{x}{12k}\right),$$

where $e(\alpha) := e^{2\pi i\alpha}$ and $\chi_{12}(x) := \left(\frac{12}{x}\right)$. An important tool used to prove the asymptotic and exact formulas for $p(n)$ is the fact that

$$P(\tau) := \sum_{n=0}^{\infty} p(n) q^{n-\frac{1}{24}} = \frac{1}{\eta(\tau)}$$

is a weight $-1/2$ modular form. Rademacher and Zuckerman [42, 52, 53] subsequently showed exact formulas for the coefficients of generic weakly holomorphic modular forms of negative weight.

The situation is more complicated if one turns to non-modular objects. Let us mention Ramanujan's mock theta functions and in particular

$$f(q)=\sum_{n=0}^{\infty}\alpha(n)q^n:=1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2}.$$

The problem of obtaining an asymptotic formula for $\alpha(n)$ is greatly complicated by the fact that f is not a modular form. Dragonette [19] in her PhD thesis confirmed a conjecture of Ramanujan concerning an asymptotic formula for $\alpha(n)$; subsequently this was improved by Andrews [1]. Infinite families of further asymptotic formulas were recently proven by the first author [6]. Andrews and Dragonette moreover conjectured an exact formula for $\alpha(n)$ which was then proved by the first author and Ono [11] using the theory of Maass Poincaré series. The authors of [13] obtained more generally exact formulas for all coefficients of mock modular forms of non-positive weight. From the above description it becomes clear that currently asymptotic/exact formulas for coefficients of modular or mock modular forms are well understood. The situation is totally different for mixed mock modular forms. The above mentioned methods cannot be applied as the space of harmonic Maass forms is not closed under multiplication. The first such example was considered by the first author and Mahlburg [10] and is related to so-called partitions without sequences [3], a partition statistic that we do not want to recall for the purpose of this paper. Developing a generalization of the Circle Method to involve certain non-modular objects, the authors managed to obtain asymptotic expansions for such partitions without sequences. So far this is the only example of an asymptotic formula for coefficients of forms in the tensor space. It is of mathematical interest to find further such examples. We note that due to the more complicated situation the authors of [10] only obtain an asymptotic and not an exact formula. In this paper we derive the first example of such an exact formula.

Turning back to an exact formula for the coefficients $\alpha_j(n)$ of f_j, we require some more notation. We let for $k\in\mathbb{N}, g\in\mathbb{Z}$, and $u\in\mathbb{R}$

$$f_{k,g}(u):=\begin{cases}\frac{\pi^2}{\sinh^2\left(\frac{\pi u}{k}-\frac{\pi i g}{2k}\right)} & \text{if } g\not\equiv 0 \pmod{2k},\\ \frac{\pi^2}{\sinh^2\left(\frac{\pi u}{k}\right)}-\frac{k^2}{u^2} & \text{if } g\equiv 0 \pmod{2k}.\end{cases}\tag{1.4}$$

Furthermore we define the Kloosterman sums

$$K_{j,\ell}(n,m;k):=\sum_{\substack{0\le h<k\\ (h,k)=1}}\psi_{j\ell}(h,h',k)e^{-\frac{2\pi i}{k}\left(hn+\frac{h'm}{4}\right)},$$

where $\psi_{j\ell}$ is a multiplier defined in (3.4), and h' is given by the congruence $hh'=-1 \pmod{k}$. Finally we let

$$\mathcal{I}_{k,g}(n):=\int_{-1}^{1}f_{k,g}\left(\frac{u}{2}\right)I_{\frac{7}{2}}\left(\frac{\pi}{k}\sqrt{(4n-(j+1))\left(1-u^2\right)}\right)\left(1-u^2\right)^{\frac{7}{4}}du.$$

Theorem 1.2. *The coefficients $\alpha_j(n)$ of f_j are given by the following exact formula:*

$$\alpha_j(n) = -\frac{\pi}{6}\left(4n-(j+1)\right)^{-\frac{5}{4}}\sum_{k=1}^{\infty}\frac{K_{j,0}(n,0;k)}{k}I_{\frac{5}{2}}\left(\frac{\pi}{k}\sqrt{4n-(j+1)}\right)$$
$$+\frac{1}{\sqrt{2}}\left(4n-(j+1)\right)^{-\frac{3}{2}}\sum_{k=1}^{\infty}\frac{K_{j,0}(n,0;k)}{\sqrt{k}}I_{3}\left(\frac{\pi}{k}\sqrt{4n-(j+1)}\right)$$
$$-\frac{1}{8\pi}\left(4n-(j+1)\right)^{-\frac{7}{4}}\sum_{k=1}^{\infty}\sum_{\substack{\ell\in\{0,1\}\\ -k<g\leq k\\ g\equiv\ell \pmod 2}}\frac{K_{j,\ell}(n,g^2;k)}{k^2}\mathcal{I}_{k,g}(n).$$

The integrals $\mathcal{I}_{k,g}(n)$ can be estimated using well-known asymptotic formulas for Bessel functions and Proposition 5.1 of [10].

Corollary 1.3. *The leading asymptotic terms of $\alpha_j(n)$ for $n\to\infty$ are:*

$$\alpha_j(n) = \left(\frac{1}{96}n^{-\frac{3}{2}} - \frac{1}{32\pi}n^{-\frac{7}{4}} + O\left(n^{-2}\right)\right)e^{2\pi\sqrt{n}}.$$

We want to make two remarks concerning Theorem 1.2 and Corollary 1.3.

(1) Firstly we note that in contrast to mock modular forms, the shadows of the mixed mock modular forms do contribute to our leading asymptotic terms. One could determine further polynomial lower order main terms.
(2) Secondly, we like to mention that the first term in the exact formula are the coefficients of a negative weight Poincaré series as described by Niebur [40]. Numerical experiments by F. Strömberg give strong evidence that generically this term does not converge to an integer. G. W. Moore and the second author [33] considered negative weight Poincaré series (which are essentially a sum over $\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})$), because of their interpretation in the context of the correspondence between 3-dimensional Anti-de Sitter space and 2-dimensional conformal field theory [18]. This led them to consider alternate functions, say $\widetilde{f}_j$, in addition to f_j, whose coefficients are given by the first term of Theorem 1.2. Interestingly, the theorem shows that $\widetilde{f}_j$ appears naturally as part of f_j. Moreover, it is possible to show that $f_j - \tilde{f}_j$ can also be written as a sum over $\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})$. We leave a precise discussion for the future.

The results of this paper might be relevant for various applications and current developments, some of which we want to list here:

(1) The appearance of Lerch sums in the generating functions of Poincaré polynomials is rather intriguing. Besides for $\mathbb{P}^2$, one can show, using the results of [49, 50], that they also appear for rank 2 sheaves on ruled surfaces. Their appearance is essentially a consequence of the contributions of stable bundles to the generating series for a specific polarization. It would be interesting to investigate whether Lerch sums play also a role for different surfaces, higher rank sheaves, and related systems like Calabi-Yau black holes.

(2) The exact formula for the coefficients of f_j can be generalized to other mixed mock modular forms. Besides the intrinsic mathematical interest, this might also prove to be very useful in physics, in particular in discussions on black hole entropy and the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence. For example the functions h_j/η^{24} are known to appear as generating functions of the degeneracies of $\mathcal{N}=4$ dyons [16].

The outline of this article is as follows. Section 2 reviews briefly the generating functions of invariants of moduli spaces of stable sheaves, relates those of rank 2 to the Lerch sum and theta function, and proves the conjecture by Vafa and Witten. Section 3 derives the exact formula for the Fourier coefficients of f_j, using the Hardy-Ramanujan Circle Method.

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2. Generating functions of topological invariants

Let us start by recalling some of the relevant background. We refer the reader who is unfamiliar with these notions from algebraic geometry to consult textbooks like [24, 39]. It is well-known that the Betti numbers of $\mathbb{P}^2$ equal $b_0=b_2=b_4=1$ and $b_1=b_3=0$, therefore $\chi(\mathbb{P}^2)=3$. The total Chern class $c(\mathbb{P}^2)$ of the tangent bundle of $\mathbb{P}^2$ is defined as:

$$c\left(\mathbb{P}^2\right):=1+c_1\left(\mathbb{P}^2\right)+c_2\left(\mathbb{P}^2\right)=(1+J)^3,$$

where J is the hyperplane class and $c_i(\mathbb{P}^2)\in H^{2i}(\mathbb{P}^2,\mathbb{Z})$. Since these cohomology groups are 1-dimensional, we will also denote the integrated forms $\int c_i(\mathbb{P}^2)$ by $c_i(\mathbb{P}^2)$, thus $c_1(\mathbb{P}^2)=c_2(\mathbb{P}^2)=3$.

Chern classes $c_i(E)$ are defined for any sheaf E on $\mathbb{P}^2$, and play a central role in the classification of sheaves. If no confusion can arise, the Chern classes $c_i(E)$ are in the following abbreviated by c_i. The complex dimension of the moduli space of stable sheaves may be written in terms of these Chern classes as

$$\dim_{\mathbb{C}}\left(\mathcal{M}(r,c_1,c_2)\right)=2rc_2-(r-1)c_1^2-r^2+1, \tag{2.1}$$

where r is the rank of the sheaf. The moduli space of stable sheaves generically depends on the choice of an ample line bundle over the surface. However, since $b_2(\mathbb{P}^2)=1$, stability does not depend on this choice.

We are interested in the generating functions of the Poincaré polynomials and Euler numbers of moduli spaces $\mathcal{M}(r,c_1,c_2)$ as functions of c_1 and c_2. Twisting a sheaf by a line bundle $E\otimes\mathcal{O}(k)$ gives an isomorphism between the moduli spaces $\mathcal{M}(r,c_1,c_2)$ and $\mathcal{M}(c_1+rk,c_2+(r-1)kc_1+\frac{1}{2}r(r-1)k^2)$. It is therefore sufficient to only consider c_1 (mod r).

The generating function of the Poincaré polynomials of $\mathcal{M}(1,0,c_2)$ for any surface S is given by [23]

$$\sum_{n\geq 0} p\left(\mathcal{M}(1,0,n),s\right)t^n = \prod_{m\geq 1}\frac{\left(1+s^{2m-1}t^m\right)^{b_1(S)}\left(1+s^{2m+1}t^m\right)^{b_1(S)}}{\left(1-s^{2(m-1)}t^m\right)^{b_0(S)}\left(1-s^{2m}t^m\right)^{b_2(S)}\left(1-s^{2(m+1)}t^m\right)^{b_0(S)}}. \tag{2.2}$$

To exhibit the modular properties for $S=\mathbb{P}^2$, we write it in terms of the Jacobi theta function $\theta_1(z;\tau)$, which has the following sum and product expansion ($w=e^{2\pi i z}$):
(2.3)

$$\theta_1(z;\tau) := i\sum_{r\in\mathbb{Z}+\frac{1}{2}}(-1)^{r-\frac{1}{2}}q^{\frac{r^2}{2}}w^r = iq^{\frac{1}{8}}\left(w^{\frac{1}{2}}-w^{-\frac{1}{2}}\right)\prod_{n=1}^{\infty}\left(1-q^n\right)\left(1-wq^n\right)\left(1-w^{-1}q^n\right).$$

This theta function transforms under the generators $S := \left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ and $T := \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ of $\mathrm{SL}_2(\mathbb{Z})$ as:

$$\begin{aligned}\theta_1\left(\frac{z}{\tau};-\frac{1}{\tau}\right) &= -i\sqrt{-i\tau}\exp\left(\frac{\pi i z^2}{\tau}\right)\theta_1(z;\tau),\\ \theta_1(z;\tau+1) &= \exp\left(\frac{\pi i}{4}\right)\theta_1(z;\tau).\end{aligned}$$

Moreover θ_1 has simple zeros at the points $z=n\tau+m$ with $n,m\in\mathbb{Z}$.

With the substitutions $q=s^2t=\exp(2\pi i\tau)$ and $w=s^2=\exp(2\pi i z)$, equation (2.2) becomes

$$q^{-\frac{1}{8}}\sum_{n\geq 0}p\left(\mathcal{M}(1,0,n),w^{\frac{1}{2}}\right)\left(qw^{-1}\right)^n = \frac{i\left(w^{\frac{1}{2}}-w^{-\frac{1}{2}}\right)}{\theta_1(z;\tau)}. \tag{2.4}$$

The Betti numbers can be obtained by first expanding equation (2.4) in q for $q\approx 0$, and then in w for $w\approx 0$. One easily sees that (2.4) has no poles for $z\in\mathbb{Z}$, but does have simple poles for $z=m\tau+n$, with $(m,n)\in\mathbb{Z}^2$, $m\neq 0$. The Fourier coefficients of (2.4) depend therefore on the choice of contour to extract the Fourier coefficients. The physical origin of these poles is however unclear, but might be related to the fact that the Poincaré polynomial is not a supersymmetric index like the Euler number.

The above substitutions for s and t are not arbitrary but compatible with the Lefshetz $sl(2)$-action on the moduli space. If J_3 is identified with the Cartan element $sl(2)$, then the action of J_3 on an harmonic form on the moduli space is given by [17, 24]

$$J_3\,\omega = \tfrac{1}{2}\left(\deg\omega - \dim\mathcal{M}\right)\omega.$$

The eigenvalue of J_3 is the exponent of w in the expansion.

The Euler characteristics are obtained by setting $s=-1$: $p(\mathcal{M}(1,0,n),-1)=\chi(\mathcal{M}(1,0,n))$. Then the generating function becomes:

$$f_{1,0}(\tau) := q^{-\frac{1}{8}}\sum_{n=0}^{\infty}\chi\left(\mathcal{M}(1,0,n)\right)q^n = \frac{1}{\eta^3(\tau)}, \tag{2.5}$$

thus a modular form of weight $-3/2$. For $r \geq 1$, one can more generally define the functions

$$f_{r,c_1}(\tau) := \sum_{c_2 \geq \frac{r-1}{2r}c_1^2} \chi\left(\mathcal{M}(r,c_1,c_2)\right) q^{r\Delta - r\chi(\mathbb{P}^2)/24}, \tag{2.6}$$

with Δ the discriminant of E: $\Delta := \frac{1}{r}\left(c_2 - \frac{r-1}{2r}c_1^2\right)$ and $0 \leq c_1 \leq r-1$. These functions are expected to exhibit transformation properties of a vector-valued modular form of length r and weight $-\chi(\mathbb{P}^2)/2$. The modular properties of this function are most straightforwardly derived from the point of view of multiple M5-branes wrapping $\mathbb{P}^2 \otimes T^2$. This leads to a generating function which also sums over all c_1 [36, 32]:

$$\mathcal{Z}_r(\rho;\tau) := \sum_{c_1,c_2\in\mathbb{Z}} \chi(\mathcal{M}(r,c_1,c_2))\bar{q}^{r\left(\Delta - \frac{\chi(\mathbb{P}^2)}{24}\right)} q^{\frac{1}{2r}\left(c_1 + \frac{rc_1\left(\mathbb{P}^2\right)}{2}\right)^2} (-\xi)^{c_1 + \frac{rc_1\left(\mathbb{P}^2\right)}{2}} \tag{2.7}$$

with $\xi := e^{2\pi i\rho}$. Physical arguments suggest that this function transforms under $\mathrm{SL}_2(\mathbb{Z})$ like a Jacobi form which is non-holomorphic in τ and has weight $(\frac{1}{2}, -\frac{3}{2})$. Moreover, the isomorphism of moduli spaces due to twisting by a line bundle implies a decomposition of $\mathcal{Z}_r(\rho;\tau)$ into theta functions $\Theta_{r,\mu}(\rho;\tau)$ and vector-valued modular forms $f_{r,\mu}(\tau)$:

$$\mathcal{Z}_r(\rho;\tau) = \sum_{\mu \pmod r} \bar{f}_{r,\mu}(\tau)\Theta_{r,\mu}(\rho;\tau), \tag{2.8}$$

with

$$\Theta_{r,\mu}(\rho;\tau) := \sum_{n=\mu \pmod r} q^{\frac{1}{2r}\left(n + \frac{rc_1\left(\mathbb{P}^2\right)}{2}\right)^2} (-\xi)^{n + \frac{rc_1\left(\mathbb{P}^2\right)}{2}}.$$

This decomposition implies that

$$D_r\left(\mathcal{Z}_r(\rho;\tau)\right) = 0$$

with $D_r := \frac{\partial}{\partial\tau} + \frac{i}{4\pi r}\frac{\partial^2}{\partial\rho^2}$. Functions which satisfy this condition together with an appropriate transformation law are known as skew (weakly) holomorphic Jacobi forms [44]. In particular for $r = 1$ we have that

$$\mathcal{Z}_1(\rho;\tau) = \frac{\theta_1(\rho;\tau)}{\bar{\eta}^3(\tau)}.$$

Already for rank 2, we will find two refinements of these physical expectations. As explained in the introduction, the $f_{2,\mu}(\tau)$ appear to be mixed mock modular forms, such that $D_2\left(\widehat{\mathcal{Z}}_2(\rho;\tau)\right) \neq 0$. This is in physics called a "holomorphic anomaly". The other refinement concerns the integrality of the Fourier coefficients.

Before returning to the functions of interest for this paper, we want to recall the precise definition of harmonic weak Maass forms. Here we only require the case of half-integral weight on $\Gamma_0(4)$.

Definition 2.1. A harmonic weak Maass form of weight $k \in \frac{1}{2} + \mathbb{Z}$ for the group $\Gamma_0(4)$ is a smooth function $f : \mathbb{H} \to \mathbb{C}$ satisfying the following:

(1) For all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$, we have that

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = \left(\frac{c}{d}\right) \epsilon_d^{-2k} (c\tau+d)^k \; f(\tau).$$

Here $\left(\frac{c}{d}\right)$ denotes the Jacobi symbol, $\epsilon_d = 1$ for $d \equiv 1 \pmod 4$ and $\epsilon_d = i$ for $d \equiv 3 \pmod 4$, and $\sqrt{\tau}$ is the principal branch of the holomorphic square root.

(2) We have $\Delta_k f = 0$, where ($\tau = x+iy$) the weight k hyperbolic Laplacian Δ_k is defined as

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

(3) The function f has at most linear exponential growth at all the cusps.

Using this notation, the function

$$\widehat{h}(\tau) := \sum_{\substack{n=0 \\ n \equiv 0,3 \pmod 4}}^{\infty} H(n) q^n + \frac{(1+i)}{16\pi} \int_{-\overline{\tau}}^{i\infty} \frac{\Theta_0(w)}{(\tau+w)^{\frac{3}{2}}}\, dw \tag{2.9}$$

is a harmonic Maass form of weight $\frac{3}{2}$ on $\Gamma_0(4)$ (see [51]). We moreover require the restrictions of $\widehat{h}$ to arithmetic progressions $0, 3 \pmod 4$ (individually). It is not hard to see that the associated harmonic weak Maass forms are given by

$$\widehat{h}_j(\tau) := h_j(\tau) + \frac{(1+i)}{8\pi} \int_{-\overline{\tau}}^{i\infty} \frac{\Theta_j(w)}{(\tau+w)^{\frac{3}{2}}}\, dw, \qquad j \in \{0,1\}, \tag{2.10}$$

where the functions h_j were defined in (1.1) and $\Theta_j(\tau) := \sum_{n\in\mathbb{Z}} q^{\frac{1}{4}(2n+j)^2}$.

Now we continue our discussion on generating functions related to semi-stable coherent sheaves of rank 2 on $\mathbb{P}^2$. Yoshioka [49, 50] computed the generating functions of the Poincaré polynomials. To present his result, define

$$Z_s\left(\mathbb{P}^2, t\right) := \frac{1}{(1-t)(1-st)\left(1-s^2t\right)}.$$

The expression for the generating function for $c_1 = -1$ is [49]:

$$\sum_{n=1}^{\infty} p\left(\mathcal{M}(2,-1,n), s\right) t^n = \frac{\prod_{d\geq 1} Z_{s^2}\left(\mathbb{P}^2, s^{4d-2}t^d\right)^2}{(s^2-1)\sum_{n\in\mathbb{Z}} s^{2n(2n-1)} t^{n^2}} \tag{2.11}$$
$$\times \sum_{b\geq 0} \left(\frac{s^{2(b+1)(2b+1)}}{1-s^{8(b+1)}t^{2b+1}} - \frac{s^{2b(2b+5)}}{1-s^{8b}t^{2b+1}}\right) t^{(b+1)^2},$$

and similarly for $c_1 = 0$ [50]:

$$\sum_{n=2}^{\infty} p\left(\mathcal{M}(2,0,n),s\right)t^n = \frac{\prod_{d\geq 1} Z_{s^2}\left(\mathbb{P}^2, s^{4d-2}t^d\right)^2}{(1-s^2)\sum_{n\in\mathbb{Z}} s^{2n(2n+1)}t^{n(n+1)}} \times \left(\sum_{b\geq 0} -\left(\frac{s^{2(b+1)(2b+3)}}{1-s^{8(b+1)}t^{2b+1}} - \frac{s^{2b(2b+7)}}{1-s^{8b}t^{2b+1}}\right)t^{b^2+3b+1} + \sum_{b\geq 0}\frac{s^{2(b+1)(2b+1)}-s^{2b(2b+1)}}{2s^2}t^{b(b+1)}\right) + \frac{\prod_{d\geq 1} Z_{s^4}\left(\mathbb{P}^2, s^{8d-4}t^{2d}\right)}{2s^2\left(1+s^2\right)}. \tag{2.12}$$

Here we have corrected a sign error in Remark 4.6 of [50].

To simplify the expressions (2.11) and (2.12), we make the substitutions $s^4 t = q$ and $s^2 = w$, analogous to the substitutions in the rank 1 case. One finds after a straightforward computation:

Proposition 2.2. *The generating functions of the Poincaré polynomials $p\left(\mathcal{M}(2,c_1,c_2),s\right)$ take the form:*

$$q^{-\frac{1}{2}}\sum_{n=1}^{\infty} p\left(\mathcal{M}(2,-1,n),w^{\frac{1}{2}}\right)\left(qw^{-2}\right)^n = -\frac{(1-w)}{w^{\frac{5}{2}}\,\theta_1^2(z;\tau)}\,\mu\left(2z-\tau,\tfrac{1}{2}-\tau-z;2\tau\right), \tag{2.13}$$

$$q^{-\frac{1}{4}}\sum_{n=2}^{\infty} p\left(\mathcal{M}(2,0,n),w^{\frac{1}{2}}\right)\left(qw^{-2}\right)^n = \frac{(1-w)}{w^2\,\theta_1^2(z;\tau)}\left(\frac{1}{2} - q^{-\frac{1}{4}}w^{\frac{3}{2}}\,\mu\left(2z-\tau,\tfrac{1}{2}-z;2\tau\right)\right) - \frac{i(1-w)}{2w^2\,\theta_1(2z;2\tau)}, \tag{2.14}$$

where $\mu(u,v;\tau)$ is the Lerch sum defined by

$$\mu(u,v;\tau) := \frac{e^{\pi i u}}{\theta_1(v;\tau)}\sum_{n\in\mathbb{Z}}\frac{(-1)^n e^{\pi i(n^2+n)\tau+2\pi i n v}}{1-e^{2\pi i n\tau+2\pi i u}}, \tag{2.15}$$

with $u,v\in\mathbb{C}$.

Moreover, we define

$$f_{2,1}(z;\tau) := \frac{(1-w)\,q^{-\frac{1}{4}}}{w\,\theta_1^2(z;\tau)\,\theta_3(z;2\tau)}\sum_{n\in\mathbb{Z}}\frac{q^{n^2}w^{-n}}{1-q^{2n-1}w^2},$$

$$f_{2,0}(z;\tau) := \frac{(1-w)}{w^2\,\theta_1^2(z;\tau)}\left(\frac{1}{2}+\frac{q^{-\frac{3}{4}}w^{\frac{5}{2}}}{\theta_2(z;2\tau)}\sum_{n\in\mathbb{Z}}\frac{q^{n^2+n}w^{-n}}{1-q^{2n-1}w^2}\right),$$

with

$$\theta_2(z;\tau) := \sum_{n\in\mathbb{Z}+\frac{1}{2}} q^{\frac{n^2}{2}}w^n, \qquad \theta_3(z;\tau) := \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}}w^n.$$

and

$$g_1(z;\tau) := \frac{q^{-\frac{1}{4}}w^{\frac{3}{2}}}{\theta_3(z;2\tau)}\sum_{n\in\mathbb{Z}}\frac{q^{n^2}w^{-n}}{1-q^{2n-1}w^2} = -\mu\left(2z-\tau,\tfrac{1}{2}-\tau-z;2\tau\right),$$

$$g_0(z;\tau) := \frac{1}{2}+\frac{q^{-\frac{3}{4}}w^{\frac{5}{2}}}{\theta_2(z;2\tau)}\sum_{n\in\mathbb{Z}}\frac{q^{n^2+n}w^{-n}}{1-q^{2n-1}w^2} = \frac{1}{2}-q^{-\frac{1}{4}}w^{\frac{3}{2}}\,\mu\left(2z-\tau,\tfrac{1}{2}-z;2\tau\right),$$

Similarly to the case of $r=1$ these functions have poles for $z\in m\tau+n$ with $(m,n)\in\mathbb{Z}^2$, $m\neq 0$.

Since we have now explicit expressions for the generating functions of Poincaré polynomials for $r=1,2$ at our disposal, it is particularly interesting to investigate their modular properties. For rank 1, equations (2.4) and (2.5) show that the generating function of Euler numbers is indeed a weakly holomorphic modular form, whereas the generating function for Poincaré polynomials transforms as a Jacobi form of weight $-\frac{1}{2}$ and index $-\frac{1}{2}$ (up to the prefactor $w^{\frac{1}{2}}-w^{-\frac{1}{2}}$).

To make the modular properties of (2.13) and (2.14) more manifest, we recall some results of Zwegers' thesis [54]. The Lerch sum (2.15) does not transform as a Jacobi form under $\mathrm{SL}_2(\mathbb{Z})$. However, the completed function [54]

$$\widehat{\mu}(u,v;\tau) := \mu(u,v;\tau)+\frac{i}{2}R(u-v;\tau)$$

transforms as a multi-variable Jacobi form of weight $\frac{1}{2}$. Here the function $R(u;\tau)$ is defined by:

$$R(u;\tau) := \sum_{r\in\mathbb{Z}+\frac{1}{2}}\left(\mathrm{sgn}(r)-E\left((r+a)\sqrt{2y}\right)\right)(-1)^{r-\frac{1}{2}}e^{-\pi i r^2\tau-2\pi i r u},$$

with $a:=\mathrm{Im}(u)/y$, and

$$E(z) := 2\int_0^z e^{-\pi u^2}\,du.$$

To be more precise, we have that:

(1) For $k,l,m,n\in\mathbb{Z}$, we have that:

$$\widehat{\mu}(u+k\tau+l,v+m\tau+n;\tau) = (-1)^{k+l+m+n}\,e^{\pi i(k-m)^2\tau+2\pi i(k-m)(u-v)}\widehat{\mu}(u,v;\tau).$$

(2) For $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$, we have that:

$$\widehat{\mu}\left(\frac{u}{c\tau+d},\frac{v}{c\tau+d};\frac{a\tau+b}{c\tau+d}\right) = v(\gamma)^{-3}(c\tau+d)^{\frac{1}{2}}\,e^{-\frac{\pi i c(u-v)^2}{c\tau+d}}\,\widehat{\mu}(u,v;\tau),$$

with $v(\gamma):=\eta\left(\frac{a\tau+b}{c\tau+d}\right)/\left((c\tau+d)^{\frac{1}{2}}\eta(\tau)\right)$.

Moreover we require the following identity, which allows us to shift parameters in the function μ ($z\in\mathbb{C}$):

$$\mu(u+z,v+z;\tau)-\mu(u,v;\tau) = \frac{i\eta^3(\tau)\theta_1(u+v+z;\tau)\theta_1(z;\tau)}{\theta_1(u;\tau)\theta_1(v;\tau)\theta_1(u+z;\tau)\theta_1(v+z;\tau)}. \tag{2.16}$$

We now turn back to the functions $g_j(z;\tau)$ and define their completions as:

$$\widehat{g}_j(z;\tau) := g_j(z;\tau) + \frac{1}{2}R_j(z;\tau),$$

with

$$R_j(z;\tau) := \sum_{n\in\mathbb{Z}+\frac{j}{2}} \left(\mathrm{sgn}(n) - E\left((2n+3a)\sqrt{y}\right)\right) q^{-n^2} w^{-3n}.$$

We note that

$$\begin{aligned} R_1(z;\tau) &= -iR\left(3z-\frac{1}{2};2\tau\right), \\ R_0(z;\tau) &= -1 - iq^{-\frac{1}{4}} w^{\frac{3}{2}} R\left(3z-\tau-\frac{1}{2};2\tau\right). \end{aligned}$$

Using the above stated transformation properties of $\widehat{\mu}$, one can show that the functions $\widehat{g}_j$ are invariant under T^4, and transform under $S^{-1}T^{-4}S \in \Gamma_0(4)$ as

$$\widehat{g}_j\left(\frac{z}{4\tau+1};\frac{\tau}{4\tau+1}\right) = (4\tau+1)^{\frac{1}{2}} \exp\left(2\pi i \frac{-9z^2}{4\tau+1}\right) \widehat{g}_j(z;\tau).$$

Moreover, one can prove with some more work that the function $\widehat{g}_j(z;\tau)$ may be viewed as components of a function that transforms like a vector valued modular form for $\mathrm{SL}_2(\mathbb{Z})$ of weight $\frac{1}{2}$ and with the same multipliers as the $\widehat{h}_j(\tau)$ defined in equation (2.10).

We observe that if $f_{2,j}(z;\tau)$ would be completed to $\widehat{f}_{2,j}(z;\tau)$ by changing $g_j(z;\tau)$ to $\widehat{g}_j(z;\tau)$, they would transform as Jacobi forms for $\Gamma_0(4)$ of weight $-\frac{1}{2}$ and index $-\frac{13}{4}$, if we ignore the prefactors $(1-w)/w^{2-j}$. The non-holomorphic parts of $\widehat{f}_{2,j}(z;\tau)$ might appear naturally in physics, but precisely how is unknown. Since the functions $\widehat{g}_j(z;\tau)$ transform as a modular vector, the function $\widehat{\mathcal{Z}}_2(z,\rho;\tau) = \sum_{j=0,1} \widehat{f}_{2,j}(z;\tau)\Theta_{2,j}(\rho;\tau)$ transforms with weight $(\frac{1}{2},-\frac{1}{2})$ under $\mathrm{SL}_2(\mathbb{Z})$ (ignoring the prefactors), which can be understood from physics.

As expected from physical arguments, the modular properties improve if one takes the limit $w^{\frac{1}{2}} \to -1$. One can derive straightforwardly that in this case the last term in equation (2.14) is equal to $\frac{1}{4}\eta^{-3}(2\tau)$, which is a modular form of $\Gamma_0(4)$ with a non-trivial multiplier. More interesting is that the limit translates to taking the derivative of the Lerch sums in Eqs. (2.13) and (2.14). Proposition 1.1 gives for these terms $f_{2,j}(\tau) := f_{2,j}(0;\tau) = 3h_j(\tau)/\eta^6(\tau)$.

Before proving the Proposition 1.1, we would like to make a couple of remarks concerning $f_{2,j}(\tau)$, and $\mathcal{Z}_2(\rho;\tau)$ defined by equation (2.7). The completions $\widehat{f}_{2,j}(\tau)$ can be obtained from $\widehat{f}_{2,j}(z;\tau)$, by computing the coefficient of z^1 in the Taylor expansion of $\widehat{g}_j(z;\tau)$. Due to the non-holomorphic term, $D_2\left(\widehat{\mathcal{Z}}_2(\rho;\tau)\right) \neq 0$. One finds

$$D_2\left(\widehat{\mathcal{Z}}_2(\rho;\tau)\right) = \frac{-3i}{16\pi y^{3/2}} \frac{\theta_1^2(\rho;\tau)}{\overline{\eta}^6(\tau)},$$

which is proportional to $\mathcal{Z}_1^2(\rho;\tau)$. The authors of [36] conjecture that such an anomaly appears generically for $r \geq 2$. We did not find such a factorization in the case of Poincaré polynomials, that is to say for $D_2\left(\widehat{\mathcal{Z}}_2(z,\rho;\tau)\right)$.

The generating functions of the Euler numbers, $\widehat{f}_{2,1}(\tau)$ and $\widehat{f}_{2,0}(\tau)+\frac{1}{4}\eta^{-3}(2\tau)$, do not combine to a vector-valued modular form because of the term $\frac{1}{4}\eta^{-3}(2\tau)$. This term disappears, if we consider the generating functions of the rational invariants

$$\overline{\chi}(\Gamma) := \sum_{m\geq 1,\, m|\Gamma} (-1)^{\dim_{\mathbb{C}}(\mathcal{M}(\Gamma/m))}\chi(\Gamma/m)/m^2,$$

where Γ represents the data of the sheaf (r, c_1, c_2). Refs. [34, 45] give also evidence that the generating function of the rational invariants $\overline{\chi}(\Gamma)$ have better modular properties, than the ones for the integer invariants $\chi(\Gamma)$. This is the second refinement, alluded to below equation (2.8).

On the other hand, the coefficients of $f_{2,0}(\tau)+\frac{1}{4}\eta^{-3}(2\tau)$ are required to be integers. This can easily be seen from the arithmetic properties of the functions. To see this, multiply the function by $\eta(\tau)^6$, which gives $3h_0(\tau)+\frac{1}{4}\Theta_0^3(\tau+\frac{1}{2})$. Integrality of the coefficients of this function is manifest, due to the properties of the class numbers $H(n)$ and $\Theta_0^3(\tau+\frac{1}{2})$.

Proof of Proposition 1.1. One could prove the proposition straightforwardly by verifying 1) that the shadows of $f_{2,j}(\tau)$, as obtained from $f_{2,j}(z;\tau)$, coincide with those of $3h_j(\tau)/\eta^6(\tau)$, and 2) that a specific number (related to the dimension of the space of associated modular forms) of coefficients agree. Since this is rather technical, we choose to prove the proposition by relating it to known expressions in the literature.

We start with the identity for $f_{2,1}$. The limit $z\to 0$ of $f_{2,1}(z;\tau)$ is finite and leads to differentation of the Lerch sum:

$$f_{2,1}(\tau) = -\frac{1}{\eta^6(\tau)}\frac{d}{dw}\left[\mu\left(2z-\tau,-z-\tau+\frac{1}{2};2\tau\right)\right]_{w=1}. \tag{2.17}$$

Using (2.16) yields that

$$\mu\left(2z-\tau,-z-\tau+\frac{1}{2};2\tau\right) = \mu\left(-\tau,-3z-\tau+\frac{1}{2};2\tau\right) + \frac{i\eta^3(2\tau)\theta_1\left(-z-2\tau+\frac{1}{2};2\tau\right)\theta_1(2z;2\tau)}{\theta_1(2z-\tau;2\tau)\theta_1\left(-z-\tau+\frac{1}{2};2\tau\right)\theta_1(-\tau;2\tau)\theta_1\left(-3z-\tau+\frac{1}{2};2\tau\right)}.$$

One can prove that the second summand contributes $\frac{1}{2}\Theta_1^3(\tau)/\eta^6(\tau)$ to (2.17). Moreover, the contribution from the first summand is given by

$$-\frac{3}{2}\frac{q^{-\frac{1}{4}}}{\eta^6(\tau)\,\Theta_0(\tau)}\sum_{n\in\mathbb{Z}}\frac{(2n-1)q^{n^2}}{1-q^{2n-1}}.$$

Using work of Kronecker [31], Mordell [38], and Watson [46], one can prove that

$$h_1(\tau) = -\frac{1}{2\Theta_0(\tau)}q^{-\frac{1}{4}}\sum_{n\in\mathbb{Z}}\frac{(2n-1)q^{n^2}}{1-q^{2n-1}} + \frac{1}{6}\Theta_1^3(\tau). \tag{2.18}$$

From this the claim may be easily concluded. We first note that by Watson (correcting a typo) we obtain that

$$\sum_{n=0}^{\infty} F(4n+3)q^{n+\frac{3}{4}} = \frac{1}{4}\Theta_1^3(\tau) - \frac{1}{\vartheta_3(0)} \sum_{n\in\mathbb{Z}} \frac{\left(n-\frac{1}{2}\right) q^{\left(n-\frac{1}{2}\right)^2}}{q^{\frac{1}{2}-n} - q^{n-\frac{1}{2}}} \tag{2.19}$$

where $F(n)$ counts the number of uneven equivalence classes of positive definite quadratic forms of discriminant $-n$. Next one can easily show (for example by using the theory of modular forms) that

$$\Theta_1^3(\tau) = \sum_{n=0}^{\infty} r(4n+3)q^{n+\frac{3}{4}},$$

where the coefficient $r(n)$ is defined by

$$\Theta_0^3(\tau) = \sum_{n=0}^{\infty} r(n)q^n.$$

Now a direct computation gives (2.18).

We next turn to $f_{2,0}$. It is not hard to see that

$$f_{2,0}(\tau) = -\frac{1}{\eta^6(\tau)} \frac{d}{dw} \left[q^{-\frac{1}{4}} w^{\frac{3}{2}} \mu\left(2z-\tau, -z+\frac{1}{2}; 2\tau\right)\right]_{w=1}. \tag{2.20}$$

We find using (2.16)

$$\mu\left(2z-\tau, -z+\frac{1}{2}; 2\tau\right) = \mu\left(-\frac{1}{2}, -3z+\tau; 2\tau\right) + \frac{i\eta^3(2\tau)\theta_1(-z;2\tau)\theta_1\left(2z-\tau+\frac{1}{2};2\tau\right)}{\theta_1(2z-\tau;2\tau)\theta_1\left(-z+\frac{1}{2};2\tau\right)\theta_1\left(-\frac{1}{2};2\tau\right)\theta_1(-3z+\tau;2\tau)}.$$

One can show that the contribution of the second summand to (2.20) equals $-\frac{1}{4}\Theta_0^3(\tau)/\eta^6(\tau)$. Moreover one can prove that the first summand gives a contribution of

$$\frac{-3}{\eta^6(\tau)\,\Theta_0\left(\tau+\frac{1}{2}\right)} \sum_{n\in\mathbb{Z}} \frac{n(-1)^n q^{n^2}}{1+q^{2n}}.$$

Now the claim easily follows using (1.2) and the identity

$$\sum_{n=0}^{\infty} H(n)q^n = -\frac{1}{2\Theta_0\left(\tau+\frac{1}{2}\right)} \sum_{n\in\mathbb{Z}} \frac{n(-1)^n q^{n^2}}{1+q^{2n}} - \frac{1}{12}\Theta_0^3(\tau). \tag{2.21}$$

Indeed, equation (2.21) may for example be concluded by combining Theorem 1.1 and Corollary 1.6 of [8] and inserting the generating function for $\overline{f}$ given in [9]. □

3. Exact formulas for $\alpha_j(n)$

The introduction motivates the derivation of an exact formula of the Fourier coefficients of $f_0(\tau)=\frac{1}{3}f_{2,0}(\tau)$ and $f_1(\tau)=\frac{1}{3}f_{2,1}(\tau)$. This will be the subject of this section. We start by providing various useful transformation formulas, after which we use the Hardy-Ramanujan Circle Method, to derive the exact formula.

3.1. **Some transformation formulas.**

In this section, we give transformation properties for the class number generating functions h_0 and h_1. Throughout, we let $z\in\mathbb{C}$ with $\mathrm{Re}(z)>0$, $k>0$, $(h,k)=1$, and h' defined via the congruence $hh'\equiv -1 \pmod k$. Moreover, we assume that $4|h'$ if k is odd. We require the transformation law of the eta-function:

$$\eta\left(\frac{1}{k}(h+iz)\right)=e^{\frac{\pi i}{12k}(h-h')}\cdot\omega_{h,k}^{-1}\cdot z^{-\frac{1}{2}}\cdot\eta\left(\frac{1}{k}\left(h'+\frac{i}{z}\right)\right), \tag{3.1}$$

where $\omega_{h,k}$ is given by

$$\omega_{h,k}:=\exp\left(\pi i\sum_{\mu \pmod k}\left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{h\mu}{k}\right)\right)\right),$$

with

$$((x)):=\begin{cases} x-\lfloor x\rfloor-\frac{1}{2} & \text{if } x\in\mathbb{R}\setminus\mathbb{Z},\\ 0 & \text{if } x\in\mathbb{Z}.\end{cases}$$

Writing Θ_0 and Θ_1 as eta-quotients

$$\Theta_0(\tau)=\frac{\eta^5(2\tau)}{\eta^2(\tau)\eta^2(4\tau)},\qquad \Theta_1(\tau)=2\frac{\eta^2(4\tau)}{\eta(2\tau)}$$

yields the following transformation law ($j\in\{0,1\}$):

$$\Theta_j\left(\frac{1}{k}(h+iz)\right)=\frac{1}{\sqrt{z}}\sum_{\ell\in\{0,1\}}\chi_{j\ell}(h,h',k)\Theta_\ell\left(\frac{1}{k}\left(h'+\frac{i}{z}\right)\right).$$

Here the multipliers $\chi_{j\ell}$ are defined as follows:

$$\chi_{00}(h,h',k):=\begin{cases}\dfrac{\omega_{h,k}^2\omega_{h,\frac{k}{4}}^2}{\omega_{h,\frac{k}{2}}^5} & \text{if } 4|k,\\ \dfrac{1}{\sqrt{2}}\dfrac{\omega_{h,k}^2\omega_{4h,k}^2}{\omega_{2h,k}^5} & \text{if } 2\nmid k,\\ 0 & \text{if } 2\|k,\end{cases}\qquad \chi_{01}(h,h',k):=\begin{cases}0 & \text{if } 4|k,\\ \dfrac{1}{\sqrt{2}}\dfrac{\omega_{h,k}^2\omega_{4h,k}^2}{\omega_{2h,k}^5} & \text{if } 2\nmid k,\\ \dfrac{\omega_{h,k}^2\omega_{2h,\frac{k}{2}}^2}{\omega_{h,\frac{k}{2}}^5}e^{-\frac{\pi ih'}{2k}} & \text{if } 2\|k,\end{cases}$$

$$\chi_{10}(h,h',k):=\begin{cases}0 & \text{if } 4|k,\\ \dfrac{1}{\sqrt{2}}\dfrac{\omega_{2h,k}}{\omega_{4h,k}^2}e^{\frac{\pi ih}{2k}} & \text{if } 2\nmid k,\\ \dfrac{\omega_{h,\frac{k}{2}}}{\omega_{2h,\frac{k}{2}}^2}e^{\frac{\pi ih}{2k}} & \text{if } 2\|k,\end{cases}\qquad \chi_{11}(h,h',k):=\begin{cases}\dfrac{\omega_{h,\frac{k}{2}}}{\omega_{h,\frac{k}{4}}^2}e^{\frac{\pi i}{2k}(h-h')} & \text{if } 4|k,\\ -\dfrac{1}{\sqrt{2}}\dfrac{\omega_{2h,k}}{\omega_{4h,k}^2}e^{\frac{\pi ih}{2k}} & \text{if } 2\nmid k,\\ 0 & \text{if } 2\|k.\end{cases}$$

We next use the well-known behavior of $\widehat{h}_j$ under inversion and translation

$$\widehat{h}_0(\tau+1)=\widehat{h}_0(\tau) \qquad \widehat{h}_1(\tau+1)=-i\widehat{h}_1(\tau)$$
$$\widehat{h}_0\left(-\frac{1}{\tau}\right)=\tau^{\frac{3}{2}}\frac{(1+i)}{2}\left(\widehat{h}_0(\tau)+\widehat{h}_1(\tau)\right) \qquad \widehat{h}_1\left(-\frac{1}{\tau}\right)=\tau^{\frac{3}{2}}\frac{(1+i)}{2}\left(\widehat{h}_0(\tau)-\widehat{h}_1(\tau)\right).$$

This gives that $\widehat{h}_j$ has multiplier dual to the one of Θ_j. To be more precise we have

$$\widehat{h}_j\left(\frac{1}{k}(h+iz)\right)=-z^{-\frac{3}{2}}\sum_{\ell\in\{0,1\}}\overline{\chi_{j\ell}(h,h',k)}\,\widehat{h}_\ell\left(\frac{1}{k}\left(h'+\frac{i}{z}\right)\right).$$

From this, a straightforward calculation shows that

$$h_j\left(\frac{1}{k}(h+iz)\right)=-z^{-\frac{3}{2}}\sum_{\ell\in\{0,1\}}\overline{\chi_{j\ell}(h,h',k)}h_\ell\left(\frac{1}{k}\left(h'+\frac{i}{z}\right)\right) - \frac{1}{4\sqrt{2}\pi}z^{-\frac{3}{2}}\int_0^\infty\frac{\sum_{\ell\in\{0,1\}}\overline{\chi_{j\ell}(h,h',k)}\Theta_\ell\left(it-\frac{h'}{k}\right)}{\left(t+\frac{1}{kz}\right)^{\frac{3}{2}}}dt. \tag{3.2}$$

Defining

$$\mathcal{I}_j(x):=\int_0^\infty\frac{\Theta_j(iw-\frac{h'}{k})}{(w+x)^{\frac{3}{2}}}\,dw$$

we may rewrite (3.2) as

$$h_j\left(\frac{1}{k}(h+iz)\right)=-\,z^{-\frac{3}{2}}\sum_{\ell\in\{0,1\}}\overline{\chi_{j\ell}(h,h',k)}h_\ell\left(\frac{1}{k}\left(h'+\frac{i}{z}\right)\right) - \frac{1}{4\sqrt{2}\pi}z^{-\frac{3}{2}}\sum_{\ell\in\{0,1\}}\overline{\chi_{j\ell}(h,h',k)}\mathcal{I}_\ell\left(\frac{1}{kz}\right).$$

Dividing by η^6 and applying (3.1) yields that

$$f_j\left(\frac{1}{k}(h+iz)\right)=z^{\frac{3}{2}}e^{\frac{\pi ih'}{2k}-\frac{\pi i(j+1)h}{2k}}\sum_{\ell\in\{0,1\}}\psi_{j\ell}(h,h',k)f_\ell\left(\frac{1}{k}\left(h'+\frac{i}{z}\right)\right) + \frac{1}{4\sqrt{2}\pi}z^{\frac{3}{2}}e^{\frac{\pi ih'}{2k}-\frac{\pi i(j+1)h}{2k}}\sum_{\ell\in\{0,1\}}\psi_{j\ell}(h,h',k)\eta^{-6}\left(\frac{1}{k}\left(h'+\frac{i}{z}\right)\right)\mathcal{I}_\ell\left(\frac{1}{kz}\right) \tag{3.3}$$

with
(3.4)

$$\psi_{00}(h,h',k) := \begin{cases} -\frac{\omega_{h,k}^4\omega_{h,\frac{k}{2}}^5}{\omega_{h,\frac{k}{4}}^2} & \text{if } 4|k, \\ -\frac{1}{\sqrt{2}}\frac{\omega_{h,k}^4\omega_{2h,k}^5}{\omega_{4h,k}^2} & \text{if } 2\nmid k, \\ 0 & \text{if } 2\|k, \end{cases} \qquad \psi_{01}(h,h',k) := \begin{cases} 0 & \text{if } 4|k, \\ -\frac{1}{\sqrt{2}}\frac{\omega_{h,k}^4\omega_{2h,k}^5}{\omega_{4h,k}^2} & \text{if } 2\nmid k, \\ -\frac{\omega_{h,k}^4\omega_{h,\frac{k}{2}}^5}{\omega_{2h,\frac{k}{2}}^2}e^{\frac{\pi i h'}{2k}} & \text{if } 2\|k, \end{cases}$$

$$\psi_{10}(h,h',k) := \begin{cases} 0 & \text{if } 4|k, \\ -\frac{1}{\sqrt{2}}\frac{\omega_{h,k}^6\omega_{4h,k}^2}{\omega_{2h,k}} & \text{if } 2\nmid k, \\ -\frac{\omega_{h,k}^6\omega_{2h,\frac{k}{2}}^2}{\omega_{h,\frac{k}{2}}^2} & \text{if } 2\|k, \end{cases} \qquad \psi_{11}(h,h',k) := \begin{cases} -\frac{\omega_{h,k}^6\omega_{h,\frac{k}{4}}^2}{\omega_{h,\frac{k}{2}}^2}e^{\frac{\pi i h'}{2k}} & \text{if } 4|k, \\ \frac{1}{\sqrt{2}}\frac{\omega_{h,k}^6\omega_{4h,k}}{\omega_{2h,k}} & \text{if } 2\nmid k, \\ 0 & \text{if } 2\|k. \end{cases}$$

For later purposes we require a different representation of $I_\ell(x)$. Similarly as in [8], one can show

Lemma 3.1. *We have for $x \in \mathbb{C}$ with $Re(x) > 0$*

$$\mathcal{I}_j(x) = \sum_{\substack{g \pmod{2k} \\ g\equiv j \pmod 2}} e\left(-\frac{g^2h'}{4k}\right)\left(\frac{2\delta_{0,g}}{\sqrt{x}} - \frac{1}{\sqrt{2}\pi k^2 x}\int_{-\infty}^{\infty} e^{-2\pi x u^2} f_{k,g}(u)\,du\right), \tag{3.5}$$

where $\delta_{0,g} = 0$ unless $g \equiv 0 \pmod{2k}$ in which case it equals 1.

Note that we corrected a sign error in the statement of Lemma 4.4 of [8].

3.2. **Proof of Theorem 1.2.** Throughout this section, we use the notation from Subsection 3.1. For the proof of Theorem 1.2, we employ the Hardy-Ramanujan Circle Method [42] and write for $j \in \{0,1\}$

$$\widetilde{f}_j(q) := q^{\frac{j+1}{4}} f_j(\tau) = \sum_{n=0}^{\infty} \alpha_j(n) q^n.$$

By Cauchy's Theorem we have for $n > 0$

$$\alpha_j(n) = \frac{1}{2\pi i}\int_C \frac{\widetilde{f}_j(q)}{q^{n+1}}\,dq,$$

where C is an arbitrary path inside the unit circle looping around 0 counterclockwise. We choose the circle with radius $r = e^{\frac{-2\pi}{N^2}}$, where we later let $N \to \infty$, and decompose it into consecutive Farey arcs of order N:

$$\alpha_j(n) = \sum_{\substack{0\le h<k\le N \\ (h,k)=1}} e^{-2\pi i n\frac{h}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} \widetilde{f}_j\left(e^{\frac{-2\pi}{N^2}+2\pi i\frac{h}{k}+2\pi i\phi}\right) e^{\frac{2\pi n}{N^2}-2\pi i n\phi} d\phi$$

with:

$$\vartheta'_{h,k} := \frac{1}{k(k_1+k)}, \qquad \vartheta''_{h,k} := \frac{1}{k(k_2+k)},$$

where $\frac{h_1}{k_1} < \frac{h}{k} < \frac{h_2}{k_2}$ are adjacent Farey fractions in the Farey sequence of order N. From the theory of Farey fractions it is known that

$$\frac{1}{k+k_j} \leq \frac{1}{N+1} \qquad (j=1,2).$$

Using the transformation law (3.3) and $z = k(\frac{1}{N^2} - i\phi)$, we obtain

$$\alpha_j(n) = \sum_{\substack{0\leq h<k\leq N\\(h,k)=1}} e^{\frac{-2\pi i h n}{k}} \sum_{\ell\in\{0,1\}} \psi_{j\ell}(h,h',k) e^{\frac{\pi i h'}{2k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} f_\ell\left(\frac{1}{k}\left(h'+\frac{i}{z}\right)\right) e^{\frac{2\pi z}{k}\left(n-\frac{j+1}{4}\right)} z^{\frac{3}{2}}\, d\phi$$

$$+ \frac{1}{4\sqrt{2}\pi} \sum_{\substack{0\leq h<k\leq N\\(h,k)=1}} e^{\frac{-2\pi i h n}{k}} \sum_{\ell\in\{0,1\}} \psi_{j\ell}(h,h',k) e^{\frac{\pi i h'}{2k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} \frac{\mathcal{I}_\ell\left(\frac{1}{kz}\right)}{\eta^6\left(\frac{1}{k}\left(h'+\frac{i}{z}\right)\right)} e^{\frac{2\pi z}{k}\left(n-\frac{j+1}{4}\right)} z^{\frac{3}{2}}\, d\phi.$$

We will abbreviate the first summand by $\sum_1$ and the second by $\sum_2$.

We first consider $\sum_1$ and split of the terms with negative exponent in the Fourier expansion as they contribute to the main term. For this, we write

$$f_0(\tau) = -\frac{1}{12} q^{-\frac{1}{4}} + \sum_{n>0} b_0(n) q^{n-\frac{1}{4}},$$
$$f_1(\tau) = \sum_{n>0} b_1(n) q^{n-\frac{1}{2}}.$$

We denote the contributions of the negative exponent to $\sum_1$ by $\sum_1^*$. Using the estimates $\vartheta'_{h,k}, \vartheta''_{h,k} \ll \frac{1}{kN}$ and $|z|^2 \ll N^{-2}$ gives that

$$\sum_1 = \sum_1^* + O\left(N^{-\frac{5}{2}} \sum_{0\leq h<k\leq N} \frac{1}{k}\right) = \sum_1^* + O\left(N^{-\frac{3}{2}}\right).$$

To estimate $\sum_2$, we use the representation of $\mathcal{I}_\ell$ given in Lemma 3.1. We start with the contribution of the first summand in the representation of $\mathcal{I}_\ell$ (only occuring for $\ell = 0$ and $g \equiv 0 \pmod{2k}$). Splitting of the non-principal terms yields as before an error of $N^{-\frac{3}{2}}$. To estimate the remaining terms of $\sum_2$, we aim to estimate integrals of the shape

$$\mathcal{I}_{k,g,b}(z) := e^{\frac{2\pi b}{kz}} z^{\frac{5}{2}} \int_{-\infty}^{\infty} e^{-\frac{2\pi u^2}{kz}} f_{k,g}(u)\, du.$$

Similarly to the case of Fourier expansions we are interested in the "principal integral part" contribution. To be more precise, we let for $b > 0$ and $g \in \mathbb{Z}$,

$$\mathcal{J}_{k,g,b}(z) := e^{\frac{2\pi b}{kz}} z^{\frac{5}{2}} \int_{-\sqrt{b}}^{\sqrt{b}} e^{-\frac{2\pi u^2}{kz}} f_{k,g}(u)\, du.$$

Similarly as in [10], we may show:

Lemma 3.2. *As $z \to \infty$ we have for $-k < g \leq k$:*

(1) *If $b \leq 0$, then*

$$|\mathcal{I}_{k,g,b}(z)| \ll |z|^{\frac{5}{2}} \times \begin{cases} \frac{k^2}{g^2} & \text{if } g \neq 0, \\ 1 & \text{if } g = 0. \end{cases} \tag{3.6}$$

(2) *If $b > 0$, then*

$$\mathcal{I}_{k,g,b}(z) = \mathcal{J}_{k,g,b}(z) + \mathcal{E}_{k,g,b}(z),$$

where the error $\mathcal{E}_{k,g,b}$ satisfies the same estimate as $\mathcal{I}_{k,g,b}$ in (3.6).

Proof. Recall that $Re\left(\frac{1}{z}\right) \geq \frac{k}{2}$. We use the estimate

$$\left|\sinh\left(\frac{\pi u}{k} - \frac{\pi i g}{2k}\right)\right| = \left|\cosh\left(\frac{\pi u}{k} - \pi i\left(\frac{g}{2k} + \frac{1}{2}\right)\right)\right| \geq \left|\sin\left(\frac{\pi g}{2k}\right)\right|.$$

Now for $0 \leq x \leq \frac{\pi}{2}$, the function $\frac{\sin(x)}{x}$ is bounded from below, thus for $-k < g \leq k$, $g \neq 0$

$$\left|\frac{1}{\sin\left(\frac{\pi g}{2k}\right)^2}\right| \ll \frac{k^2}{g^2}.$$

Moreover

$$\left|\frac{1}{\sinh^2(x)} - \frac{1}{x^2}\right| = \frac{1}{x^2} - \frac{1}{\sinh^2(x)} \leq 1.$$

We now define $h_{g,k}$ as

$$h_{g,k} := \begin{cases} \frac{k^2}{g^2} & \text{if } -k < g \leq k, g \neq 0, \\ 1 & \text{if } g = 0. \end{cases}$$

Then by the above

$$|f_{k,g}(u)| \leq h_{g,k}.$$

We now first assume that $b \leq 0$. Then

$$|I_{k,g,b}(z)| \leq |z|^{\frac{5}{2}} h_{g,k} \int_{-\infty}^{\infty} e^{-\frac{2\pi u^2}{k} Re\left(\frac{1}{z}\right)} du \ll |z|^{\frac{5}{2}} h_{g,k} \sqrt{\frac{k}{Re\left(\frac{1}{z}\right)}} \ll |z|^{\frac{5}{2}} h_{g,k}$$

which gives the claim for $b \leq 0$. The case $b > 0$ works similarly. □

The contribution of non-principal part of the remaining terms of $\sum_3$ may be estimated against a constant times

$$\sum_{h,k} \frac{1}{k} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} \sum_g |\mathcal{I}_{k,g,0}(z)|\, d\phi \ll \sum_{h,k} \frac{1}{k} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} |z|^{\frac{5}{2}} \left(1 + \sum_{g=1}^{k} \frac{k^2}{g^2}\right) d\phi \ll N^{-\frac{3}{2}}.$$

In the terms coming from the principal part, we may similarly truncate the integral to lead $\mathcal{J}_{k,g,\frac{1}{4}}(z)$. Combining the above, we have shown that

$$\alpha_j(n) = S_1 + S_2 + S_3 + O\left(N^{-\frac{3}{2}}\right)$$

with

$$S_1 := -\frac{1}{12}\sum_{\substack{0\le h<k\le N\\(h,k)=1}} e^{\frac{-2\pi i h n}{k}}\psi_{j0}(h,h',k)\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{2\pi z}{k}\left(n-\frac{(j+1)}{4}\right)+\frac{\pi}{2kz}} z^{\frac{3}{2}}d\phi,$$

$$S_2 := \frac{1}{2\sqrt{2}\pi}\sum_{\substack{0\le h<k\le N\\(h,k)=1}} \sqrt{k}e^{\frac{-2\pi i h n}{k}}\psi_{j0}(h,h',k)\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{2\pi z}{k}\left(n-\frac{j+1}{4}\right)+\frac{\pi}{2kz}} z^{2}d\phi,$$

$$S_3 := -\frac{1}{8\pi^2}\sum_{\substack{0\le h<k\le N\\(h,k)=1}} \frac{1}{k}\, e^{\frac{-2\pi i h n}{k}}\sum_{\substack{\ell\in\{0,1\}\\-k<g\le k\\ g\equiv \ell \pmod 2}}\psi_{j\ell}(h,h',k)e\left(\frac{-g^2h'}{4k}\right)\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{2\pi z}{k}\left(n-\frac{j+1}{4}\right)}\mathcal{J}_{k,g,\frac{1}{4}}(z)d\phi.$$

We next write the path of integration in a symmetrized way

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} = \int_{-\frac{1}{kN}}^{\frac{1}{kN}} - \int_{-\frac{1}{kN}}^{-\frac{1}{k(k+k_1)}} - \int_{\frac{1}{k(k+k_2)}}^{\frac{1}{kN}} .$$

The second and third term contribute to the error term and may be estimated as before. To finish the proof, we require estimates for integrals of the form ($r>0$)

$$\mathcal{I}_{k,r,n,m} := \int_{-\frac{1}{kN}}^{\frac{1}{kN}} z^r e^{\frac{2\pi}{k}\left(nz+\frac{m}{z}\right)}d\phi.$$

In a standard way (we refer the reader to [5] for the details) one may show that

$$\mathcal{I}_{k,r,n,m} = \frac{2\pi}{k}\left(\frac{m}{n}\right)^{\frac{r+1}{2}} I_{r+1}\left(\frac{4\pi}{k}\sqrt{nm}\right) + O\left(\frac{1}{kN^{r+1}}\right).$$

Inserting this bound into the S_i and letting $N\to\infty$ now easily gives the claim. □

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Mathematical Institute, University of Cologne, Weyertal 86-90, 50931 Cologne, Germany
E-mail address: kbringma@math.uni-koeln.de

Institut de Physique Théorique, CEA Saclay, CNRS-URA 2306, 91191 Gif sur Yvette, France
E-mail address: manschot@uni-bonn.de