PARTITION STATISTICS AND QUASIHARMONIC MAASS FORMS

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Abstract. Andrews recently introduced k-marked Durfee symbols, which are a generalization of partitions that are connected to moments of Dyson's rank statistic. He used these connections to find identities relating their generating functions as well as to prove Ramanujan-type congruences for these objects and find relations between.

In this paper we show that the hypergeometric generating functions for these objects are natural examples of *quasimock theta functions*, which are defined as the holomorphic parts of harmonic Maass forms and their derivatives. In particular, these generating functions may be viewed as analogs of Ramanujan's mock theta functions with arbitrarily high weight. We use the automorphic properties to prove the existence of infinitely many congruences for the Durfee symbols. Furthermore, we show that as k varies, the modularity of the k-marked Durfee symbols is precisely dictated by the case $k = 2$. Finally, we use this relation in order to prove the existence of general congruences for rank moments in terms of level one modular forms of bounded weight.

1. Introduction and Statement of results

Modular and automorphic forms play an important role in many different areas, including mathematical physics, representation theory, the theory of elliptic curves, quadratic forms, and partitions, just to mention a few. Many important generating functions are modular forms, and the modular transformation properties can often be used to prove arithmetic properties for the underlying combinatorial objects [17]. There are many other examples of modular forms to be found in the realm of hypergeometric q-series, such as the infinite products in the Rogers-Ramanujan identities. Until recently, however, there were few known examples of more general automorphic forms arising from similar "arithmetic" generating functions. Work of the first author and Ono [9, 10] (see also [5, 7, 8, 11]) constructed several infinite families of harmonic Maass forms of weights $1/2$ and $3/2$, whose holomorphic parts were based on Ramanujan's mock theta functions and also more general hypergeometric functions.

With the benefit of retrospect, we may view the error terms in the mock theta transformations as suggesting that the appropriate functions to consider were not modular forms, but the more general harmonic Maass forms. In this paper we study higher weight analogs that are based on Andrews' work on Durfee symbols [1]. The main result of this paper is that the associated generating functions can be written in terms of automorphic functions of higher weights, namely the derivatives of Maass forms. Although these derivatives are difficult to understand on their own, we show that there are cancellations among certain linear combinations of derivatives of different orders, and are able to successfully describe the analytic behavior of the moment functions. Furthermore, this allows us to better understand the arithmetic of the coefficients; as a sample application we prove the existence of congruences.

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In order to define the new objects at hand, first recall the generating function for the partition function,

$$P(q) := \sum_{n=0}^{\infty} p(n)\, q^n = q^{\frac{1}{24}} \eta(z)^{-1}, \tag{1.1}$$

where $\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$ is Dedekind's η-function, a weight $\frac{1}{2}$ modular form, and $q := e^{2\pi i z}$. Of the the many consequences of the modularity properties of $P(q)$, some of the most striking are the three congruences due to Ramanujan, namely

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

To explain the congruences with modulus 5 and 7, Dyson [12] introduced the *rank* of a partition, which is defined to be its largest part minus the number of its parts. Dyson conjectured that the partitions of $5n+4$ (resp. $7n+5$) form 5 (resp. 7) groups of equal size when sorted by their ranks modulo 5 (resp. 7). This conjecture was proven by Atkin and Swinnerton-Dyer [3]. If $N(m,n)$ denotes the number of partitions of n with rank m, then we have the generating function

$$R(w;q) := 1 + \sum_{m\in\mathbb{Z}} \sum_{n=1}^{\infty} N(m,n) w^m q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(wq;q)_n (w^{-1}q;q)_n} = \frac{(1-w)}{(q;q)_\infty} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\frac{n}{2}(3n+1)}}{1-wq^n},$$

where $(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j)$ and $(a;q)_\infty := \lim_{n\to\infty}(a;q)_n$. In particular

$$\begin{aligned} R(1;q) &= \mathcal{P}(q), \\ R(-1;q) &= f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}. \end{aligned}$$

The function $f(q)$ is one of the *mock theta functions* defined by Ramanujan in his last letter to Hardy. The first author and Ono shed light on their mysteries by showing that if w is a root of unity, then the rank generating functions $R(w;q)$ (and in particular $f(q)$) are the "holomorphic parts" of harmonic Maass forms [10] (we say more on these results and recall the notion of a harmonic Maass form in Section 3). The theory of harmonic Maass forms proved to be very useful for understanding the arithmetic of the coefficients, leading to many notable results. These include, for example, an exact formula for the coefficients of $f(q)$ [9], asymptotics for $N(m,n)$ [6], identities for rank differences [11], and congruences for certain partition statistics [10].

Here we consider infinite families of harmonic Maass forms of arbitrarily high half-integer weight that also arise from combinatorial hypergeometric functions. To state those results recall that Andrews introduced in [1] the *symmetrized k-th rank moment function*

$$\eta_k(n) := \sum_{m=-\infty}^{\infty} \binom{m + \left[\frac{k-1}{2}\right]}{k} N(m,n), \tag{1.2}$$

which are linear combinations of the *k-th rank moments*

$$N_k(n) := \sum_{m=-\infty}^{\infty} m^k N(m,n) \tag{1.3}$$

considered by Atkin and the second author [2]. Using the rank symmetry $N(-m,n) = N(m,n)$, Andrews showed that $\eta_{2k+1}(n) = 0$, and thus we need only consider even rank moments. For these we define the rank generating function

$$R_{k+1}(q) := \sum_{n=0}^{\infty} \eta_{2k}(n)\, q^n.$$

The function $R_2(q)$ was studied in detail by the first author in [5]. One of the key results relates $R_2(q)$ to a certain harmonic Maass form (see Section 3), but the connection is more complicated than in the case of usual ranks due to double poles in the generating function. This leads to expressions involving *quasimodular forms*, which are meromorphic functions $f : \mathbb{H} \to \mathbb{C}$ that can written as a linear combination of derivatives of modular forms. Furthermore, asymptotics and congruences for $\eta_2(n)$ are obtained as applications of the modularity of the generating function.

In the present work we consider the case of general k. The functions that arise in this setting require yet a more general analytic definition; we say that $f : \mathbb{H} \to \mathbb{C}$ is a *quasimock theta function* if there exists a quasimodular form $h(q)$ such that $f(q) + h(q)$ is a linear combination of derivatives of the holomorphic parts of harmonic Maass forms. Moreover we call linear combinations of derivatives of harmonic Maass forms *quasiharmonic Maass form.*

Remark. The authors thank Don Zagier for pointing out that the forms considered in this paper lie in the differential closure containing E_2 and F_2, where F_2 is defined in [20]. This can be understood analogous to the classical situation of quasimodular forms which can be defined as differential closure of the ring containing modular forms and E_2 [16].

Theorem 1.1. *The function $q^{-1}R_{k+1}(q^{24})$ is a quasimock theta function.*

Remark. The highest weight component includes a harmonic Maass form of weight $2k - 1/2$.

The idea of the proof of Theorem 1.1 is to relate certain rank and crank moments via a differential equation (see Section 4), and then argue inductively, using the fact that the crank moment generating functions are quasimodular forms. The base case $k = 1$ is considered in [5], although our induction step actually requires a new "twisted" version of those results.

Theorem 1.1 has many applications. We only address some of these here. We first consider congruences for partition statistics. For this we let $NF_k(r,t;n)$ be the number of k-marked Durfee symbols of size n with full rank congruent to r modulo t (see Section 2). We show that the full rank satisfies infinitely many congruences, just as the first author and Ono proved for Dyson's original rank [10]. The case $k = 2$ of the following theorem was proven in [5].

Theorem 1.2. *Let t be a positive odd integer, suppose that $j \in \mathbb{N}$, $k \geq 3$, and let $\mathcal{Q} \nmid 6t$ a prime. Then there exist infinitely many arithmetic progressions $An + B$, such that for every $0 \leq r < t$, we have*

$$NF_k(r,t;An+B) \equiv 0 \pmod{\mathcal{Q}^j}.$$

For the proof of Theorem 1.2, we employ the fact that the rank generating functions are holomorphic parts of harmonic Maass forms, and also the conclusion from Theorem 1.1 for R_k. Additional complications arise in our proof if t has a prime divisor p_t that is small relative to k (specifically, $p_t \leq 2k$). To resolve this case, we extend a result from [5] and prove the modularity of a certain "twist" of the second moment function (see Sections 4 and 5).

One nice consequence of Theorem 1.2 is a combinatorial decomposition of congruences for $\eta_{2k}(n)$.

Corollary 1.3. *Let $j \in \mathbb{N}$ and $Q > 3$ a prime. Then there exist infinitely many arithmetic progressions $An + B$ such that*

$$\eta_{2k}(An + B) \equiv 0 \pmod{\mathcal{Q}^j}.$$

To illustrate the nature of these arithmetic progressions, we give some of the simpler examples

$$\begin{aligned} \eta_2(11^3 n + 479) &\equiv 0 \pmod{11}, \\ \eta_4(11n) &\equiv 0 \pmod{11}, \\ \eta_6(49n + 19) &\equiv 0 \pmod{7}, \\ \eta_8(13^2 n + 162) &\equiv 0 \pmod{13}. \end{aligned}$$

See [13] for a detailed explanation of the method used to find these explicit congruences. A significant part of the technique is in showing that the rank moment generating functions are congruent modulo $\mathcal{Q}$ to modular forms over a restricted set of coefficients. This is of independent interest, and is described precisely in the following theorem.

Theorem 1.4. *Suppose $\ell > 3$ is prime. Define $1 \leq \beta_\ell \leq \ell - 1$ such that $24\beta_\ell \equiv 1 \pmod{\ell}$ and let $r_\ell := \frac{24\beta_\ell - 1}{\ell}$.*

(1) *The generating function for the second rank moment satisfies*

$$\sum_{n=0}^{\infty} N_2(\ell n + \beta_\ell) q^{24n + r_\ell} \equiv \eta^{r_\ell}(24z) G_{\ell,2}(24z) \pmod{\ell},$$

where $G_{\ell,2}(z)$ is a sum of level 1 modular forms with ℓ-integral coefficients, each of weight at most $\frac{\ell(\ell+3) - r_\ell - 1}{2}$.

(2) *For $2 \leq k \leq \frac{\ell - 3}{2}$,*

$$\sum_{n=0}^{\infty} N_{2k}(\ell n + \beta_\ell) q^{24n + r_\ell} \equiv c_k \sum_{n=0}^{\infty} N_2(\ell n + \beta_\ell) q^{24n + r_\ell} + \eta^{r_\ell}(24z) G_{\ell,2k}(24z) \pmod{\ell},$$

where c_k is an integer, and $G_{\ell,2k}(z)$ is a sum of level 1 modular forms with ℓ-integral coefficients and weight at most $k(\ell + 1) - 1 + \frac{1}{2}(\ell - r_\ell)$.

(3) *Finally,*

$$\begin{aligned} \sum_{n=0}^{\infty} N_{\ell-1}(\ell n + \beta_\ell) q^{24n + r_\ell} \equiv c_{\ell-1} &\sum_{n=0}^{\infty} N_2(\ell n + \beta_\ell) q^{24n + r_\ell} + \eta^{r_\ell}(24z) G_{\ell,\ell-1}(24z) \\ &+ \frac{1}{\ell} \eta^{r_\ell}(24z) \left(H_{1,\ell}(24z) - H_{2,\ell}(24z) \right) \pmod{\ell}, \end{aligned}$$

where $G_{\ell,\ell-1}(z)$ is a sum of level 1 integral modular forms of weight at most $\frac{\ell(\ell+1) - r_\ell - 3}{2}$; $c_{\ell-1}$ is some integer; and $H_{1,\ell}(z)$, $H_{2,\ell}(z)$ are integral modular forms of weight $\frac{\ell(\ell-1) - r_\ell - 1}{2}$ and $\frac{\ell(\ell+1) - r_\ell - 3}{2}$ respectively such that

$$H_{1,\ell}(z) \equiv H_{2,\ell}(z) \pmod{\ell}.$$

We illustrate parts (1) and (2) of this theorem for the case $\ell = 11$ (see [13])

$$\sum_{n=0}^{\infty} N_2(11n+6)q^{24n+13} \equiv 3\eta^{13}(24z) \pmod{11},$$
$$\sum_{n=0}^{\infty} N_4(11n+6)q^{24n+13} \equiv 7\eta^{13}(24z) \pmod{11},$$
$$\sum_{n=0}^{\infty} N_6(11n+6)q^{24n+13} \equiv \eta^{13}(24z)(4+E_4(24z)) \pmod{11},$$
$$\sum_{n=0}^{\infty} N_8(11n+6)q^{24n+13} \equiv \eta^{13}(24z)(5+6E_4(24z)+6E_6(24z)) \pmod{11}.$$

The theory of harmonic Maass forms can also be employed to show identities for differences of rank moments. Relations between non-holomorphic parts are responsible for the existence of such identities. To state our results, let

$$R^{(k)}_{r,s,t,d}(q) := \sum_{n=1}^{\infty} \left(NF_k(r,t;tn+d) - NF_k(s,t;tn+d)\right) q^{24(tn+d)-1}.$$

We show that in certain cases this function is a weakly holomorphic modular form whose poles (if there are any) are supported on the cusps. Similar results for Dyson's rank were shown in [11].

Theorem 1.5. *Assume that $t \geq 5$ is a prime, $0 \leq r, s < t$, and $0 \leq d < t$. Then the following are true.*

(1) *If $\left(\frac{1-24d}{t}\right) = -1$, then $R^{(k)}_{r,s,t,d}(q)$ is a quasimodular form on $\Gamma_1(576t^6)$. If $k \leq \frac{p_t}{2}$, then it is weakly holomorphic.*
(2) *If $\left(\frac{1-24d}{t}\right) = 1$, and $2r, 2s \not\equiv 3(1-d+2u) \pmod{2t}$, $r, s \not\equiv 2+3u, 1+3u, 2-3d+3u, 1-3d+3u \pmod{t}$ for all $0 \leq u \leq d-1$, then $R^{(2)}_{r,s,t,3d}(q)$ is a weakly holomorphic modular form on $\Gamma_1(576t^6)$.*

One can use Theorem 1.5 along with the valence formula to prove concrete identities.

Remark. For each t there exists at least one $0 \leq d < t$ such that the statement of Theorem 1.5 is nontrivial.

The paper is organized as follows. In Section 2 we recall facts about marked Durfee symbols. In Section 3, we give the connection between rank generating functions and harmonic Maass forms. We then relate rank and crank moments via a differential equation in Section 4. Next, in Section 5 we introduce a certain twisted moment function that shows up in the case $k \geq \frac{p_t}{2}$. Section 6 is devoted the proofs of Theorems 1.1 and 1.2. In Section 7 we prove Theorem 1.4 by first making a detailed ℓ-adic analysis of the rank-crank moment relation. Section 8 provides identities for rank differences.

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2. Combinatorial results on marked Durfee symbols

Here we give some of the facts about marked Durfee symbols shown in [1]. Recall that the largest square of nodes in the Ferrers graph of a partition is called the *Durfee square.* The *Durfee symbol* consists of 2 rows and a subscript, where the top row consists of the columns to the right of the Durfee square, the bottom row consists of the rows below the Durfee square and the subscript denotes the side length of the Durfee square. The number being partitioned is equal to the sum of the rows of the symbol plus the number of nodes in the Durfee square. Note that the parts in both rows must be non-increasing. As an example, the Durfee symbol

$$\begin{pmatrix} 2 & & \\ 3 & 3 & 1 \end{pmatrix}_4$$

represents a partition of $2 + 3 + 3 + 1 + 4^2 = 25$.

Andrews defined *k-marked Durfee symbols* by using k distinct copies (or colors) of the integers designated by $\{1_1, 2_1, \cdots\}$, $\{1_2, 2_2, \cdots\}, \cdots, \{1_k, 2_k, \cdots\}$. We form Durfee symbols as before and use the k copies of integers for parts in both rows. We additionally demand that:

(1) The sequence of subscripts in each row are non-increasing.
(2) Each of the subscript $1, \cdots, k-1$ occurs at least once in the top row.
(3) If $M_1, \cdots, M_{k-1}$ are the largest parts with their respective subscripts in the top row, then all parts in the bottom row with subscript 1 lie in $[1, M_1]$, with subscript 2 lie in $[M_1, M_2], \cdots$, and with subscript k lie in $[M_{k-1}, S]$, where S is the side of the Durfee square.

The *size* of a k-marked Durfee symbol is simply the size of the partition that is obtained by ignoring the colors; we let $\mathcal{D}_k(n)$ denote the number of k-marked Durfee symbols of size n.

In [1] Andrews showed that k-marked Durfee symbols arise naturally in the combinatorial study of the rank moment functions; in particular, for $k \geq 1$,

$$\mathcal{D}_{k+1}(n) = \eta_{2k}(n).$$

He also proved some striking congruences for Durfee symbols, including

$$\begin{aligned} \mathcal{D}_2(5n + a) &\equiv 0 \pmod 5 \qquad a \in \{1, 4\}, \\ \mathcal{D}_2(7n + a) &\equiv 0 \pmod 7 \qquad a \in \{1, 5\}, \\ \mathcal{D}_3(7n + a) &\equiv 0 \pmod 7 \qquad a \in \{1, 5\}. \end{aligned}$$

Following Dyson's lead, Andrews next associated a collection of "ranks" to k-marked Durfee symbols. For such a Durfee symbol δ, we define the *full rank* $FR(\delta)$ by

$$FR(\delta) := \rho_1(\delta) + \cdots + k\rho_k(\delta),$$

where the i-th rank $\rho_i(\delta)$ is given by

$$\rho_i(\delta) := \begin{cases} \tau_i(\delta) - \beta_i(\delta) - 1 & \text{for } 1 \leq i < k, \\ \tau_i(\delta) - \beta_i(\delta) & \text{for } i = k. \end{cases}$$

Here $\tau_i(\delta)$ (resp. $\beta_i(\delta)$) denotes the number of entries in the top (resp. bottom) row of δ with subscript i. We let $NF_k(m, n)$ denote the number of k-marked Durfee symbols of size n with full rank m, and $NF_k(r, t; n)$ denote the number of k-marked Durfee symbols of size n with full rank congruent to r modulo t. Finally, define the generating function

$$R_k(w;q) := \sum_{n=1}^{\infty} \sum_{m \in \mathbb{Z}} NF_k(m,n)\, w^m\, q^n.$$

In particular

$$R_k(1;q) = R_k(q).$$

3. Ranks and harmonic Maass forms

Here we recall results of [10] and [5]. Let us first give the definition of a harmonic Maass form. If $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, $z = x+iy$ with $x, y \in \mathbb{R}$, then the weight k hyperbolic Laplacian is given by

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right). \tag{3.1}$$

If v is odd, then define ϵ_v by

$$\epsilon_v := \begin{cases} 1 & \text{if } v \equiv 1 \pmod 4, \\ i & \text{if } v \equiv 3 \pmod 4. \end{cases} \tag{3.2}$$

Moreover, we let χ be a Dirichlet character.

An *harmonic Maass form of weight k with Nebentypus χ on a subgroup* $\Gamma \subset \Gamma_0(4)$ is any smooth function $g : \mathbb{H} \to \mathbb{C}$ satisfying the following:

(1) For all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and all $z \in \mathbb{H}$, we have

$$g(Az) = \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} \chi(d)\, (cz+d)^k\ g(z).$$

(2) We have that $\Delta_k g = 0$.

(3) The function $g(z)$ has at most linear exponential growth at all the cusps of Γ.

Now let $0 < a < c$ and define

$$\begin{aligned} D\left(\frac{a}{c};q\right) &:= -S\left(\frac{a}{c};z\right) + q^{-\frac{\ell_c}{24}} R\left(\zeta_c^a; q^{\ell_c}\right), \\ S\left(\frac{a}{c};z\right) &:= -\frac{i \sin\left(\frac{\pi a}{c}\right) \ell_c^{\frac{1}{2}}}{\sqrt{3}} \int_{-\bar{z}}^{i\infty} \frac{\Theta\left(\frac{a}{c};\ell_c\tau\right)}{\sqrt{i(\tau+z)}}\, d\tau, \end{aligned}$$

where $\ell_c := \mathrm{lcm}(2c^2, 24)$, $\zeta_c := e^{\frac{2\pi i}{c}}$, and $\Theta\left(\frac{a}{c};\tau\right)$ is a certain weight $\frac{3}{2}$ cuspidal theta function (for the exact definition see [10]).

Theorem 3.1. *If $0 < a < c$, then $D\left(\frac{a}{c};q\right)$ is a harmonic Maass form of weight $\frac{1}{2}$ on $\Gamma_c := \left\langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ \ell_c^2 & 1 \end{smallmatrix}\right) \right\rangle$. If c is odd, then it is on $\Gamma_1\left(6f_c^2 l_c\right)$, where $f_c := \frac{2c}{\gcd(c,6)}$. Its non-holomorphic part has the expansion*

$$-\frac{2}{\sqrt{\pi}} \sin\left(\frac{\pi a}{c}\right) \sum_{m \pmod{f_c}} (-1)^m \sin\left(\frac{\pi a(6m+1)}{c}\right) \sum_{n \equiv 6m+1 \pmod{6f_c}} \Gamma\left(\frac{1}{2}; \frac{\ell_c n^2 y}{6}\right) q^{-\frac{\ell_c n^2}{24}},$$

where

$$\Gamma(\alpha;x) := \int_x^{\infty} e^{-t}\, t^{a-1}\, dt.$$

We next turn to $R_2(q)$. Define

$$\mathcal{R}(q) := R_2\left(q^{24}\right) q^{-1},$$

$$\mathcal{N}(z) := \frac{i}{4\sqrt{2}\pi}\int_{-\bar{z}}^{i\infty} \frac{\eta(24\tau)}{(-i(\tau+z))^{\frac{3}{2}}}\, d\tau, \tag{3.3}$$

and

$$\mathcal{M}(z) := \mathcal{R}(q) - \mathcal{N}(z) - \frac{1}{24\eta(24z)} + \frac{E_2(24z)}{8\eta(24z)}, \tag{3.4}$$

where as usual

$$E_2(z) := 1 - 24\sum_{n=1}^{\infty} \sigma_1(n)\, q^n$$

with $\sigma_1(n) := \sum_{d|n} d$. This function is a quasimodular form.

The following result is shown in [5].

Theorem 3.2. *The function $\mathcal{M}(z)$ is a harmonic Maass form of weight $\frac{3}{2}$ on $\Gamma_0(576)$ with Nebentypus character $\chi_{12} := \left(\frac{12}{\cdot}\right)$. Its non-holomorphic part has the expansion*

$$\mathcal{N}(z) = \frac{1}{4\sqrt{\pi}}\sum_{k\in\mathbb{Z}}(-1)^k(6k+1)\Gamma\left(-\frac{1}{2}; 4\pi(6k+1)^2 y\right)\, q^{-(6k+1)^2}.$$

4. Relation between rank and crank moments

In this section we recall certain relations between rank and crank moments and consider twisted generalisations. For details we refer the reader to [2]. The j-th rank moment N_j is defined in (1.3). Note that the symmetrized moment $\eta_{2k}(n)$ can easily be written as a linear combination of $N_{2j}(n)$ with $j \leq k$ (again, $N_{2j+1}(n) = 0$ due to symmetry). Define the generating function

$$\mathcal{R}_j(q) \;\; := \;\; \sum_{n\geq 1} N_j(n) q^n.$$

We will see that these functions are related to certain quasiharmonic Maass forms.

We next consider crank moments. Recall that the *crank* of a partition is defined to be the largest part if the partition contains no ones, and is otherwise the difference between the number of parts larger than the number of ones and the number of ones. For $n > 1$, we denote by $M(m,n)$ the number of partitions of n with crank equal to m, and define the boundary values by $M(0,1) := -1$, $M(-1,1) := M(1,1) := 1$, with $M(m,1) := 0$ otherwise. The generating function for the crank is then

$$C(w;q) := \sum_{n\geq 0}\sum_{m\in\mathbb{Z}} M(m,n)\, w^m\, q^n = \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-wq^n)\,(1-w^{-1}q^n)}.$$

This function is essentially a modular form when w is the root of unity ζ_c^a. The numerator is $\eta(z)q^{-1/24}$, and the denominator is an algebraic integer times a weight zero *Siegel function* of level $2c^2$ (see [14]); this implies that $q^{-1}C\left(\zeta_c^a; q^{24}\right)$ is a weight $\frac{1}{2}$ weakly holomorphic modular form on $\Gamma_1(2\cdot \mathrm{lcm}(c^2, 288))$.

Analogous to the development above, define the *j-th crank moment* as

$$M_j(n) := \sum_{k\in\mathbb{Z}} k^j M(k,n),$$

which again satisfies $M_{2j+1}(n) = 0$. Denote the crank moment generating function by

$$C_j(q) \ := \ \sum_{n\geq 1} M_j(n) q^n.$$

Now define the differential operators

$$\begin{aligned} \delta_q = \delta_z \ &:= \ q\,\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{dz}, \\ \delta_w \ &:= \ w\frac{d}{dw}. \end{aligned}$$

In [2], Atkin and the second author derived a recurrence relation for the functions C_a:

$$C_a(q) = 2\sum_{j=1}^{\frac{a}{2}-1} \binom{a-1}{2j-1} \Phi_{2j-1}(q) C_{a-2j}(q) + 2\Phi_{a-1}(q)P(q). \tag{4.1}$$

Here

$$\Phi_{2j-1}(q) := \sum_{n=1}^{\infty} \sigma_{2j-1}(n) q^n,$$

where $\sigma_j(n) := \sum_{d|n} d^j$. These functions are simply a rescaling of the classical weight j Eisenstein series minus their constant terms, since

$$E_j(z) := 1 - \frac{2j}{B_j}\Phi_{j-1}(q),$$

where B_j is the jth Bernoulli number. For even $j > 2$, the function $E_j(z)$ is a modular form of level 1, whereas $E_2(z)$ is a quasimodular form. Thus we conclude inductively from (4.1) that $q^{-1}C_a(q^{24})$ is a quasimodular form.

Atkin and the second author also proved a differential equation for the crank and rank generating functions, called the "rank-crank PDE":

$$w(q;q)_\infty^2 C(w;q)^3 = \left(3(1-w)^2\delta_q + \frac{1}{2}(1-w)^2\delta_w^2 - \frac{1}{2}(w^2-1)\delta_w + w\right) R(w;q). \tag{4.2}$$

For our current purposes, we are most interested in an identity that they derived by repeatedly applying δ_w to (4.2) and setting $w = 1$, namely that for $a \geq 2$,

$$\sum_{i=0}^{a/2-1} \binom{a}{2i} \sum_{\substack{\alpha+\beta+\gamma=a-2i \\ \alpha,\beta,\gamma\geq 0 \text{ even}}} \binom{a-2i}{\alpha,\beta,\gamma} C_\alpha(z)C_\beta(z)C_\gamma(z)P^{-2}(z) - 3(2^{a-1}-1)C_2(z)$$
$$= \frac{1}{2}(a-1)(a-2)\mathcal{R}_a(z) + 6\sum_{i=1}^{a/2-1}\binom{a}{2i}(2^{2i-1}-1)\delta_q\left(\mathcal{R}_{a-2i}(z)\right)$$
$$+ \sum_{i=1}^{a/2-1}\left[\binom{a}{2i+2}(2^{2i+1}-1) - 2^{2i}\binom{a}{2i+1} + \binom{a}{2i}\right]\mathcal{R}_{a-2i}(z). \tag{4.3}$$

Therefore modularity properties of $\mathcal{R}_a$ can be inductively concluded from modularity properties of C_a and $\mathcal{R}_2$. In particular, the functions $\mathcal{R}_a$ are related to the quasiharmonic Maass forms that we describe in more detail later.

We next turn to twisted rank and crank moments. We have

$$\delta_w\left(C(w;q)\right) = L(w;q)\,C(w;q),$$

where

$$L(w;q) := \sum_{n,m\geq 1}\left(w^m q^{nm} - w^{-m}q^{nm}\right).$$

Thus

$$\left[\delta_w^j L(w;q)\right]_{w=\zeta} = \sum_{n,m\geq 1}\left(m^j\zeta^m q^{nm} - (-m)^j\zeta^{-m}q^{nm}\right). \tag{4.4}$$

Now recall the theory of Eisenstein series on congruence subgroups (see section III.3 in [15]). The $(0,a)$ Eisenstein series of weight $j+1$ and level c is given by

$$G_{j+1}^{(0,a)}(z) := b_{j+1}^{(0,a)} + c_{j+1}\sum_{n\geq 1}\left(\sum_{d|n} d^j\left(\zeta_c^{ad} - (-1)^j\zeta_c^{-ad}\right)\right)q^n,$$

where

$$b_{j+1}^{(0,a)} = \sum_{\substack{n\geq 1\\ n\equiv a \pmod{c}}} n^{-j-1} + \sum_{\substack{n\geq 1\\ n\equiv -a \pmod{c}}} (-n)^{-j-1}.$$

and

$$c_{j+1} = \frac{2(j+1)(-1)^j\zeta(j+1)}{c^{j+1}B_{j+1}}.$$

These Eisenstein series are in $M_{j+1}(\Gamma_1(c))$ for $j \geq 2$ (as before, they may be quasimodular at weight 2), and thus the series from (4.4) is again a rescaled quasimodular form minus its constant coefficient. Note also that the constant $b_{j+1}^{(0,a)}/c_{j+1}$ must be an algebraic integer since all of the other terms in the rescaled series are.

Using (4.3) and results from Section 5, one can prove modularity properties of the *twisted moment functions*

$$\mathcal{R}_{j,a,c}(q) := \sum_{n\in\mathbb{N}} \mathcal{R}_j\left(\frac{a}{c};n\right) q^n,$$
$$C_{j,a,c}(q) := \sum_{n\in\mathbb{N}} C_j\left(\frac{a}{c};n\right) q^n,$$

where

$$\mathcal{R}_j\left(\frac{a}{c};n\right) := \sum_{k\in\mathbb{Z}} k^j \zeta_c^{ak} N(k,n),$$
$$C_j\left(\frac{a}{c};n\right) := \sum_{k\in\mathbb{Z}} k^j \zeta_c^{ak} M(k,n).$$

5. A twisted moment function

Define for coprime integers $0 < a < c$ the twisted second moment rank generating function

$$R_2\left(\frac{a}{c};q\right) := \frac{\zeta_{2c}^a}{2(q;q)_\infty}\sum_{n\in\mathbb{Z}} \frac{(-1)^{n+1}q^{\frac{n}{2}(3n+1)}}{(1-\zeta_c^a q^n)} + \frac{\zeta_{2c}^{3a}}{(q;q)_\infty}\sum_{n\in\mathbb{Z}} \frac{(-1)^{n+1}q^{\frac{3n}{2}(n+1)}}{(1-\zeta_c^a q^n)^2}. \tag{5.1}$$

We relate this function to a harmonic Maass form. For this, let

$$\mathcal{M}_{\frac{a}{c}}(z) = \mathcal{R}\left(\frac{a}{c};q\right) + \mathcal{N}_{\frac{a}{c}}(z).$$

Here

$$\mathcal{R}\left(\frac{a}{c};q\right) := q^{-1}R_2\left(\frac{a}{c};q^{24}\right),$$
$$\mathcal{N}_{\frac{a}{c}}(z) := -\frac{i}{64\sqrt{3\pi}}\int_{-\bar{z}}^{i\infty} \frac{\Theta_{a,c}\left(-\frac{1}{16d_c^2\tau}\right)(-i\tau)^{-\frac{1}{2}}}{(-i(\tau+z))^{\frac{3}{2}}}\, d\tau,$$

where $d_c := \operatorname{lcm}(6,c)$, and where

$$\Theta_{a,c}(\tau) := \sum_{m\equiv \frac{d_c}{6}\pm\frac{d_c a}{c} \pmod{d_c}} (-1)^m e^{2\pi i m^2\tau}.$$

Theorem 5.1. *The function $\mathcal{M}_{\frac{a}{c}}(z)$ is a weight $\frac{3}{2}$ harmonic Maass form on $\Gamma_1\left(96d_c^2\right)$.*

The first step in proving Theorem 5.1 is to show a transformation law for $R_2\left(\frac{a}{c};q\right)$. Due to double poles, we cannot work directly with this function, but use a function of an additional parameter w that is related and only has single poles. Define

$$R_2\left(\frac{a}{c},q;w\right) := \zeta_{2c}^a\frac{e^{\pi i w}}{(q;q)_\infty}\sum_{n\in\mathbb{Z}} \frac{(-1)^{n+1}q^{\frac{n}{2}(3n+1)}}{(1-\zeta_c^a e^{2\pi i w}q^n)}.$$

This function is connected to $R_2\left(\frac{a}{c};q\right)$ by

$$L\left(R_2\left(\frac{a}{c},q;w\right)\right) = R_2\left(\frac{a}{c};q\right),$$

where for a function g that is differentiable in some neighborhood of 0, we define

$$L(g) := \left[\frac{1}{2\pi i}\frac{\partial}{\partial w} g(w)\right]_{w=0}.$$

We prove a transformation law for $R_2\left(\frac{a}{c}, q; w\right)$ and then apply L. Some of the calculations are similar to those in [5], therefore we skip some of the details here and refer the reader to that paper.

We begin with some useful notation. For $\frac{a}{c} \notin \left\{0, \frac{1}{6}, \frac{1}{2}, \frac{5}{6}\right\}$, let

$$s = s(a,c) := \begin{cases} 0 & \text{if } 0 < \frac{a}{c} < \frac{1}{6}, \\ 1 & \text{if } \frac{1}{6} < \frac{a}{c} < \frac{1}{2}, \\ 2 & \text{if } \frac{1}{2} < \frac{a}{c} < \frac{5}{6}, \\ 3 & \text{if } \frac{5}{6} < \frac{a}{c} < 1. \end{cases}$$

Moreover define

$$\omega_{h,k} := \exp\left(\pi i t(h,k)\right),$$

where we have denoted the standard Dedekind sum by

$$t(h,k) := \sum_{\mu \pmod{k}} \left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{h\mu}{k}\right)\right),$$

with

$$((x)) := \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

Let

$$\begin{aligned} T_2\left(\frac{a}{c}, q; w\right) &:= \frac{1}{(q;q)_\infty} \sum_{\pm} \pm e^{\pm \pi i w}\, q^{\pm \frac{a}{2c}} \sum_{m \geq 0} (-1)^m \frac{q^{\frac{m}{2}(3m+1) \pm ms}}{1 - e^{\pm 2\pi i w} q^{\pm \frac{a}{c} + m}}, \\ I^{\pm}_{k,\nu,a,c}(z;w) &:= \pm \int_{\mathbb{R}} \frac{e^{-\frac{3\pi z x^2}{k}}}{\sinh\left(\frac{\pi z x}{k} + \frac{\pi i}{6k} - \frac{\pi i \nu}{k} \mp \pi i \left(w + \frac{a}{c}\right)\right)}\, dx. \end{aligned}$$

Theorem 5.2. *Assume the notation above. Moreover for coprime integers h and k, with $k > 0$ and either $k = 1$ or $2c^2|k$, let $q := e^{\frac{2\pi i}{k}(h + iz)}$ and $q_1 := e^{\frac{2\pi i}{k}\left(h' + \frac{i}{z}\right)}$, with $z \in \mathbb{C}$, $Re(z) > 0$, where $h' = 0$ for $k = 1$, and $hh' \equiv -1 \pmod{2k}$ and $h' \equiv 1 \pmod{2c^2}$ for $2c^2|k$. Then*

$$R_2\left(\frac{a}{c}, q; w\right) = S_1 - \frac{z^{\frac{1}{2}}}{2k}\omega_{h,k}\, e^{-\frac{\pi z}{12k}} \sum_{\substack{\nu \pmod{k} \\ \pm}} (-1)^\nu e^{\frac{\pi i h'}{k}(-3\nu^2 + \nu)} I^{\pm}_{k,\nu,a,c}(z;w),$$

where

$$S_1 := \begin{cases} -iz^{-\frac{1}{2}} e^{\frac{\pi}{12}\left(z^{-1} - z\right) - \frac{2\pi s}{z}\left(w + \frac{a}{c}\right) + \frac{3\pi}{z}\left(w + \frac{a}{c}\right)^2} T_2\left(\frac{a}{c}, q_1; \frac{iw}{z}\right) & \text{if } k = 1, \\ -\frac{i}{z^{\frac{1}{2}}}\, \omega_{h,k} e^{\frac{3\pi k w^2}{z} + \frac{\pi}{12k}\left(z^{-1} - z\right)} R_2\left(\frac{a}{c}, q_1; \frac{w}{iz}\right) & \text{if } 2c^2|k. \end{cases}$$

Proof. Let

$$\widetilde{R}_2\left(\frac{a}{c}, q; w\right) := \zeta_{2c}^a\, e^{\pi i w} \sum_{n \in \mathbb{Z}} \frac{(-1)^{n+1} q^{\frac{n}{2}(3n+1)}}{\left(1 - \zeta_c^a e^{2\pi i w} q^n\right)}.$$

Poisson summation yields

$$\widetilde{R}_2\left(\frac{a}{c},q;w\right)=-\frac{1}{2k}\sum_{\substack{\nu \pmod k\\ \pm}}\pm(-1)^{\nu}e^{\frac{3\pi ih\nu^2}{k}}\sum_{n\geq 0}\int_{\mathbb{R}}\frac{e^{-\frac{3\pi zx^2}{k}+\frac{\pi i}{k}(2n+1)(x-\nu)}}{\sinh\left(\frac{\pi zx}{k}-\frac{\pi ih\nu}{k}\mp\pi i\left(w+\frac{a}{c}\right)\right)}\,dx. \tag{5.2}$$

We shift the path of integration through $\omega_n:=\frac{(2n+1)i}{6z}$. Using the residue theorem yields

$$\widetilde{\mathcal{R}}_2\left(\frac{a}{c},q;w\right)=\sum_1+\sum_2.$$

Here

$$\sum_1:=2\pi i\sum_{\text{residues}},$$

where the sum runs over all residues of the integrand in (5.2), and where

$$\sum_2 \;:=\; -\frac{1}{2k}\sum_{\substack{\nu \pmod k\\ \pm}}\pm(-1)^{\nu}e^{\frac{3\pi ih\nu^2}{k}}\sum_{n\geq 0}\int_{\mathbb{R}+\omega_n}\frac{e^{-\frac{3\pi zx^2}{k}+\frac{\pi i}{k}(2n+1)(x-\nu)}}{\sinh\left(\frac{\pi zx}{k}-\frac{\pi ih\nu}{k}\mp\pi i\left(w+\frac{a}{c}\right)\right)}\,dx.$$

To compute $\sum_2$, we observe that for the computation of $\sum_2$ in [5] one does not need the fact that w is small, therefore we may change $w\mapsto w+\frac{a}{c}$. This yields

$$\sum_2=-\frac{(q_1;q_1)_\infty}{2k}\,e^{-\frac{\pi}{12kz}}\sum_{\substack{\nu \pmod k\\ \pm}}(-1)^{\nu}\,e^{\frac{\pi ih'}{k}\left(-3\nu^2+\nu\right)}I^{\pm}_{k,\nu,a,c}(z;w).$$

We next turn to $\sum_1$. First we consider the case $k=1$. In this case poles of the integrand can only lie in points

$$x_m^{\pm}:=\frac{i}{z}\left(m\pm\left(\frac{a}{c}+w\right)\right).$$

If we shift the path of integration through ω_n, we have to take those $x_m^{\pm}$ into account for which $n\geq 3m\pm s$ and $m\geq\frac{1}{2}(1\mp 1)$. We denote the residues of each summand by $\lambda_{n,m}^{\pm}$. Then one can easily see that

$$\lambda_{n,m}^{\pm}=\mp\frac{e^{-3\pi zx_m^{\pm 2}+\pi i(2n+1)x_m^{\pm}}}{2\pi z\,\cosh\left(\pi zx_m^{\pm}\mp\pi i\left(w+\frac{a}{c}\right)\right)}.$$

Using that $\lambda_{n+1,m}^{\pm}=e^{2\pi ix_m^{\pm}}\lambda_{n,m}^{\pm}$, one can compute that

$$\sum_1=\frac{1}{iz}\,e^{\frac{3\pi}{z}\left(w+\frac{a}{c}\right)^2-\frac{2\pi s}{z}\left(w+\frac{a}{c}\right)}\left(\sum_{m\geq 0}(-1)^m\frac{e^{-\frac{3\pi m^2}{z}-\frac{2\pi sm}{z}-\frac{\pi m}{z}-\frac{\pi}{z}\left(w+\frac{a}{c}\right)}}{1-e^{-\frac{2\pi}{z}\left(w+\frac{a}{c}\right)-\frac{2\pi m}{z}}}\right.$$
$$\left.-\sum_{m>0}(-1)^m\frac{e^{-\frac{3\pi m^2}{z}+\frac{2\pi sm}{z}-\frac{\pi m}{z}+\frac{\pi}{z}\left(w+\frac{a}{c}\right)}}{1-e^{\frac{2\pi}{z}\left(w+\frac{a}{c}\right)-\frac{2\pi m}{z}}}\right).$$

We next consider the case $2c^2|k$. Define the entire function

$$S_w^{\pm}(x):=\frac{\sinh(x\pm\pi ikw)}{\sinh\left(\frac{x}{k}\pm\pi iw\right)}.$$

From this one can see that poles of the integrand in (5.2) only lie in points

$$x_m^{\pm} := \frac{i}{z}(m \pm kw),$$

and a non-trivial residue occurs for at most one ν modulo k, which we may chose as

$$\nu_m^{\pm} := -h'\left(m \mp \frac{ak}{c}\right).$$

Shifting the path of integration through ω_n, we have to take those m into account for which $n \geq 3m \geq \frac{1}{2}(1 \mp 1)$. A lengthy calculation using the same methods as before gives

$$\sum_1 := \frac{1}{iz}\, \zeta_{2c}^a\, e^{\frac{3\pi k w^2}{z} + \frac{\pi w}{z}} \sum_{m \in \mathbb{Z}} \frac{(-1)^{m+1}\, q_1^{\frac{m}{2}(3m+1)}}{1 - \zeta_c^a\, e^{\frac{2\pi w}{z}}\, q_1^m}.$$

Now the theorem follows using

$$(q_1; q_1)_\infty = \omega_{h,k} z^{\frac{1}{2}} e^{\frac{\pi}{12k}\left(z^{-1} - z\right)} (q;q)_\infty.$$

□

Next we realize the integrals occurring in Theorem 5.2 as theta integrals. For this let

$$I_{a,c}^{\pm}(w;z) := \zeta_{2c}^a e^{\pi i w} e^{\mp \frac{\pi i}{6}} \int_{\mathbb{R}} e^{-\frac{3\pi i x^2}{z}} \frac{e^{\mp \frac{\pi i x}{z}}}{1 - e^{\mp \frac{\pi i}{3} + 2\pi i\left(w + \frac{a}{c}\right) \mp \frac{2\pi i x}{z}}}\, dx.$$

As in [5], we show.

Lemma 5.3. *We have*

$$L\left(I_{a,c}^{+}(w;z) + I_{a,c}^{-}(w;z)\right) = \frac{\sqrt{3}(-iz)^2}{4\pi} \int_0^\infty \frac{\Theta_{a,c}\left(\frac{3iu}{2d_c^2}\right)}{\left(-i(iu+z)\right)^{\frac{3}{2}}}\, du.$$

To finish the proof of Theorem 5.1, we change in Theorem 5.2 $z \mapsto \frac{i}{z}$ and apply the operator L on both sides. Observe that

$$L\left(\frac{e^{2\pi i s\left(w + \frac{a}{c}\right)z - 3\pi i z\left(w + \frac{a}{c}\right)^2 \pm \pi i z w}}{1 - e^{\pm 2\pi i w z} q_1^{\pm \frac{a}{c} + m}}\right) = z\, e^{2\pi i s \frac{a}{c} z - 3\pi i \frac{a}{c} z} \left(\frac{\left(s - \frac{3a}{c} \pm \frac{1}{2}\right)}{1 - q_1^{m \pm \frac{a}{c}}} \pm \frac{q_1^{m \pm \frac{a}{c}}}{\left(1 - q_1^{m \pm \frac{a}{c}}\right)^2}\right).$$

In the sum over the $-$ sign, we change $m \mapsto -m$, yielding

$$L\left(e^{2\pi i s\left(w + \frac{a}{c}\right)z - 3\pi i z\left(w + \frac{a}{c}\right)^2} T_2\left(\frac{a}{c}, q; zw\right)\right) = z\, q^{\frac{sa}{c} - \frac{3a^2}{2c^2}} T_2\left(\frac{a}{c}; q\right),$$

where

$$T_2\left(\frac{a}{c}; q\right) := \frac{1}{(q;q)_\infty}\left(\left(s - \frac{3a}{c}\right) q^{\frac{a}{2c}} \sum_{m \in \mathbb{Z}} (-1)^m \frac{q^{\frac{m}{2}(3m+1) + ms}}{1 - q^{m + \frac{a}{c}}} + q^{\frac{3a}{2c}} \sum_{m \in \mathbb{Z}} (-1)^m \frac{q^{\frac{3m}{2}(m+1) + ms}}{\left(1 - q^{m + \frac{a}{c}}\right)^2}\right).$$

Now Lemma 5.3 implies the following decomposition of the second rank moment; the subsequent lemma describes the corresponding theta integral component.

Corollary 5.4. *We have*

$$\mathcal{R}\left(\frac{a}{c};q_1\right)=\frac{(-iz)^{\frac{3}{2}}}{48\sqrt{6}}e^{-\frac{\pi iz}{288}+\frac{\pi isaz}{12c}-\pi i\frac{a^2z}{8c^2}}T_2\left(\frac{a}{c};q^{\frac{1}{24}}\right)+\frac{(-iz)^{\frac{3}{2}}}{64\sqrt{3}}\int_0^\infty\frac{\Theta_{a,c}\left(\frac{iu}{16d_c^2}\right)}{(-i(iu+z))^{\frac{3}{2}}}du.$$

Lemma 5.5.

$$\begin{aligned}\mathcal{N}_{\frac{a}{c}}(z+1) &= \mathcal{N}_{\frac{a}{c}}(z),\\ \mathcal{N}_{\frac{a}{c}}\left(-\frac{1}{z}\right) &= -\frac{i}{64\sqrt{3}\pi}\int_{-\bar\tau}^{i\infty}\frac{\Theta_{a,c}\left(\frac{\tau}{16d_c^2}\right)}{(-i(\tau+z))^{\frac{3}{2}}}d\tau+\frac{i}{64\sqrt{3}\pi}\int_0^\infty\frac{\Theta_{a,c}\left(\frac{it}{16d_c^2}\right)}{(-i(z+it))^{\frac{3}{2}}}dt.\end{aligned}$$

By work of Shimura it now follows that $\Theta_{a,c}\left(-\frac{1}{16d_c^2z}\right)(-iz)^{\frac{1}{2}}$ is a modular form of weight $\frac{1}{2}$ on $\Gamma_1\left(64d_c^2\right)$. Moreover observe that

$$L\left(e^{-3\pi i\gamma w^2z}R_2\left(\frac{a}{c},q;-zw\right)\right)=-zR_2\left(\frac{a}{c};q\right).$$

That $\mathcal{M}_{\frac{a}{c}}$ is annihilated under $\Delta_{\frac{3}{2}}$ can be seen as in [5]. Combining the above now easily gives the theorem.

6. Proof of Theorems 1.1 and 1.2

6.1. **Proof of Theorem 1.1.** First observe that the function $R_k(q)$ is a linear combination of the functions $\mathcal{R}_j(q)$ given in Section 4. We use the rank-crank relation (4.3) and the modularity of the occurring functions. The functions $C_\alpha(q^{24})q^{-1}$ are quasimodular forms and $\eta(24z)$ is a modular form. Since $R_2\left(q^{24}\right)q^{-1}$ is a quasimock theta function, it follows inductively that the functions $R_\alpha\left(q^{24}\right)q^{-1}$ are also quasimock theta functions.

For the readers convenience we give more details in the case $k=3$. We obtain from (4.3)

$$3\mathcal{R}_4(q)=-2\left(3\delta_q+1\right)C_2(q)+8C_4(q)+3\left(-12\delta_q+1\right)\mathcal{R}_2(q).$$

Since

$$\begin{aligned}\eta_2(n) &= \frac{1}{2}N_2(n), \qquad \text{and}\\ \eta_4(n) &= \frac{1}{24}(N_4(n)-N_2(n)),\end{aligned}$$

this implies that

$$\begin{aligned}&36q^{-1}R_3\left(q^{24}\right)\\ &=-\frac{9}{8}q^{-1}C_2\left(q^{24}\right)-\frac{1}{8}\delta_q\left(q^{-1}C_2\left(q^{24}\right)\right)+4q^{-1}C_4\left(q^{24}\right)-\frac{3}{2}q^{-1}R_2\left(q^{24}\right)-\frac{3}{2}\delta_q\left(q^{-1}R_2\left(q^{24}\right)\right).\end{aligned}\tag{6.1}$$

From Section 4, we know that the functions $q^{-1}C_j\left(q^{24}\right)$ are also quasimodular forms. This further implies that the function $\delta_q\left(q^{-1}C_2(q^{24})\right)$ is as well, since differentiation preserves the space of quasimodular forms. Moreover

$$q^{-1}R_2\left(q^{24}\right)=\mathcal{R}(q)=\mathcal{M}(z)+\mathcal{N}(z)+\frac{1}{24\eta(24z)}-\frac{E_2(24z)}{8\eta(24z)}.$$

Therefore $q^{-1}R_3\left(q^{24}\right)$ can be written as the sum of a quasimodular form and the derivative of the holomorphic part of a harmonic Maass form, and is thus a quasimock theta function.

6.2. **Proof of Theorem 1.2.** Denote by $\mathcal{D}_k(m_1, m_2, \cdots, m_k; n)$ the number of k-marked Durfee symbols arising from partitions of n with ith rank equal to m_i. Let

$$R_k(z_1, \cdots, z_k; q) := \sum_{m_1, \cdots, m_k \in \mathbb{Z}} \sum_{n=0}^{\infty} \mathcal{D}_k(m_1, m_2, \cdots, m_k; n)\, z_1^{m_1} z_2^{m_2} \cdots z_k^{m_k} q^n.$$

In particular

$$R_k(w; q) = R_k\left(w, w^2, \cdots, w^k; q\right).$$

If $x_i \neq x_j$ and $x_i x_j \neq 1$, then Andrews showed that the generating function for the Durfee symbols is actually a linear combination of rank functions [1]:

$$R_k(x_1, x_2, \cdots, x_k; q) = \sum_{i=1}^{k} \frac{R(x_i; q)}{\prod_{\substack{j=1 \\ j \neq i}}^{k} (x_i - x_j)\left(1 - \frac{1}{x_i x_j}\right)}. \tag{6.2}$$

Standard techniques for dissections of q-series then give

$$\sum_{n=0}^{\infty} NF_k(r, t; n)\, q^{24n-1} = \frac{1}{t}\left(R_k\left(q^{24}\right) + \sum_{j=1}^{t-1} \zeta_t^{-rj} q^{-1} R_k\left(\zeta_t^j; q^{24}\right)\right). \tag{6.3}$$

Thus, if $\zeta_t^{lj} \neq \zeta_t^{mj}$ and $\zeta_t^{j(l+m)} \neq 1$ for all $0 < l \neq m \leq k$, which is guaranteed if $k \leq \frac{p_t}{2}$, then we obtain by (6.2)

$$R_k\left(\zeta_t^j; q\right) = \sum_{l=1}^{k} \frac{R\left(\zeta_t^{lj}; q\right)}{\prod_{\substack{m=1 \\ m \neq l}}^{k} \left(\zeta_t^{jm} - \zeta_t^{jl}\right)\left(1 - \zeta_t^{-j(l+m)}\right)}. \tag{6.4}$$

Otherwise, for "large" k we need a modified version of (6.2). Namely if $x_i = x_j$ or $x_i x_j = 1$, then $R_k(x_1, x_2, \cdots, x_k; q)$ can be related to $R(x; q)$ via analytic continuation of (6.2). One can show that the new function is a linear combination of $\left[\frac{\partial^r R(y;q)}{\partial y^r}\right]_{y=x_i}$. For example if $k = 2$, $x_1 = x_2$, and $x_1 x_2 \neq 1$, then

$$R_2(x_1, x_2; q) = \lim_{x_2 \to x_1} \left(\frac{R(x_1; q)}{(x_1 - x_2)\left(1 - \frac{1}{x_1 x_2}\right)} + \frac{R(x_2; q)}{(x_2 - x_1)\left(1 - \frac{1}{x_1 x_2}\right)}\right) = \frac{\left[\frac{\partial}{\partial y} R(y; q)\right]_{y=x_1}}{\left(1 - x_1^{-2}\right)}.$$

In general, $R_k(x_1, \ldots, x_k; q)$ will be a linear combination of various derivatives of $R(y; q)$ for any values assigned to the x_i. This can be seen by comparing the two sides of (6.2); since the left side has no poles, all of the singularities on the right side must be removable, and thus L'Hospital's rule can be applied (at most $2k$ times).

We largely restrict our argument to the case that $k \leq \frac{p_t}{2}$, and then make some comments on the general case. Fix a prime $p > 3$ with $p \nmid t$. We treat the two summands in (6.4) separately. From Subsection 6.1, we know that $R_\alpha\left(q^{24}\right) q^{-1}$ is a quasimock theta function. Moreover one can conclude from Theorem 3.2 that the holomorphic part of the associated harmonic Maass form is supported on negative squares. Thus the restriction to coefficients lying in

$$S_p := \left\{n \in \mathbb{Z} : \left(\frac{24n - 1}{p}\right) = -\left(\frac{-1}{p}\right)\right\}$$

is a weakly holomorphic modular form on $\Gamma_1\left(96d_t^2p^2\right)$. Moreover from the work of the first author and Ono [10], we know that the restriction of $R\left(\zeta_t^j; q^{\ell_t}\right) q^{-\frac{\ell_t}{24}}$ to those coefficients lying in S_p is a weakly holomorphic modular form on $\Gamma_1\left(6f_t^2l_tp^2\right)$. Finally, work of Serre implies that quasimodular forms are actually p-adic modular forms, and the proof concludes as in [5].

To prove the general case, we first consider the function $\left[\frac{\partial}{\partial w}R(w;q)\right]_{w=\zeta_c^a}$ which is again the base case for induction. We have

$$\left[\frac{\partial}{\partial w}R(w;q)\right]_{w=\zeta_c^a} = \frac{1}{(q;q)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^{n+1}q^{\frac{n}{2}(3n+1)}}{(1-\zeta_c^aq^n)} - \frac{\zeta_c^a\left(1-\zeta_c^a\right)}{(q;q)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^{n+1}q^{\frac{3n}{2}(n+1)}}{(1-\zeta_c^aq^n)^2}.$$

Since we know that $R\left(\frac{a}{c};q\right)$ is related to a harmonic Maass form, it is enough to consider the function $R_2\left(\frac{a}{c};q\right)$ defined in (5.1). Theorem 3.2 implies that $\mathcal{R}\left(\frac{a}{c};q\right)$ is the holomorphic part of a harmonic Maass form. Moreover as in [10], one can compute that the non-holomorphic part is supported on negative squares. Now we argue as before, utilizing the easy shown technical result that for these Maass forms, the quadratic twist operator commutes with differentiation (due to the form of the non-holomorphic coefficients). For higher derivatives, it is enough to relate the functions $\mathcal{R}_{j,a,c}(q)$ to quasiharmonic Maass forms. For this we apply ∂_w^j to (4.3), set $w=\zeta_c^a$, and then argue inductively. The case $k=2$ again begins the induction. The claim now follows, using the relation of the functions $C_{j,a,c}(q)$ to quasimodular forms, and the modularity of the twisted Eisenstein series $G_{j+1}^{(0,a)}(z)$.

7. Explicit differential operators and rank moment differences

In this section, we show more explicitly how the modularity and holomorphicity of the higher rank moments $\mathcal{R}_{2k}$ are determined by the derivatives of $\mathcal{R}_2$. We describe the differential operator that naturally acts on $\mathcal{R}_2$, and then prove Theorem 1.4 by considering the operator ℓ-adically. Throughout this section we assume that $\ell > 3$ is prime.

7.1. Higher rank moments and derivatives of $\mathcal{R}_2$.

We begin by defining by what we mean by an "ℓ-integral quasimodular form of level 1." We consider functions $E_2^a(z)\,F_b(z)$, where $F_b(z)\in M_b(1)$, the coefficients in the q-expansion of $F_b(z)$ are ℓ-integral and have bounded denominators, and a and b are nonnegative integers. We call such a function an ℓ-integral quasi-modular form of weight $2a+b$. Let k be a nonnegative integer. In general, an *ℓ-integral quasi-modular form of weight k and level* 1 is sum of such functions where $2a+b=k$. For k an even nonnegative integer, let $\mathcal{X}_k$ denote the set of functions that are sums of ℓ-integral quasi-modular forms of weight $\le k$. Let

$$P\mathcal{X}_k = \{G\,P\,:\,G\in\mathcal{X}_k\}.$$

In equation (4.3), replace a by $2k$ and denote the left side by Y_{2k}, so that

$$Y_{2k} := \sum_{i=0}^{k-1}\binom{2k}{2i}\sum_{\substack{\alpha+\beta+\gamma=2k-2i\\ \alpha,\beta,\gamma \text{ even } \ge 0}}\binom{2k-2i}{\alpha,\beta,\gamma}C_\alpha\,C_\beta\,C_\gamma\,P^{-2} - 3\left(2^{2k-1}-1\right)C_2. \tag{7.1}$$

With this notation, the relation (4.3) can be solved for $\mathcal{R}_{2k}$, yielding

$$\mathcal{R}_{2k} = \frac{1}{(2k-1)(k-1)} Y_{2k} - \frac{1}{(2k-1)(k-1)} \left(\sum_{i=1}^{k-1} 6 \binom{2k}{2i} \left(2^{2i-1} - 1\right) \delta_q(\mathcal{R}_{2k-2i}) + \sum_{i=1}^{k-1} \left[\binom{2k}{2i+2} \left(2^{2i+1} - 1\right) - 2^{2i} \binom{2k}{2i+1} + \binom{2k}{2i} \right] \mathcal{R}_{2k-2i} \right). \tag{7.2}$$

An easy induction argument shows that the differential operator that acts on $\mathcal{R}_2$ is a polynomial $P_k(\delta_q)$ with rational coefficients and degree $k-1$, so that

$$\mathcal{R}_{2k} = P_k(\delta_q)\, \mathcal{R}_2 + \sum_{j=2}^{k} Q_{k,j}(\delta_q)\, Y_{2j}, \tag{7.3}$$

where $Q_{k,j} \in \mathbb{Q}[x]$ has degree $k-j$.

We now focus on $P_k(x)$; we will see shortly that the other terms in (7.3) may be absorbed into the quasimodular component of $\mathcal{R}_{2k}$.

Proposition 7.1. *Let $P_0(x) := 0$ and $P_1(x) := 1$. For $k \geq 2$ we have the recurrence relation*

$$P_k(x) = (1 - 12x)\, P_{k-1}(x) - 36x^2\, P_{k-2}(x),$$

and the explicit formula

$$P_k(x) = 2^{1-2k} \sum_{j=0}^{k-1} \binom{2k}{2j+1} (1-24x)^j = \frac{1}{\sqrt{1-24x}} \left(\left(\frac{1+\sqrt{1-24x}}{2} \right)^{2k} - \left(\frac{1-\sqrt{1-24x}}{2} \right)^{2k} \right).$$

Proof. From (7.2) and (7.3) we see that for $k \geq 0$,

$$P_k(x) = -\frac{1}{(2k-1)(k-1)} \sum_{i=1}^{k-1} \left(6x \binom{2k}{2i} \left(2^{2i-1} - 1\right) + \binom{2k}{2i+2} \left(2^{2i+1} - 1\right) - 2^{2i} \binom{2k}{2i+1} + \binom{2k}{2i} \right) P_{k-i}(x).$$

For $k \geq 0$ we define

$$V_k(z) := P_k\left(\frac{1-z^2}{24} \right).$$

Using some elementary (though lengthy) binomial sum evaluations and an induction argument, one can show that

$$V_k(z) = \frac{1}{z} \left(\left(\frac{1+z}{2} \right)^{2k} - \left(\frac{1-z}{2} \right)^{2k} \right).$$

From this one can conclude that for $k \geq 2$,

$$V_k(z) = \left(\frac{1+z^2}{2} \right) V_{k-1}(z) - \left(\frac{1-z^2}{4} \right)^2 V_{k-2}(z).$$

Letting $z = \sqrt{1-24x}$ (taking any fixed branch of the square root), we obtain the results. □

By letting $2k = \ell + 1$ and reducing the formula from Proposition 7.1 modulo ℓ we obtain the following result.

Corollary 7.2. *For $k \geq 2$ the polynomial $P_k(x)$ has integer coefficients. If $\ell > 3$ is prime then*

$$P_{\frac{\ell+1}{2}}(x) \equiv \frac{\ell+1}{2}\left(1+(1-24x)^{\frac{\ell-1}{2}}\right) \pmod{\ell}.$$

We will need precise expansions of the rank and crank moments into ℓ-integral components.

Proposition 7.3. *Let $\ell > 3$ be prime.*

1. *For $k \geq 0$ even we have $\delta_q(\mathcal{X}_k) \subset \mathcal{X}_{k+2}$.*
2. *For $k \geq 0$ even and $m \geq 0$ we have $\delta_q^m(P\mathcal{X}_k) \subset P\mathcal{X}_{k+2m}$.*
3. *For $1 \leq j \leq \frac{\ell+1}{2}$ except $j = \frac{\ell-1}{2}$ we have $C_{2j} \in P\mathcal{X}_{2j}$.*
4. *$C_{\ell-1} = 2P\Phi_{\ell-2} + PG$ for some $G \in \mathcal{X}_{\ell-1}$.*
5. *For $1 \leq j \leq \frac{\ell-3}{2}$ we have $Y_{2j} \in P\mathcal{X}_{2j}$.*
6. *$Y_{\ell-1} = 6P\Phi_{\ell-2} + PG$ for some $G \in \mathcal{X}_{\ell-1}$.*

Proof. Suppose $\ell > 3$ is prime and $k \geq 0$ is even. Our goal is to write the crank moments C_{2j} and hence the Y_{2j} in terms of ℓ-integral quasimodular forms. Throughout the proof, we will only write the components that are *not* obviously ℓ-integral, with trailing ellipses representing the remaining terms which are ℓ-integral quasimodular forms.

(1) This result is well known. It follows from the fact that

$$12\delta_q E_2 = E_2^2 - E_4 \quad \text{and} \quad 12\delta_q F - kE_2F \in M_{k+2} \quad \text{if } F \in M_k.$$

(2) From [2, (2.11)] we have

$$\delta_q P = \Phi_1 P = \tfrac{1}{24}(1-E_2)P \in P\mathcal{X}_2. \tag{7.4}$$

The result follows from (1) and (7.4) by an induction argument.

(3) Suppose $1 \leq j \leq \frac{\ell-3}{2}$. Then $\Phi_{2j-1} = \frac{B_{2j}}{4j}(E_{2j}-1) \in \mathcal{X}_{2j}$ by the von-Staudt and Kummer congruences [4], [18, p.20]. Hence $C_{2j} \in P\mathcal{X}_{2j}$ by (4.1). Similarly, $C_{\ell+1} \in P\mathcal{X}_{\ell+1}$ by (4.1), since

$$\begin{aligned} C_{\ell+1} &= 2\Phi_\ell P + 2\ell C_{\ell-1}\Phi_1 + \cdots + 2\Phi_{\ell-2}C_2, \\ &= 2\Phi_\ell P + 8\ell\Phi_{\ell-2}\Phi_1 + \cdots \end{aligned}$$

and

$$\ell\Phi_{\ell-2}\Phi_1 = \frac{\ell B_{\ell-1}}{2(\ell-1)}(1-E_{\ell-1})\Phi_1 \in \mathcal{X}_{\ell+2},$$

again by the von-Staudt and Kummer congruences.

(4) The result follows from (4.1) and (3).

(5) The result follows from (7.1) and (3).

(6) The result follows from (7.1), (3), (4) and since

$$Y_{\ell-1} = 3C_{\ell-1} + \cdots = 6\Phi_{\ell-2}P + \cdots.$$

□

We can now prove that the ℓ-adic behavior of the higher rank moments comes from that of $\mathcal{R}_2$.

Theorem 7.4. *Let $\ell > 3$ be prime. Then*

$$\mathcal{R}_{\ell+1} - P_{\frac{\ell+1}{2}}(\delta_q)\mathcal{R}_2 \in P\mathcal{X}_{\ell+1}. \tag{7.5}$$

Proof. The idea of the proof is to use (7.3), and rewrite the Y_{2j} in terms of ℓ-integral quasi-modular forms, keeping track of when ℓ occurs in denominator of a coefficient. We consider the equations (7.2) for $2 \le k \le \frac{\ell+1}{2}$. In these equations the only time ℓ occurs in a denominator is when $k = \frac{\ell+1}{2}$, and only the term of the right side which is not ℓ-integral is the term involving $Y_{\ell+1}$. We note that all coefficients in the sum are ℓ-integral. The only term which could rise to a non-ℓ-integral quasi-modular form is the term with $i = 1$. Equation (7.1) gives $Y_{\ell+1}$ in terms of crank moments. We wish to write $\frac{1}{\ell}Y_{\ell+1}$ in terms of quasi-modular forms identifying which terms that are not ℓ-integral. As usual, we will only write the components that are *not* obviously ℓ-integral, with trailing ellipses representing the remaining ℓ-integral portion. We find that

$$\begin{aligned}\tfrac{1}{\ell}Y_{\ell+1} &= \tfrac{3}{\ell}\left(C_{\ell+1} - C_2\right) + 3(\ell+1)C_{\ell-1}C_2P^{-1} + \tfrac{3}{2}(\ell+1)C_{\ell-1} + \cdots \\ &= \tfrac{6}{\ell}P\left(\Phi_\ell - \Phi_1\right) + 18(\ell+1)\Phi_{\ell-2}\Phi_1P + 3(\ell+1)\Phi_{\ell-2}P + \cdots .\end{aligned}$$

Therefore by (7.1), (7.2), (7.3), and Proposition 7.3

$$\begin{aligned}\mathcal{R}_{\ell+1} - P_k(\delta_q)\mathcal{R}_2 &= \frac{2}{\ell(\ell-1)}Y_{\ell+1} - \frac{(\ell+1)(72\delta_q + 7\ell^2 - 37\ell + 42)}{6(\ell-1)(\ell-2)(\ell-3)}Y_{\ell-1} + \cdots \\ &= F_1 + \sum_{j=2}^{(\ell+1)/2} \widetilde{Q}_{\ell,j}(\delta_q)PZ_{2j},\end{aligned} \tag{7.6}$$

where the $\widetilde{Q}_{\ell,j}$ are polynomials of degree $\ell - j$ with integer coefficients, the $Z_{2j} \in \mathcal{X}_{2j}$ $(2 \le j \le \frac{\ell+1}{2})$, and

$$\begin{aligned}F_1 &= \frac{2}{\ell-1}\left(\tfrac{6}{\ell}P\left(\Phi_\ell - \Phi_1\right) + 18(\ell+1)\Phi_{\ell-2}\Phi_1P + 3(\ell+1)\Phi_{\ell-2}P\right) \\ &\quad - \frac{(\ell+1)(72\delta_q + 7\ell^2 - 37\ell + 42)}{6(\ell-1)(\ell-2)(\ell-3)}\left(6\Phi_{\ell-2}P\right) \\ &= F_2P + \frac{1}{\ell}F_3,\end{aligned}$$

where $F_2 \in \mathcal{X}_{\ell+1}$ and after some calculation we find that

$$F_3 = (\tfrac{1}{2} - 12\delta_q)\widetilde{E}_{\ell-1}P + 12\widetilde{E}_{\ell+1}P - \tfrac{3}{2}E_2\widetilde{E}_{\ell-1}P.$$

Here we have defined the ℓ-integral modular forms $\widetilde{E}_{\ell-1}(z) := \frac{\ell B_{\ell-1}}{2(\ell-1)} - \ell\Phi_{\ell-2}$ and $\widetilde{E}_{\ell+1}(z) := \frac{B_{\ell+1}}{2(\ell+1)} - \Phi_\ell$, and we have used the von-Staudt and Kummer congruences $\ell B_{\ell-1} \equiv -1 \pmod{\ell}$, and $12B_{\ell+1} \equiv 1 \pmod{\ell}$. Since $\widetilde{E}_{\ell-1} \in M_{\ell-1}(1)$,

$$V_\ell = 12\delta_q\widetilde{E}_{\ell-1} - (\ell-1)E_2\widetilde{E}_{\ell-1}$$

is an ℓ-integral modular form of weight $(\ell+1)$. Now

$$12\delta_q\widetilde{E}_{\ell-1}P = 12\left(\delta_q\widetilde{E}_{\ell-1}\right)P + 12\widetilde{E}_{\ell-1}\delta_qP = V_\ell P + (\ell - \tfrac{3}{2})E_2\widetilde{E}_{\ell-1}P + \tfrac{1}{2}\widetilde{E}_{\ell-1}P.$$

Hence

$$F_3 = (-V_\ell + 12\widetilde{E}_{\ell+1})P - \ell E_2\widetilde{E}_{l-1}P.$$

Now

$$-V_\ell + 12\widetilde{E}_{\ell+1} \equiv -E_2\widetilde{E}_{\ell-1} + 12\widetilde{E}_{\ell+1} \equiv 0 \pmod{\ell},$$

by the well known congruences $2\widetilde{E}_{\ell-1} \equiv 1$ and $24\widetilde{E}_{\ell+1} \equiv E_2 \pmod{\ell}$, and we note that $-V_\ell + 12\widetilde{E}_{\ell+1} \in M_{\ell+1}(1)$. Hence

$$F_1 = F_4 P,$$

where $F_4 \in \mathcal{X}_{\ell+1}$. The function

$$\sum_{j=2}^{(\ell+1)/2} \widetilde{Q}_{\ell,j}(\delta_q) P Z_{2j}$$

has the same property by Proposition 7.3, and the result (7.5) follows from (7.6). ☐

For $\ell > 3$ prime and $\epsilon \in \{-1, 0, 1\}$ we define the operator $U^*_{\epsilon,\ell}$, which acts on q-series by

$$U^*_{\epsilon,\ell}\left(\sum_n a(n)q^n\right) := \sum_{\left(\frac{1-24n}{\ell}\right)=\epsilon} a(n)q^n.$$

The following corollary of Theorem 7.4 follows from Corollary 7.2.

Corollary 7.5. *Let $\ell > 3$ be prime and suppose $\epsilon = -1$ or 0. Then*

$$U^*_{\epsilon,\ell}(\mathcal{R}_2) \equiv U^*_{\epsilon,\ell}(G_\ell P) \pmod{\ell},$$

where $G_\ell \in \mathcal{X}_{\ell+1}$.

Proof. Let $\epsilon = -1$ or 0. We define

$$c_{\epsilon,\ell} := \begin{cases} \frac{\ell+1}{2} & \text{if } \epsilon = 0, \\ 1 & \text{if } \epsilon = -1. \end{cases}$$

If $\left(\frac{1-24n}{\ell}\right) = \epsilon$ we have

$$\left(1 - P_{\frac{\ell+1}{2}}(n)\right) \equiv c_{\epsilon,\ell} \pmod{\ell}, \tag{7.7}$$

by Corollary 7.2. Since $\mathcal{R}_{\ell+1} \equiv \mathcal{R}_2 \pmod{\ell}$, from Theorem 7.4 we have

$$U^*_{\epsilon,\ell}\left(\left(1 - P_{\frac{\ell+1}{2}}(\delta_q)\right)\mathcal{R}_2\right) \equiv U^*_{\epsilon,\ell}(G_\ell P) \pmod{\ell},$$

where $G_\ell \in \mathcal{X}_{\ell+1}$. Finally by (7.7) we have

$$\begin{aligned} U^*_{\epsilon,\ell}\left(\left(1 - P_{\frac{\ell+1}{2}}(\delta_q)\right)\mathcal{R}_2\right) &= \sum_{\left(\frac{1-24n}{\ell}\right)=\epsilon} \left(1 - P_{\frac{\ell+1}{2}}(n)\right) N_2(n)q^n \\ &\equiv c_{\epsilon,\ell} \sum_{\left(\frac{1-24n}{\ell}\right)=\epsilon} N_2(n)q^n \equiv c_{\epsilon,\ell} U^*_{\epsilon,\ell}(\mathcal{R}_2) \pmod{\ell}, \end{aligned}$$

and the result follows since $c_{\epsilon,\ell} \not\equiv 0 \pmod{\ell}$. ☐

The following theorem describes the relation between $\mathcal{R}_{2k}$ and $\mathcal{R}_2$ for other k. The proof is entirely analogous to Theorem 7.4, with an extra term that is not ℓ-integral appearing when $k = (\ell-1)/2$.

Theorem 7.6. *Let $\ell > 3$ be prime.*

(1) *For $1 \le k \le \frac{\ell-3}{2}$ we have $C_{2k} \in P\mathcal{X}_{2k}$, whereas $C_{\ell-1} = 2P\Phi_{\ell-2} + PG_C$ for some $G_C \in \mathcal{X}_{\ell-1}$.*

(2) *For $2 \leq k \leq \frac{\ell-3}{2}$ we have*

$$\mathcal{R}_{2k} - P_k(\delta_q)\mathcal{R}_2 \in P\mathcal{X}_{2k},$$

whereas

$$\mathcal{R}_{\ell-1} - P_{\frac{\ell-1}{2}}(\delta_q)\mathcal{R}_2 = 2P\Phi_{\ell-2} + PG_R$$

for some $G_R \in \mathcal{X}_{\ell-1}$.

7.2. **Proof of Theorem 1.4.** We are now ready to prove Theorem 1.4 by combining the above results with the theory of ℓ-adic modular forms.

(1) By Corollary 7.5,

$$U^*_{0,\ell}(\mathcal{R}_2) \equiv U^*_{0,\ell}(G_\ell P) \pmod{\ell}$$

for some $G_\ell \in \mathcal{X}_{\ell+1}$. By reduction mod ℓ we may assume that all forms involved have integer coefficients. The function $G_\ell P$ is a sum of functions of the form

$$\widetilde{E}_2^a FP \equiv \widetilde{E}_{\ell+1}^a FP \equiv q^{\frac{1}{24}} \frac{\widetilde{F}_a(z)}{\eta(z)} \pmod{\ell},$$

where $\widetilde{E}_2 = \frac{1}{24}E_2$ and $F(z)$ is an integral modular form $F \in M_b(1)$, with total weight $2a+b \leq \ell+1$. Define the associated ℓ-adic modular form $\widetilde{F}_{a,b}(z) := \widetilde{E}_{\ell+1}^a(z)F(z) \in M_{b+a(\ell+1)}(1)$, which has weight at most $\frac{1}{2}(\ell+1)^2$. Define the coefficients $p\left(\widetilde{F}, n\right)$ by $\sum_n p\left(\widetilde{F}, n\right) q^n = \frac{\widetilde{F}(z)}{(q;q)_\infty}$. By standard arguments using Hecke operators (see [13]), we have

$$\sum_{n=0}^{\infty} N_2(\ell n + \beta_\ell) q^{24n+r_\ell} \equiv \sum_{2a+b\leq \ell+1} \sum_{n=0}^{\infty} p\left(\widetilde{F}_{a,b}, \ell n + \beta_\ell\right) q^{24n+r_\ell} \equiv \eta^{r_\ell}(24z) G_{\ell,2}(24z) \pmod{\ell},$$

where $G_{\ell,2} \in \mathcal{X}_{(\ell^2+3\ell-r_\ell-1)/2}$.

(2) This claim follows immediately using the same arguments as above and the relation between $\mathcal{R}_{2k}$ and $\mathcal{R}_2$ found in Theorem 7.6.

(3) To find ℓ-adic modular forms associated with $\mathcal{R}_{\ell-1}$, we again use Theorem 7.6. The difference is that there is now an extra term that is not ℓ-integral, namely $2P\Phi_{\ell-2} = \frac{1}{\ell}\left(\frac{\ell B_{\ell-1}}{(\ell-1)} - 2\widetilde{E}_{\ell-1}\right)P$.

Define α_ℓ such that $\alpha_\ell \equiv \frac{\ell B_{\ell-1}}{(\ell-1)} \pmod{\ell^2}$, so that the first part of the contribution from $2P\Phi_{\ell-2}$ is

$$\alpha_\ell \sum_{n=0}^{\infty} p(n) q^{24n-1} \equiv \alpha_\ell \frac{1}{\eta(24z)} \equiv \alpha_\ell \frac{1}{\eta(24z)} \left(\frac{\eta^\ell(24z)}{\eta(24\ell z)}\right)^\ell \equiv \alpha_\ell \frac{\Delta^{\frac{\ell^2-1}{24}}(24z)}{\eta^\ell(24\ell z)} \pmod{\ell^2}.$$

This implies that

$$\alpha_\ell \sum_{n=0}^{\infty} p(\ell n + \beta_\ell) q^{24n+r_\ell} \equiv \alpha_\ell \frac{\Delta^{(\ell^2-1)/24}(24z) \mid U(\ell)}{\eta^\ell(24z)} \pmod{\ell^2}.$$

Observe that $\Delta^{(\ell^2-1)/24}(z) \in S_{\frac{1}{2}(\ell^2-1)}(1)$, and recall that $T(\ell) \equiv U(\ell) \pmod{\ell}$, so

$$f_\ell(z) := \alpha_\ell \Delta^{(\ell^2-1)/24}(z) \mid T(\ell) = (q^{(\ell^2-1)/24} + \cdots) \mid T(\ell) = c_3 q^{\lambda_\ell} + \cdots,$$

where $\lambda_\ell := \frac{\ell^2+24\beta_\ell-1}{24\ell}$, and c_3 is an integer. Since Hecke operators preserve spaces of modular forms, it must be that $f_\ell(z) = \Delta^{\lambda_\ell}(z)H_1(z)$ for some $H_1(z) \in M_{\frac{1}{2}(\ell(\ell-1)-r_\ell-1)}(1)$. We conclude that

$$\alpha_\ell \sum_{n=0}^{\infty} p(\ell n+\beta_\ell)q^{24n+r_\ell} \equiv \frac{\Delta^{\lambda_\ell}(24z)H_1(24z)}{\eta^\ell(24z)} \equiv \eta^{r_\ell}(24z)H_1(24z) \pmod{\ell^2}.$$

We proceed in a similar fashion for the term $2\widetilde{E}_{\ell-1}P$. Define the coefficients $e_{\ell-1}(n)$ so that $2\widetilde{E}_{\ell-1}P \equiv \sum_{n=0}^{\infty} e_{\ell-1}(n)q^n \pmod{\ell^2}$. As before, we find that

$$\sum_{n=0}^{\infty} e_{\ell-1}(n)q^{24n-1} \equiv 2\frac{\Delta^{(\ell^2-1)/24}(24z)\widetilde{E}_{\ell-1}(24z)}{\eta^\ell(24\ell z)} \pmod{\ell^2},$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} e_{\ell-1}(\ell n+\beta_\ell)q^{24n+r_\ell} &\equiv 2\frac{\Delta^{(\ell^2-1)/24}(24z)\widetilde{E}_{\ell-1}(24z) \mid U(\ell)}{\eta^\ell(24z)} \pmod{\ell^2}\\ &\equiv \eta^{r_\ell}(24z)H_2(24z) \pmod{\ell^2},\end{aligned}$$

where $H_2(z) \in M_{\frac{1}{2}(\ell(\ell+1)-r_\ell-3)}(1)$ and

$$\Delta^{\lambda_\ell}(z)H_2(z) \equiv 2\Delta^{(\ell^2-1)/24}(z)\widetilde{E}_{\ell-1}(z) \mid U(\ell) \pmod{\ell^2}.$$

Since $2\widetilde{E}_{\ell-1} \equiv \alpha_\ell \equiv 1 \pmod{\ell}$, we have $H_1(24z) \equiv H_2(24z) \pmod{\ell}$, so the overall the contribution to the congruence for $U^*_{0,\ell}(\mathcal{R}_{\ell-1})$ is

$$\frac{1}{\ell}\eta^{r_\ell}(24z)\left(H_1(24z) - H_2(24z)\right)$$

as claimed, completing the proof.

8. Proof of Theorem 1.5

Using (6.3), we observe that

$$\sum_{n=0}^{\infty} \left(NF_k(r,t;n) - NF_k(s,t;n)\right) q^{24n-1} = \frac{1}{t}\sum_{j=1}^{t-1}\left(\zeta_t^{-rj} - \zeta_t^{-sj}\right) q^{-1}R_k\left(\zeta_t^j; q^{24}\right).$$

Without loss of generality, we assume that $k \le \frac{p_t}{2}$, the general case is proven similarly. In this case we may use (6.4). The functions $R\left(\zeta_t^j; q^{\ell_t}\right) q^{-\frac{\ell_t}{24}}$ are the holomorphic parts of harmonic Maass forms on $\Gamma_1\left(576t^4\right)$. One can generalize the usual Atkin $U(t^2)$-operator to harmonic Maass forms. This gives that $q^{-1}R\left(\zeta_t^j; q^{24}\right)$ are the holomorphic parts of harmonic Maass forms on $\Gamma_1\left(576t^4\right)$. Moreover by Theorem 3.1, the non-holomorphic parts of those forms are supported on negative squares. Generalizing the theory of twists of modular forms to twists of harmonic Maass forms, we obtain that the restriction of those form to the coefficients supported on arithmetic progression congruent to d modulo t satisfying $\left(\frac{1-24d}{t}\right) = -1$ is a weakly holomorphic modular form on $\Gamma_1\left(576t^6\right)$. Thus (1) follows.

In order to conclude (2), we have to show that the restriction of the function

$$\sum_{j=1}^{t-1}\left(\zeta_t^{-rj}-\zeta_t^{-sj}\right)\frac{\zeta_t^{2j}}{(1-\zeta_t^{j})(\zeta_t^{3j}-1)}\left(q^{-1}R\left(\zeta_t^{j};q^{24}\right)-q^{-1}R\left(\zeta_t^{2j};q^{24}\right)\right)$$

to those arithmetic progressions stated in the theorem doesn't have a non-holomorphic part. The correct group follows as in (1). Using Theorem 3.1, we see that this is equivalent to the identity

$$0=\sum_{j=1}^{t-1}\left(\zeta_t^{-rj}-\zeta_t^{-sj}\right)\frac{\zeta_t^{2j}}{(1-\zeta_t^{j})(\zeta_t^{3j}-1)}\left(\sin\left(\frac{\pi j}{t}\right)\sin\left(\frac{3\pi jd}{t}\right)-\sin\left(\frac{2\pi j}{t}\right)\sin\left(\frac{6\pi jd}{t}\right)\right). \tag{8.1}$$

Since $\sin(x)=\frac{1}{2i}\left(e^{ix}+e^{-ix}\right)$, identity (8.1) is equivalent to

$$0=\sum_{j=1}^{t-1}\frac{\left(\zeta_t^{-rj}-\zeta_t^{-sj}\right)\left(1-\zeta_t^{3dj}\right)}{\left(1-\zeta_t^{3j}\right)}\zeta_{2t}^{3j(1-d)}\left(1-\left(\zeta_{2t}^{j}+\zeta_{2t}^{-j}\right)\left(\zeta_{2t}^{3dj}+\zeta_{2t}^{-3dj}\right)\right). \tag{8.2}$$

This is further equivalent to

$$0=\sum_{j=1}^{t-1}\left(\zeta_t^{-rj}-\zeta_t^{-sj}\right)\zeta_{2t}^{3j(1-d)} \left(1+\zeta_t^{3j}+\cdots+\zeta_t^{3j(d-1)}\right)\left(1-\zeta_{2t}^{j(3d+1)}-\zeta_{2t}^{j(-3d+1)}-\zeta_{2t}^{j(3d-1)}-\zeta_{2t}^{-j(3d+1)}\right). \tag{8.3}$$

Identity (8.3) can be verified using the conditions in the theorem and the standard orthogonality of roots of unity.

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