RAMANUJAN TYPE CONGRUENCES AND JACOBI FORMS

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1. Introduction and Statement of results

Finding congruences between different arithmetic objects has always been a challenging task. From Kummer we have beautiful relations between the Bernoulli numbers B_n which have a wide range of important applications to different areas of mathematics. If n, m are even positive integers and p is a prime with $(p-1) \nmid n$ and $n \equiv m \pmod{p-1}$, then $\frac{B_n}{n} \equiv \frac{B_m}{m} \pmod{p}$. This leads directly to congruences for the Riemann zeta function $\zeta(s)$

$$\zeta(1-n) \equiv \zeta(1-m) \pmod{p} \tag{1.1}$$

and was the foundation for the construction of the p-adic L-function. Using Kummer congruences for generalized Bernoulli numbers this has been generalized to the abelian Dedekind zeta function [Le, Ca].

Another type of congruences involves the coefficients of modular forms. The most famous ones are due to Ramanujan, and they assert that if n is a non-negative integer, then the following congruences hold for $p(n)$, the number of partitions of n,

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

These congruences can be explained by the fact that certain Hecke operators annihilate specific modular forms modulo p. Moreover Ramanujan showed congruences involving the coefficients of the Δ-function

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n := q \prod_{n=1}^{\infty} (1-q^n)^{24},$$

where $q := e^{2\pi i z}$ with $z \in \mathbb{H}$, the Poincaré upper half-plane. For example Ramanujan proved that

$$\begin{aligned} \tau(n) &\equiv n\sigma_9(n) \pmod{5}, \\ \tau(n) &\equiv n\sigma_3(n) \pmod{7}, \\ \tau(n) &\equiv \sigma_{11}(n) \pmod{691}, \end{aligned}$$

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where σ_k is the usual k-th divisor sum. For a complete list of these congruences we refer the reader to [BO, SD]. One can show these congruences using identities between modular forms. For example the congruence modulo 691 follows from the identity

$$E_6^2(z) = E_{12}(z) - \frac{762048}{691}\Delta(z),$$

where for $k \geq 4$,

$$E_k(z) := 1 - \frac{2k}{B_k}\sum_{l=1}^{\infty}\sigma_{k-1}(l)\cdot e(lz)$$

is the usual weight k Eisenstein series. Here $e(z) := e^{2\pi i z}$. Moreover these congruences can be explained by the theory of ℓ-adic Galois representation developed by Deligne, Serre, and Swinnerton-Dyer. For details we refer the reader for example to [Ka2, Ri, Se, SD]. It is known that these type of congruences are also related to special values of the standard L-function (which will also play an important role in our paper) and the relations between the denominators of certain L-functions. For example the famous Ramaujan 691 congruence is related to the special value of the Rankin-Selberg zeta function attached to the Hecke eigenform Δ. The above congruences can be used to prove that for allmost all integers n, $5\cdot 7\cdot 691$ divides $\tau(n)$ (see for example [Mo]). In 1978 Doi and Hida [DH1] discovered nontrivial congruences among the Hecke eigenform (see also [DH2]). Recently progress has been made also in the context of Siegel modular forms by Katsurada [Ka1], Mizumoto [Mi2], and Böcherer and Nagaoka [BN].

In certain special cases congruences between modular forms of half-integer weight can also be deduced from congruences of modular forms of integral weight (see e.g. [Ko, AK]) by "inverting" the Shimura map. To the authors knowledge there exists no general theory which can be used to obtain congruences for half-integral weight modular forms from integer weight ones in each single case. In the situation of Jacobi forms one could similarly as in [Ko] find single congruences. Here we choose a different approach which leads to infinitely many congruences. In Theorem 1.1 and Theorem 1.3 we obtain congruences between Jacobi Eisenstein series and Jacobi cusp forms. Using results of [Tr] and the correspondence between Jacobi forms of half-integer weight, one could also obtain that infinitely many coefficients of Jacobi cusp forms are congruent to 0 modulo p for each prime $p \neq 2, 3$. Congruences for Jacobi forms were also for example considered in [G1, G2]. In Theorem 1.4 we show that the congruences for Jacobi forms considered in Theorem 1.1 are "optimal". In Theorem 1.5 we prove congruences between special values of different kind of L-functions (Hecke- and Rankin type) attached to elliptic cusp forms of different weights .

Let us illustrate our results with an example. Let f be the primitive elliptic cusp form of weight 22 with Fourier coefficents $a^f(n)$ and $\chi_{12,1}$ the Jacobi cusp form of weight 12 and index 1, normalized such that the Fourier coefficient associated to the fundamental discriminant -3 is 1. Then Theorem 1.5 implies that

$$\frac{\zeta(22)\sum_{n=1}^{\infty}\tau(n)^2\,n^{-22}}{\zeta(11)\,\pi^{33}\parallel\Delta\parallel^2} \equiv 394\cdot 593\frac{\sum_{n=1}^{\infty}a^f(n)\,n^{-21}}{\pi^{21}\parallel\chi_{12,1}\parallel^2} \pmod{593}. \tag{1.2}$$

Moreover

$$\frac{\zeta(22)\,\sum_{n=1}^{\infty}\tau(n)^2\,n^{-22}}{\zeta(11)\,\pi^{33}\parallel\Delta\parallel^2}\in\mathbb{Z}^*_{131\cdot 593}.$$

See also [Ka1].

In the following assume that m is a positive square-free integer. We let k and g be positive integers with k even, and $k>g+2$ and let $(\tau,z)\in\mathbb{H}_g\times\mathbb{C}^g$, where $\mathbb{H}_g$ denotes the Siegel upper half plane of genus g. We define the Jacobi Eisenstein series of weight k, index m, and degree g as

$$E^g_{k,m}(\tau,z):=\sum_{\gamma\in\Gamma^J_\infty\backslash\Gamma^J_g}1|_{k,m}\gamma(\tau,z),$$

where Γ^J_g denotes the usual Jacobi group, $|_{k,m}$ is the slash operator for Jacobi forms, and Γ^J_∞ is the stabilizer group of the function 1 in Γ^J_g. Moreover for a non-positive discriminant $D=D_0f^2$ ($f\in\mathbb{N}_0$, D_0 a fundamental discriminant), we define for $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1$

$$L_D(s):=\begin{cases}\zeta(2s-1) & \text{if } D=0,\\ L_{D_0}(s)\sum_{d|f}\mu(d)\chi_{D_0}(d)\,d^{-s}\,\sigma_{1-2s}(f/d) & \text{if } D\neq 0.\end{cases}\tag{1.3}$$

Here $L_{D_0}(s):=\sum_{n>0}\chi_{D_0}(n)\,n^{-s}$, where χ_{D_0} is the primitive Dirichlet character associated to D_0, and $\mu(\cdot)$ is the usual Möbius function. The function $L_D(s)$ has an analytic continuation to the whole complex plane and the values $L_D(2-k)$ for k even are known to be rational and non-zero (see [Za] page 130). If $g=1$, then we drop the index.

We now take the following basis for $J^{\mathrm{cusp}}_{k,m}$, the vector space of Jacobi cusp forms on $\mathbb{H}\times\mathbb{C}$ (see also [He]). For $d|m$, we let $(\Phi_{j_d,d})_{j_d}$ be a primitive Hecke eigenbasis for the space of new forms $J^{\mathrm{new}}_{k,d}$. As is well-known, this subspace is isomorphic as a Hecke module to a subspace $M^{\mathrm{new},-}_{2k-2}(m)$ of elliptic cusp forms (see [SZ] page 138 (ii)). Define the function $\Psi_{j_d,d}:=\Phi_{j_d,d}|T_-\left(\frac{m}{d}\right)$, where $T_-(\cdot)$ is a certain Hecke type operator defined in [He]. Then $(\Psi_{j_d,d})_{j_d,d}$ is a Hecke eigenbasis for $J^{\mathrm{cusp}}_{k,m}$. Each $\Phi_{j_d,d}$ corresponds to an elliptic cusp form f_{j_d} under the above described correspondence. In [He] (see Definition 2.4), the L-series $L(s,\Phi_{j_d,d})$ is defined in such a way that it coincides with the Hecke L-series $L(s,f_{j_d})$, where for a primitive new form $f(z)=\sum_{n=1}^{\infty}a(n)q^n$, we define $L(s,f):=\sum_{n=1}^{\infty}a(n)n^{-s}$. If $m=1$, then we also write $(\Phi_j)_j$ and f_j, where now f_j is the primitive Hecke eigenform that corresponds to Φ_j under the Saito-Kurokawa correspondence. In the following we mean by $\alpha\equiv\beta \pmod{p^e}$ for $\alpha,\beta\in\mathbb{Q}_p, e\in\mathbb{N}$ that $p^e|(\alpha-\beta)$. More generally we consider congruences modulo ideals $\wp$ lying above p in the sense of ideals in $\mathcal{O}_K$, the ring of integers of some number field K. Usually we take $\mathcal{O}_{K_f}$ where for we denote for a Hecke eigenform $f\in S_k$ by K_f the totally real number field generated by the Hecke eigenvalues λ_f of f. We now state our first theorem.

Theorem 1.1. *Assume that m is a square-free integer and that $k>8$ is even. Then there exists a prime p and $\alpha\in\mathbb{N}$ such that $p^\alpha\parallel\frac{B_{2k-2}\cdot\sigma_{k-1}(m)}{2k-2}$ and $p|\frac{B_{2k-2}}{2k-2}$, where B_k is the $k-$th*

Bernoulli number. Now define

$$c_{k,m,d} := \frac{(-1)^{\frac{k}{2}}(2k-4)!\sigma_{k-2}(m/d)\cdot d^{k-2}}{2^{3k-5}}.$$

Then we have for $r^2-4nm \leq 0$:

$$\sum_{d|(n,r,m)} d^{k-1} L_{\frac{r^2-4nm}{d^2}}(2-k)p^{\alpha}E_{k,m}(\tau,z) \tag{1.4}$$

$$\equiv \sum_{d|m} c_{k,m,d} \sum_{i_d=1}^{dim\ J_{k,d}^{new}} p^{\alpha}\, \frac{L(2k-3,\Phi_{i_d,d})}{\pi^{2k-3}\ \|\ \Psi_{i_d,d}\ \|^2}\, c^{i_d,d}(n,r)\cdot\Psi_{i_d,d}(\tau,z) \pmod{p^{\alpha}},$$

where $c^{i_d,d}(\cdot)$ denote the Fourier coefficients of $\Psi_{i_d,d}$. Here $\|\cdot\|$ is the usual Petersson norm for Jacobi forms. Note that both sides of (1.4) are in $\mathbb{Z}_p$.

From this we conclude.

Corollary 1.2. *If $m=1$, then we have for $D_0 := r^2-4n<0$*

$$L_{D_0}(2-k)\cdot p^{\alpha}E_{k,1}(\tau,z)$$

$$\equiv (-1)^{k/2}\,2^{5-3k}(2k-4)! \sum_{j=1}^{dim\ J_{k,1}^{cusp}} \frac{p^{\alpha}L(2k-3,f_j)}{\pi^{2k-3}\ \|\ \Phi_j\ \|^2}\, c^j(n,r)\cdot\Phi_j(\tau,z) \pmod{p^{\alpha}}.$$

We now illustrate Theorem 1.1 with an example (for details see Section 5). We let $\chi_{10,1}(\tau,z)\in J_{k,m}^{\text{cusp}}$ be the unique element with $(1,1)$-th Fourier coefficient equal to 1. Then Theorem 1.1 implies the congruence

$$43867E_{10,1}(\tau,z)\equiv 16564\cdot\chi_{10,1}(\tau,z) \pmod{43867}.$$

Following McGraw and Ono [MO] we call a prime p a *congruence prime for a primitive Hecke eigenform* $f(z)\in \mathrm{S}_k$ if there is another primitive Hecke eigenform $f_1(z)\in S_k$ for which

$$f(z)\equiv f_1(z) \pmod{\wp}$$

for some prime ideal $\wp$ above p in the ring of algebraic integers of a suitable large number field. Congruence primes are rather rare. According to numerical data of W. Stein one has for example that the only prime $p<10^4$ that is a congruent prime for any primitive Hecke eigenform in S_{p+1} is $p=389$. If p is not a congruence prime for any $f\in \mathrm{S}_k$, we also call p a *non-congruence prime* for S_k.

Let us first consider the case of elliptic modular forms. We put $G_k(z) := \frac{B_k}{-2k}E_k(z)$. Let p a prime number which divides B_k/k and let $\wp$ be a prime ideal above p. Then there exist a primitive Hecke eigenform $f\in S_k$ such that

$$G_k(z)\equiv f(z) \pmod{\wp}.$$

This depends mainly on the fact, that Fourier coefficients are equal to the eigenvalues and that the algebra of elliptic modular forms is generated be E_4 and E_6. In the setting of

Jacobiforms or modular forms of half-integral weight, the situation is different. It is an open question in the theory of Jacobi forms and Siegel modular forms (this is related to the Maass conjecture of Siegel modular forms [Ma]) if $p^\alpha E_{k,1}$ is not congruent to zero modulo p^α in general. On the other hand if if $p^\alpha E_{k,1}$ is congruent to zero modulo p^α we can by trivial reasons always find an eigen cuspform $\phi \in J_{k,1}^{\mathrm{cusp}}$ which is congruent to the "deformed" Jacobi Eisenstein series. Therefore, we assume in the following that $p^\alpha E_{k,1}(\tau, z) \not\equiv 0 \pmod{p^\alpha}$. In contrast to the case of elliptic modular forms where for a Hecke eigenform one can choose the first coefficient to be 1, there is no uniqe normalization for coefficients for Jacobi forms. Let $D_0 := r^2 - 4n < 0$ then we fix a certain normalization attached to D_0 of Jacobi Eisenstein series of index 1:

$$G_{k,1}^{D_0}(\tau, z) := L_{D_0}(2-k) \cdot p^\alpha E_{k,1}(\tau, z).$$

Theorem 1.3. *Let k be an even positive integer. Let p a prime number which divides $\frac{B_{2k-2}}{2k-2}$ and which is a non-congruence prime for S_{2k-2}. Let $\wp$ be an ideal above p in the number field generated by all the Hecke eigenvalues of the primitive Hecke eigenbasis of S_{2k-2}. Then there exists a Hecke eigenform with totally real coefficients $\Phi \in J_{k,1}^{cusp}$ such that we have for all D_0*

$$G_{k,1}^{D_0}(\tau, z) \equiv \widetilde{L}(2k-3, f)^{p, D_0} \cdot c^\Phi(n, r)\Phi(\tau, z) \pmod{\wp}, \tag{1.5}$$

where $f \in S_{2k-2}$ is the form corresponding to Φ, and where we put for simplification

$$\widetilde{L}(2k-3, f)^{p, D_0} := (-1)^{k/2}\, 2^{5-3k}(2k-4)! \frac{p^\alpha L(2k-3, f)}{\pi^{2k-3} \parallel \Psi \parallel^2} \in K_f.$$

In particular there exists a D_0 such that

$$G_{k,1}^{D_0}(\tau, z) \equiv \widetilde{L}(2k-3, f)^{p, D_0} \cdot \Phi(\tau, z) \pmod{\wp},$$

Moreover if we evaluate the congruence (1.5) at the Fourier coefficient $c(n, r)$ we get a congruence for the special value $\widetilde{L}$. It is maybe worthwile to note that the normalization constant to get an explicit congruence is given by a certain special value of a L-function of Hecke type. For Jacobi forms of general index a similar statement would be possible. Since this would lead to more technical complications, we chose not to address this question here.

We next show that for infinitely many k, m, and $D = r^2 - 4nm$ that Theorem 1.1 is optimal, i.e., the (n, r)-th coefficient of the left-hand side of (1.4) is an element of $\mathbb{Z}_p^*$.

Theorem 1.4. *There exist infinitely many positive integers k and m, such that for a prime p with $p^\alpha \parallel \frac{B_{2k-2}}{2k-2}$ we have for infinitely many discriminants $D = r^2 - 4nm < 0$*

$$(2k-4)! \sum_{d|m} \sum_{i_d} p^\alpha \frac{L(2k-3, \Phi_{i_d,d})}{2^{\omega(d)} \parallel \Psi_{i_d,d} \parallel^2 \pi^{2k-3}} c^{i_d,d}(n, r)^2 \equiv \epsilon \pmod{p^\alpha}, \tag{1.6}$$

where $\epsilon \in \mathbb{Z}_p^*$ is explicitely computable, and where $\omega(d)$ denotes the number of prime divisors of d.

To state our next theorem which gives congruences between L-functions, we define for $f(z)=\sum_{n=1}^{\infty} a(n)q^n \in S_k$ the Dirichlet series

$$D_f(s) := \frac{\zeta(2s-2k+2)}{\zeta(s-k+1)} \sum_{n=1}^{\infty} a(n)^2 n^{-s}.$$

Theorem 1.5. *Assume that $k>8$ is an even integer and p a prime with $p^\alpha \parallel \frac{B_{2k-2}}{2k-2}$, where $\alpha \in \mathbb{N}$ and $\left(p, \frac{B_k}{k}\right) = 1$. Let further $D_0 < 0$ be a fundamental discriminant such that $\nu_p(L_{D_0}(2-k)) = 0$. Let $\{\Phi_j\}_j$ be a primitive Hecke-Jacobi eigenbasis of $J_{k,1}^{cusp}$, which is ordered and normalized in such a way that for $d'_k \in \mathbb{N}$ we have that $c^j(D_0) = 1$ if and only if $1 \le j \le d'_k$ and 0 otherwise. Here $c^j(D_0)$ is the Fourier coefficient of Φ_f corresponding to the discriminant D_0. Moreover let $(g_j)_j$ be a primitive Hecke eigenbasis of S_k. Then we have*

$$\begin{aligned}(1.7)\quad \varepsilon_{k,p,D_0}\,(k-2)!(2k-3)! &\sum_{j=1}^{\dim S_k} \frac{D_{g_j}(2k-2)}{\pi^{3k-3} \parallel g_j \parallel^2} \\ &\equiv (2k-4)! \sum_{j=1}^{d'_k} p^\alpha \frac{L(2k-3, f_j)}{\pi^{2k-3} \parallel \Phi_j \parallel^2}\; a^{\mathbb{W}\Phi_j}(1) \qquad \pmod{p^\alpha},\end{aligned}$$

where $a^{\mathbb{W}\Phi_j}(1)$ is the first coefficient of $\Phi(\tau,0)$, and

$$\varepsilon_{k,p,D_0} := (-1)^{\frac{k}{2}+1} p^\alpha\, 2^{1-k}\, \zeta(3-2k)^{-1} L_{D_0}(2-k) \in \mathbb{Z}_p^*.$$

Remark. Formulae of type (1.7) which involve congruences between special values of L-functions of different type seem to be new. From the proof of the theorem the existence can be expected without going into concrete calculations. But the pattern behind our result is not clear and will hopefully give inspiration to further study of such congruences. We believe that applying the method used in the proof of theorem 1.3 will reduce the sum on the left side of (1.7) to single special value by going to congruences to prime ideals above p.

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2. Basic facts about Jacobi forms

Let us recall some basic facts about Jacobi forms. For details we refer the reader to [EZ] and [Zi]. The Jacobi group $\Gamma_g^J := \Gamma_g \ltimes (\mathbb{Z}^g \times \mathbb{Z}^g)$, where $g \in \mathbb{N}$ and $\Gamma_g := \mathrm{Sp}_g(\mathbb{Z})$, acts on $\mathbb{H}_g \times \mathbb{C}^g$ via

$$(M, (\lambda, \mu)) \circ (\tau, z) := \left((A\tau + B)(C\tau + D)^{-1}, (z + \lambda\tau + \mu)(C\tau + D)^{-1}\right).$$

Let k and m be positive integers, and Φ a complex valued function on $\mathbb{H}_g \times \mathbb{C}^g$. If $\gamma = \left(\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), (\lambda, \mu) \right) \in \Gamma_g^J$, we define

$$\begin{aligned} &\Phi|_{k,m}\gamma(\tau, z) \\ &:= \det(C\tau + D)^{-k} e\left(-m(C\tau+D)^{-1}C[(z+\lambda\tau+\mu)^t] + m\tau[\lambda^t] + 2m\lambda z^t\right) \cdot \Phi\left(\gamma \circ (\tau, z)\right), \end{aligned}$$

where $A[B] := B^t AB$ for matrices of compatible sizes and $e(A) := e^{2\pi i \mathrm{tr}\, A}$ for a square matrix A. Let $J_{k,m}^g$ be the space of Jacobi form of weight k, degree g, and index m with respect to Γ_g^J, i.e., the space of holomorphic functions $\Phi : \mathbb{H}_g \times \mathbb{C}^g \to \mathbb{C}$ satisfying $\Phi|_{k,m}\gamma = \Phi$ for all $\gamma \in \Gamma_g^J$ that have a Fourier expansion of the form

$$\Phi(\tau, z) = \sum_{\left(\begin{smallmatrix} n & r/2 \\ r^t/2 & m \end{smallmatrix}\right) \in \mathbb{A}_{g+1}} c(n, r) e\left(n\tau + rz\right), \tag{2.1}$$

where $\mathbb{A}_g$ denote the set of half-integral, symmetric, semi-positive $g \times g$ matrices. Moreover $J_{k,m}^{g,\mathrm{cusp}}$ denotes the vector space of those Jacobi forms for which in (2.1) the sum only runs over positive definite matrices. If $g = 1$, then we drop the index.

The space $J_{k,m}^{g,\mathrm{cusp}}$ is a finite dimensional Hilbert space with the Petersson scalar product

$$\langle \Phi, \Psi \rangle := \int_{\Gamma_g^J \backslash \mathbb{H}_g \times \mathbb{C}^{(1,g)}} \Phi(\tau, z) \cdot \overline{\Psi(\tau, z)} \cdot (\det v)^k \exp\left(-4\pi \mathrm{tr}\left(mv^{-1}[y^t]\right)\right) dV_g^J,$$

where $\tau = u + iv$, $z = x + iy$, and $dV_g^J := (\det v)^{-(g+2)} dx\, dy\, du\; dv$.

Let $\Phi \in J_{k,m}^{\mathrm{cusp}}$ be a Hecke Jacobi eigenform and for simplificity we let m be square-free. Let K_Φ be the totally real field generated by its Hecke eigenvalues. Then by standard arguments (see for example [Ga] and [Mi1]) one shows that it is possible to normalize Φ in such a way that all its Fourier coefficients lie in K_Φ. Such Hecke-Jacobi eigenforms we call primitive. In this setting we say that K_Φ is the coefficient field of Φ.

3. Jacobi and Siegel Eisenstein series

Here we recall basic facts on Jacobi and Siegel Eisenstein series and explain their connection. For details we refer the reader to [Kl] and [Zi].

Define for $Z \in \mathbb{H}_g$ and even $k > g + 1$ the Siegel Eisenstein series

$$E_k^g(Z) := \sum_{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma_g} \det(CZ + D)^{-k},$$

where $\Gamma_\infty := \left\{ \left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) \in \Gamma_g \right\}$. This series is absolutely and locally uniformly convergent and an element of $M_k(\Gamma_g)$, the vector space of Siegel modular forms of weight k and genus g. We denote its Fourier coefficients by $A^{E_k^g}(T)$, where $T \in \mathbb{A}_g$. Here $A^{E_k^g}(0) = 1$. Siegel [S1, S2] proved that the coefficients $A^{E_k^g}(T)$ are rational and have bounded denominators. If $g = 1$, then we drop the index.

Let

$$\begin{aligned} N^*_{2k-g} &:= \prod_{p | D_{2k-g}} p^{1+\nu_p(k-g/2)}, \\ N^{**}_{2k-g} &:= \prod_{\substack{p | D_{2k-g} \\ p \equiv 3 \pmod 4}} p^{1+\nu_p(k-g/2)}, \end{aligned}$$

where D_k denotes the denominator of B_k and where $\nu_p(n)$ for a rational number n is the p-order of n. Moreover let

$$\mathcal{D}_{k,g} := \begin{cases} 2^g \frac{k}{B_k} \prod_{i=1}^{(g-1)/2} \frac{k-i}{B_{2k-2i}} & \text{if } g \text{ is odd,} \\ 2^g \frac{k}{B_k} \frac{1}{N^*_{2k-g}} \prod_{i=1}^{g/2} \frac{k-i}{B_{2k-2i}} & \text{if } g \equiv 0 \pmod 4, \\ 2^{g-1} \frac{k}{B_k} \frac{1}{N^{**}_{2k-g}} \prod_{i=1}^{g/2} \frac{k-i}{B_{2k-2i}} & \text{if } g \equiv 2 \pmod 4. \end{cases}$$

Böcherer showed that $\mathcal{D}_{k,g}$ is a common divisor of all $a^{E_k^g}(T)$ with T positive definite. This generalizes the case $g = 2$ done by [Ma]. For odd g it is known that $\mathcal{D}_{k,g}$ is the greatest common divisor of all $a^{E_k^g}(T)$ with T positive definite. The case that g is even is still open but numerical evidence suggests that $\mathcal{D}_{k,g}(T)$ is also a greatest common divisor of $A^{E_k^g}(T)$ with T positive definite. We also need the common divisors of $A^{E_k^g}(T)$ with $\mathrm{rank}(T) < g$. If $\mathrm{rank}(T) = r < g$, then there exists a unimodular matrix U such that $T[U] = \left(\begin{smallmatrix} T_1 & 0 \\ 0 & 0_{g-r} \end{smallmatrix}\right)$ with T_1 positive definite and we have

$$A^{E_k^g}(T) = A^{E_k^r}(T_1).$$

In particular $D_{k,r}$ is a common divisor of $A^{E_k^g}(T)$ with $\mathrm{rank}(T) = r$.

Let us next consider Jacobi Eisenstein series. Define for $k, m, g \in \mathbb{N}$ with k even and $k > g + 2$ the Jacobi Eisenstein series of weight k, index m, and degree g as

$$E^g_{k,m}(\tau, z) := \sum_{\gamma \in \Gamma^J_\infty \backslash \Gamma^J_g} 1|_{k,m} \gamma(\tau, z),$$

where $\Gamma^J_\infty := \left\{ \left(\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right), (0, \mu) \right) \in \Gamma^J_g \right\}$. This series is absolutely and locally uniformly convergent and an element of $J^g_{k,m}$. We denote its Fourier coefficients by $A^{E^g_{k,m}}(n, r)$.

To see the connection between the Jacobi and Siegel Eisenstein series we consider the Fourier-Jacobi expansion of the Siegel Eisenstein series. Writing $\left(\begin{smallmatrix} \tau & z^t \\ z & \widetilde{\tau} \end{smallmatrix}\right)$, we get

$$E_k^{g+1}(Z) =: \sum_{m=0}^{\infty} e^g_{k,m}(\tau, z) \cdot \widetilde{q}^{\,m},$$

where $\widetilde{q} := e(\widetilde{\tau})$. Clearly $e^g_{k,m}(\tau, z) \in J^g_{k,m}$. For square-free m and n, r such that $\begin{pmatrix} n & r/2 \\ r^t/2 & m \end{pmatrix} \in \mathbb{A}_{g+1}$, we have (see [Zi] Section 4.4)

$$A^{E^g_{k,m}}(n,r) = \left(-\frac{2k}{B_k}\,\sigma_{k-1}(m)\right)^{-1} A^{E^{g+1}_k}\left(\begin{pmatrix} n & r/2 \\ r^t/2 & m \end{pmatrix}\right). \tag{3.1}$$

In this case, we have (see [EZ] Section 6)

$$A^{E_{k,m}}(n,r) = -\frac{2k-2}{B_{2k-2}\cdot\sigma_{k-1}(m)} \sum_{d|(n,r,m)} d^{k-1} L_{\frac{r^2-4nm}{d^2}}(2-k). \tag{3.2}$$

4. Pullback formula of Jacobi Eisenstein series

Define the Witt operator $\mathbb{W} : J^2_{k,m} \to Sym^2 J_{k,m}$ by

$$(\mathbb{W}\Phi)\left((\tau,z),(\widetilde{\tau},\widetilde{z})\right) := \Phi\left(\begin{pmatrix} \tau & 0 \\ 0 & \widetilde{\tau} \end{pmatrix}, (z,\widetilde{z})\right).$$

If $\Phi(\tau,z) = \sum c(n,r)\cdot e\left(n\tau + rz\right)$, then the $((n_1,r_1),(n_2,r_2))$-th Fourier coefficient of $\mathbb{W}\Phi$ related to $\left((\tau,z),(\widetilde{\tau},\widetilde{z})\right)$ equals $\sum_l c(n_l,r)$, where $r=(r_1,r_2)$ and l runs through all integers such that $n_l = \begin{pmatrix} n_1 & l/2 \\ l/2 & n_2 \end{pmatrix}$ and $\begin{pmatrix} n_l & r^t \\ r & m \end{pmatrix} \in \mathbb{A}_3$.

Proposition 4.1. *Let $k>4$ be even and let $m \in \mathbb{N}$ be square-free. Let $(\Phi_i)_i$ be an orthogonal normalized Hecke eigenbasis of $J^{cusp}_{k,m}$. Then we have*

$$\mathbb{W}E^2_{k,m} = E_{k,m}\otimes E_{k,m} + \sum_i \alpha_i \Phi_i \otimes \Phi_i, \tag{4.1}$$

where

$$\alpha_i = \frac{(-1)^{k/2}\pi\, 2^{1-k}}{m\ (k-3/2)}\ \frac{Z^J(k,\Phi_i)}{\parallel \Phi_i \parallel^2}.$$

Here $Z^J(s,\Phi_i) := \sum_n \lambda(\Phi_i,n)\ n^{-s}$ with $\lambda(\Phi_i,n)$ the eigenvalue of Φ_i under the n-th Hecke-Jacobi operator.

Proof. We recall the basic fact that $\mathbb{W}E^2_{k,m} \in Sym^2 J_{k,m}$. This was first observed by Witt in the context of Siegel modular forms of genus 2. Projecting $\mathbb{W}E^2_{k,m}$ against Φ_j shows by a result of Arakawa (Theorem 2.8 of [Ar])

$$\left\langle \Phi_i(\cdot)\,,\,\mathbb{W}E^2_{k,m}\left((-\overline{\tau},\overline{z}),\cdot\right)\right\rangle = \frac{(-1)^{k/2}\pi\, 2^{1-k}}{m\ (k-3/2)}\ Z^J(k,\Phi_i)\ \Phi_i(\tau,z). \tag{4.2}$$

This gives that the proposition is true up to the Eisenstein series part. Here we used the technical assumption that the involved Jacobi forms have totally real Fourier coefficients which leads to $\overline{\Phi(-\overline{\tau},\overline{z})} = \Phi(\tau,z)$. Comparing the constant term shows the proposition. □

In [He] it has been shown that $\alpha_i \neq 0$, hence we have

Corollary 4.2. *Let Φ_i be as in the proposition with coefficient field K_{Φ_i}. Then*

$$\frac{\pi Z^J(k,\Phi_i)}{\parallel \Phi_i \parallel^2} \in K^*_{\Phi_i}.$$

If Φ_i is a primitive new form with related elliptic cusp form f_i of weight $2k-2$, then

$$\frac{\pi L(2k-3,f_i)}{\zeta(2k-2) \parallel \Phi_i \parallel^2} \in K^*_{f_i}.$$

For the case $m=1$ see also Yasushi Tokuno [To].

5. Proof of Theorem 1.1

5.1. **Proof of the Theorem.**

Proof. Since $\left|\frac{B_k}{k}\right| > 1$ for $k > 14$ there exists always a prime p such that $p|\left|\frac{B_k}{k}\right|$. We apply Proposition 4.1 and use as a Hecke eigenbasis the functions $(\Psi_{i_d,d})_{i_d,d}$ defined in the introduction. Then $\alpha_{i_d,d} = \frac{(-1)^{k/2}\pi\cdot Z^J(k,\Psi_{i_d,d})}{2^{k-2}\cdot m\cdot(2k-3)\|\Psi_{i_d,d}\|^2}$, where we write $\alpha_{i_d,d}$ instead of α_d in Proposition 4.1. Moreover we conclude from [He] Theorem 3.1

$$Z^J(k,\Psi_{i_d,d}) = \frac{\prod_{p|\frac{m}{d}}\left(1+p^{-(k-2)}\right)}{\zeta(2k-2)\prod_{p|m}\left(1+p^{-(k-1)}\right)} L(2k-3,\Phi_{i_d,d}).$$

This gives that

$$\alpha_{i_d,d} = \frac{(-1)^{k/2}\,(2k-2)!\cdot d^{k-2}\cdot \sigma_{k-2}(m/d)\cdot L(2k-3,\Phi_{i_d,d})}{2^{3k-5}\cdot \pi^{2k-3}\cdot(2k-3)\cdot\sigma_{k-1}(m)\cdot B_{2k-2}\cdot \parallel \Psi_{i_d,d}\parallel^2}.$$

Now formula (4.1) leads to

$$\begin{aligned} &A^{\mathbb{W}E^2_{k,m}}\left((n_1,r_1),(n_2,r_2)\right) \\ &\qquad = A^{E_{k,m}}\left(n_1,r_1\right) A^{E_{k,m}}\left(n_2,r_2\right) + \sum_{d|m}\sum_{i_d=1}^{\dim J^{\text{new}}_{k,d}} \alpha_{i_d,d}\cdot c^{i_d,d}(n_1,r_1)\cdot c^{i_d,d}(n_2,r_2), \end{aligned}$$

where $A^{\mathbb{W}E^2_{k,m}}(\cdot)$ denote the Fourier coefficients of $\mathbb{W}E^2_{k,m}$. Now let $p^\alpha \parallel \frac{B_{2k-2}\cdot\sigma_{k-1}(m)}{2k-2}$ with $p|\frac{B_{2k-2}}{2k-2}$ be given. From [Ma] and [Bo] one can see that

$$p^\alpha A^{\mathbb{W}E^2_{k,m}}\left((n_1,r_1),(n_2,r_2)\right) \in \mathbb{Z}_p. \tag{5.1}$$

For this we distinguish the possible ranks of $T_l := \begin{pmatrix} n_1 & l/2 & r_1 \\ l/2 & n_2 & r_2 \\ r_1 & r_2 & m\end{pmatrix} \in \mathbb{A}_3$ and show that

$$p^\alpha A^{E^2_{k,m}}\left(\begin{pmatrix} n_1 & l/2 \\ l/2 & n_2\end{pmatrix},(r_1,r_2)\right) \in \mathbb{Z}_p. \tag{5.2}$$

If $\mathrm{rank}(T_l) = 1$, then $T_l[U] = \left(\begin{smallmatrix} t & 0 \\ 0 & 0_2 \end{smallmatrix}\right)$ for some unimodular matrix U and $t \in \mathbb{N}$ and

$$A^{E^2_{k,m}}\left(\left(\begin{smallmatrix} n_1 & l/2 \\ l/2 & n_2 \end{smallmatrix}\right), (r_1, r_2)\right) = -\frac{B_k}{2k\sigma_{k-1}(m)}\, A^{E_k}(t) = \frac{\sigma_{k-1}(t)}{\sigma_{k-1}(m)},$$

which directly implies that (5.2) holds. If $\mathrm{rank}(T_l) = 2$, then $T_l[U] = \left(\begin{smallmatrix} T & 0 \\ 0 & 0_1 \end{smallmatrix}\right)$ for some unimodular matrix U and a positive definite matrix T, and we have

$$A^{E^2_{k,m}}\left(\left(\begin{smallmatrix} n_1 & l/2 \\ l/2 & n_2 \end{smallmatrix}\right), (r_1, r_2)\right) = -\frac{B_k}{2k\sigma_{k-1}(m)}\, A^{E^2_k}(T). \tag{5.3}$$

From [Bo] it follows that (5.3) equals

$$u_p\, \frac{(2k-2)\cdot B(k-1,\eta_T)}{\sigma_{k-1}(m)\cdot B_{2k-2}(k-1)},$$

where u_p is in $\mathbb{Z}_p$, η_T is some character, and $B(k-1,\eta_T)$ is a generalized Bernoulli number. From the properties of generalized Bernoulli numbers [Ca, Le] it is well-known that $\frac{B(k-1,\eta_T)}{(k-1)} \in \mathbb{Z}_p$. Here we need that $(p-1) \nmid 2(k-1)$ which follows from the assumptions in the theorem. This gives (5.2). If $\mathrm{rank}(T) = 3$, then (5.2) follows since $\mathcal{D}_{k,3}^{-1} \cdot A^{E^3_k}(T) \in \mathbb{Z}_p$ if T is positive definite.

Using (3.2) and (5.1) gives that

$$\begin{aligned} p^{\alpha}\, \frac{2k-2}{B_{2k-2}\cdot\sigma_{k-1}(m)} \sum_{d\mid(n_1,r_1,m)} d^{k-1}\cdot p^{\alpha} L_{\frac{r_1^2-4n_1m}{d^2}}(2-k)\cdot A^{E_{k,m}}(n_2,r_2) \\ \equiv p^{2\alpha} \sum_{d\mid m} \sum_{i_d=1}^{\dim J^{\text{new}}_{k,d}} \alpha_{i_d,d}\cdot c^{i_d,d}(n_1,r_1)\cdot c^{i_d,d}(n_2,r_2) \pmod{p^{\alpha}}. \end{aligned}$$

This leads to the Theorem. □

Proof of Corollary 1.2. It follows from Arakawa [Ar] Section 6.1 that

$$Z^J(k,\Phi_j) = \frac{L(2k-3,f_j)}{\zeta(2k-2)}.$$

Thus we get

$$\alpha_j = \frac{(-1)^{k/2}(2k-2)!\cdot L(2k-3,f_j)}{2^{3k-5}\cdot\pi^{2k-3}\cdot(2k-3)\cdot B_{2k-2}\cdot \parallel \Phi_j \parallel^2} \tag{5.4}$$

which implies the corollary. □

5.2. **Example.** We apply Theorem 1.1 with $k = 10$ and $m = 1$. We have that $\dim J^{\text{cusp}}_{10,1} = 1$ and we let $\chi_{10,1}(\tau,z)$ be the unique element in $J^{\text{cusp}}_{10,1}$ normalized with $c^{\chi_{10,1}}(1,1) = 1$, where $c^{\chi_{10,1}}(\cdot)$ denote the Fourier coefficients of $\chi_{10,1}$ (see [EZ]). We have that $B_{18} = \frac{43867}{798}$, and let $p = 43867$, $\alpha = 1$. Theorem 1.1 gives that

$$43867\cdot L_{-3}(-8)\cdot A^{E_{10,1}}(n,r) \equiv 43867\cdot c_{10,1,1}\frac{L(17,\chi_{10,1})}{\parallel \chi_{10,1} \parallel^2 \pi^{17}}\, c^{\chi_{10,1}}(n,r) \pmod{43867}. \tag{5.5}$$

We have in particular for $n = r = 1$, using that $A^{E_{10,1}}(1,1) = -\frac{18 L_{-3}(-8)}{B_{18}}$,

$$-43867\,\frac{18}{B_{18}}\,L_{-3}^2(-8) \equiv 43867 \cdot c_{10,1,1}\,\frac{L(17,\chi_{10,1})}{\parallel \chi_{10,1} \parallel^2\,\pi^{17}} \pmod{43867}.$$

Plugging this back into (5.5) gives that

$$-18 \cdot 798 \cdot L_{-3}(-8)\chi_{10,1}(\tau,z) \equiv 43867 \cdot E_{10,1}(\tau,z) \pmod{43867}.$$

Here $L_{-3}(-8)$ can be calculated with Pari and gives the value $\frac{2\cdot 809}{33}$. Thus

$$16564 \cdot \chi_{10,1}(\tau,z) \equiv 43867 \cdot E_{10,1}(\tau,z) \pmod{43867}.$$

6. Proof of Theorem 1.3

Proof. Assume the notation in Theorem 1.3. Corollary 1.2 implies that

$$G_{k,1}^{J,D_0}(\tau,z) \equiv \sum_{j=1}^{d} \widetilde{L}(2k-3,f_j)^{p,D_0} c^{\Phi_j}(n,r) \cdot \Phi_j(\tau,z) \pmod{\wp}, \tag{6.1}$$

where $d := \dim S_{2k-2}$. Without loss of generality we may now assume that $f_1 \equiv G_{k,1}^{J,D_0}(\tau,z)$ (mod $\wp$), i.e., we have for all n

$$\lambda_{f_1}(n) \equiv \sigma_{2k-3}(n) \pmod{\wp}.$$

Moreover, since p is a non-congruence prime for S_{2k-2}, there exists $l_1 \in \mathbb{N}$ such that

$$\lambda_{f_d}(l_1) \not\equiv \lambda_{f_1}(l_1) \pmod{\wp}.$$

Applying the $T^J(l_1)$ Hecke operator to equation (6.1) gives

$$\begin{aligned} 0 &\equiv (\sigma_{2k-3}(l_1) - \lambda_{f_1}(l_1)) \cdot G_{k,1}^{J,D_0}(\tau,z) \\ &\equiv \sum_{j=2}^{d} \left(\lambda_{f_j}(l_1) - \lambda_{f_1}(l_1)\right) \widetilde{L}(2k-3,f_j)^{p,D_0} \cdot c^{\Phi_j}(n,r)\Phi_j(\tau,z) \pmod{\wp}. \end{aligned}$$

Next choose $l_j \in \mathbb{N}$ $(2 \le j \le d)$ such that

$$\lambda_{f_d}(l_j) \not\equiv \lambda_{f_j}(l_j) \pmod{\wp}.$$

Succesively applying the $T^J(l_j)$ Hecke operator and subtracting $\lambda_{f_j}(l_j)$ times the previous equation gives

$$\prod_{j=1}^{d-1} \left(\lambda_{f_d}(l_j) - \lambda_{f_j}(l_j)\right) \widetilde{L}(2k-3,f_d)^{p,D_0} \cdot c^{\Phi_d}(n,r)\Phi_d(\tau,z) \equiv 0 \pmod{\wp}.$$

By the choice of the l_j this leads to

$$\widetilde{L}(2k-3,f_d)^{p,D_0} \cdot c^{\Phi_d}(n,r)\Phi_d(\tau,z) \equiv 0 \pmod{\wp}.$$

Plugging this back into (6.1) gives

$$G_{k,1}^{J,D_0}(\tau,z) \equiv \sum_{j=1}^{d-1} \widetilde{L}(2k-3,f_j)^{p,D_0} c^{\Phi_j}(n,r)\cdot\Phi_j(\tau,z) \pmod{\wp}.$$

In the same way, we show that for $2 \le j \le d-1$

$$\widetilde{L}(2k-3,f_j)^{p,D_0}\cdot c^{\Phi_j}(n,r)\Phi_j(\tau,z) \equiv 0 \pmod{\wp}.$$

This yields

$$G_{k,1}^{J,D_0}(\tau,z) \equiv \widetilde{L}(2k-3,f_1)^{p,D_0} c^{\Phi_1}(n,r)\cdot\Phi_1(\tau,z) \pmod{\wp}$$

as claimed. □

7. Proof of Theorem 1.4

As before we have for $n_1,n_2,r_1,r_2 \in \mathbb{Z}$ with $r_i^2-4n_im \le 0$ $(i=1,2)$

$$\text{(7.1)}\quad A^{\mathbb{W}E_{k,m}^2}\left((n_1,r_1),(n_2,r_2)\right) = A^{E_{k,m}}(n_1,r_1)\,A^{E_{k,m}}(n_2,r_2) + \sum_{d|m}\sum_{i_d=1}^{\dim J_{k,d}^{\text{new}}} \alpha_{i_d,d}\cdot c^{i_d,d}(n_1,r_1)\cdot c^{i_d,d}(n_2,r_2).$$

Now fix a fundamental discriminant D_0 and a weight k_0 such that $\frac{B_{2k_0-2}}{2k_0-2}$ has a prime divisor p that does not divide $L_{D_0}(2-k_0)$ and construct from this a sequence of infinitely many integers k with this property. For example we can take $k_0=12$ and $D_0=-3$ which can be checked using Pari. Indeed: let p be a prime dividing $\frac{B_{22}}{22}$, then p is either 131 or 593. Moreover $L_{-3}(-10) = -\frac{2\cdot 1847}{3}$, thus $L_{-3}(-10)\in\mathbb{Z}_p^*$. Now assume that $p^\alpha \parallel \frac{B_{2k_0-2}}{2k_0-2}$. Let $k := k_0 + rp^{\alpha-1}(p-1)$ with $r\in\mathbb{Z}$. We claim that $p^\alpha \parallel \frac{B_{2k-2}}{2k-2}$ and $p \nmid L_{D_0}(2-k)$. Indeed, since $p|\frac{B_{2k_0-2}}{2k_0-2}$ and $k\equiv k_0 \pmod{p^{\alpha-1}(p-1)}$ we have (see [Ca, Le])

$$\begin{aligned} L_{D_0}(2-k) &\equiv L_{D_0}(2-k_0) \pmod{p},\\ \left(1-p^{2k-3}\right)\frac{B_{2k-2}}{2k-2} &\equiv \left(1-p^{2k_0-3}\right)\frac{B_{2k_0-2}}{2k_0-2} \pmod{p^\alpha}. \end{aligned}$$

In particular, we have that $p^\alpha \parallel \frac{B_{2k-2}}{2k-2}$ and $p\nmid L_{D_0}(2-k)$ as claimed. For a fixed triple (k,p,D_0) with this property we now construct infinitely many discriminants D and positive integers m such that the claim of the theorem holds. In order to do so, let f and m be coprime integers that are the product of distinct prime divisors congruent to 1 modulo $p^\alpha|D_0|$ such that $d(m)$, the number of divisors of m, is not divisible by p. Clearly there are infinitely many such m and f. We write now $D_0 =: r_0^2-4n_0$ and $D := D_0f^2m^2 = r^2-4nm$, where $n := f^2mn_0$ and $r_0 := r_0fm$. We have by (3.2)

$$\text{(7.2)}\qquad A^{E_{k,m}}(n_0f^2m, r_0fm) = -\frac{2k-2}{B_{2k-2}\cdot\sigma_{k-1}(m)}\sum_{d|m} d^{k-1}L_{\frac{D_0f^2m^2}{d^2}}(2-k).$$

Since m is square-free, we have

$$\sigma_{k-1}(m) = \prod_{\substack{q|m \\ q \text{ prime}}} \sigma_{k-1}(q) \equiv 2^{\omega(m)} \pmod{p^\alpha}.$$

Moreover by (1.3) we have

$$L_{\frac{D_0 f^2 m^2}{d^2}}(2-k) = L_{D_0}(2-k) \sum_{l|\frac{fm}{d}} \mu(l)\chi_{D_0}(l)\, l^{k-2}\, \sigma_{2k-3}\left(\frac{fm}{dl}\right). \tag{7.3}$$

Now the sum over l is multiplicative in $\frac{fm}{d}$ since $\frac{fm}{d}$ is square-free. Moreover for a prime $\frac{fm}{d} = q$ with $q \equiv 1 \pmod{p^\alpha |D_0|}$ we have

$$\sum_{l|q} \mu(l)\chi_{D_0}(l)\, l^{k-2}\, \sigma_{2k-3}\left(\frac{q}{l}\right) = \sigma_{2k-3}(q) - 1 \equiv 1 \pmod{p^\alpha}.$$

Thus we get from (7.3)

$$L_{\frac{D_0 f^2 m^2}{d^2}}(2-k) \equiv L_{D_0}(2-k) \pmod{p^\alpha}.$$

Thus (7.2) gives

$$A^{E_{k,m}}(n_0 f^2 m, r_0 f m) \equiv -\frac{(2k-2)\cdot d(m) \cdot L_{D_0}(2-k)}{2^{\omega(m)} \cdot B_{2k-2}} \pmod{p^\alpha}.$$

Moreover since $p \nmid \sigma_{k-1}(m)$, we see as in the proof of Theorem 1.1 that

$$p^\alpha A^{\mathbb{W}E^2_{k,m}}\big((n_0 f^2 m, r_0 m f), (n_0 f^2 m, r_0 f m)\big) \in \mathbb{Z}_p.$$

Using the above results, we can conclude from (7.1)

$$\begin{aligned} &-2^{-2\omega(m)} d^2(m) p^{2\alpha} \left(\frac{2k-2}{B_{2k-2}}\right)^2 L^2_{D_0}(2-k) \\ &\equiv p^{2\alpha} \frac{(-1)^{k/2}(2k-2)!}{2^{3k-5}(2k-3)\cdot B_{2k-2}} \sum_{d|m}\sum_{i_d} \frac{\left(c^{i_d}(n_0 f^2 m, m f r_0)\right)^2}{2^{\omega(d)} \parallel \Psi_{i_d,d} \parallel^2 \pi^{2k-3}} L(2k-3, \Phi_{i_d,d}) \pmod{p^\alpha}. \end{aligned}$$

From this the claim follows directly.

8. Proof of Theorem 1.5

Proof. We let p and α be chosen as in the theorem. Similar as before we get

$$-p^\alpha E_{k,1} \otimes p^\alpha E_{k,1} \equiv p^{2\alpha} \sum_j \alpha_j \Phi_j \otimes \Phi_j \qquad \pmod{p^\alpha}.$$

Using (3.2) and the fact that $p \nmid L_{D_0}(2-k)$, this leads to

$$u^2_{p,k} \equiv \sum_j p^{2\alpha} \alpha_j c^j(D_0)^2 \qquad \pmod{p^\alpha}, \tag{8.1}$$

where $u_{p,k} \in \mathbb{Z}_p^*$. Since the left hand side of (8.1) is in $\mathbb{Z}_p^*$ there exists at least one j_0 such that $c^{j_0}(D_0) = 1$. Thus

$$p^\alpha \frac{L_{D_0}(2-k)(2k-2)}{B_{2k-2}} p^\alpha E_{k,1} \equiv p^{2\alpha} \sum_{j=1}^{d'_k} \alpha_j \Phi_j \qquad \pmod{p^\alpha}.$$

For a Jacobi form Φ on $\mathbb{H} \times \mathbb{C}$ there is also a Witt operator $\mathbb{W}$ defined by

$$(\mathbb{W}\Phi)(\tau) := \Phi(\tau, 0).$$

We first show that

$$\mathbb{W}E_{k,1} = E_k + \sum_i \beta_i^J \, g_i, \tag{8.2}$$

where $(g_i)_i$ is a primitive Hecke eigenbasis of S_k and

$$\beta_i^J := \frac{(k-2)!(2k-2)! D_{g_j}(2k-2)}{\parallel g_j \parallel^2 \, 2^{4k-6} \pi^{3k-3} B_{2k-2}}.$$

It can be deduced that the β_j^J are algebraic (see also [Za]). Clearly

$$\mathbb{W}E_{k,1} = aE_k + \sum_{j=1}^{\dim S_k} \widetilde{\beta}_j \, g_j$$

with some constant a and with $\widetilde{\beta}_j$ given by

$$\langle \mathbb{W}E_{k,1}, g_j \rangle = \widetilde{\beta}_j \parallel g_j \parallel^2 .$$

Since $A^{E_{k,m}}(0,0) = A^{E_k}(0) = 1$ we have that $a = 1$. Moreover from [Ga] it is known that

$$E_k^2 \left(\begin{pmatrix} \tau & 0 \\ 0 & \widetilde{\tau} \end{pmatrix} \right) = E_k(\tau) E_k(\widetilde{\tau}) + \sum_j \beta_j \, g_j(\tau) g_j(\widetilde{\tau}).$$

The β_j can be explicitely computed (for details see also [Bo]) as

$$\beta_j = -2^{7-4k} \cdot (k-2)!(2k-3)! \, \frac{k(2k-2)}{B_k B_{2k-2}} \frac{D_{g_j}(2k-2)}{\pi^{3k-3} \parallel g_j \parallel^2}.$$

This directly implies that

$$\left\langle \left\langle E_k^2 \begin{pmatrix} \cdot & 0 \\ 0 & \cdot \end{pmatrix}, g_j \right\rangle, g_j \right\rangle = \beta_j \parallel g_j \parallel^4 . \tag{8.3}$$

Moreover, for $Z \in \mathbb{H}_2$ (see [EZ] Section 6), we have the Fourier Jacobi expansion

$$E_k^2(Z) = \sum_{l=0}^\infty e_{k,1}(\tau, z) | V_l \, \widetilde{q}^{\,l} ,$$

where V_l is defined in [EZ] page 41, and where $e_{k,1}|V_0 = E_k$. Here the Jacobi Eisenstein series shows up:

$$E_k^2(Z) = -\frac{2k}{B_k}\sum_{l=0}^{\infty} E_{k,1}(\tau,z)|V_l\,\widetilde{q}^{\,l}\,.$$

It is well-known how l-th Hecke operator for M_k ($l\in\mathbb{N}$) T_l and $\mathbb{W}$ interchange

$$\mathbb{W}\left(E_{k,1}|V_l\right) = \left(\mathbb{W}E_{k,1}\right)|T_l.$$

Using that T_l is self-adjoint and g_j is a Hecke eigenform, gives

$$\left\langle E_k^2\begin{pmatrix}\cdot & 0\\ 0 & \widetilde{\tau}\end{pmatrix}, g_j(\cdot)\right\rangle = -\frac{2k}{B_k}\sum_{l=1}^{\infty}\widetilde{q}^{\,l}\left\langle \mathbb{W}E_{k,1}|T_l, g_j\right\rangle = -\frac{2k}{B_k} g_j(\widetilde{\tau})\left\langle \mathbb{W}E_{k,1}, g_j\right\rangle.$$

Here we used the orthogonality of Eisenstein series and cusp forms. Thus

$$\left\langle\left\langle E_k^2\begin{pmatrix}\cdot & 0\\ 0 & \cdot\end{pmatrix}, g_j\right\rangle, g_j\right\rangle = -\frac{2k}{B_k}\parallel g_j\parallel^2\left\langle \mathbb{W}E_{k,1}, g_j\right\rangle. \tag{8.4}$$

Setting (8.4) and (8.3) equal gives that

$$\left\langle \mathbb{W}E_{k,1}^2, g_j\right\rangle = -\frac{B_k}{2k}\beta_j\parallel g_j\parallel^2$$

which implies (8.2). Writing $\mathbb{W}\Phi_j = \sum_i \gamma_{i,j}\, g_i$ gives

$$-p^\alpha\frac{L_{D_0}(2-k)}{\zeta(3-2k)}\sum_i p^\alpha\beta_i^J\, g_i \equiv \sum_i\left(\sum_{j=1}^{d_k'} p^{2\alpha}\alpha_j\gamma_{i,j}\right) g_i \pmod{p^\alpha}.$$

Plugging in α_i and β_i^J we conclude:

$$-p^\alpha\frac{L_{D_0}(2-k)}{\zeta(3-2k)}\sum_i p^\alpha\frac{(k-2)!(2k-2)!}{2^{4k-6}B_{2k-2}}\frac{D_{g_i}(2k-2)}{\pi^{3k-3}\parallel g_i\parallel^2}\, g_i$$
$$\equiv \sum_i\left(\sum_{j=1}^{d_k'} p^{2\alpha}\frac{(-1)^{k/2}(2k-2)!\cdot L(2k-3,f_j)}{2^{3k-5}\cdot\pi^{2k-3}\cdot(2k-3)\cdot B_{2k-2}\cdot\parallel\Phi_j\parallel^2}\gamma_{i,j}\right) g_i \pmod{p^\alpha}$$

A short calculation now implies

$$-p^\alpha(-1)^{k/2}\frac{L_{D_0}(2-k)\cdot(k-2)!(2k-3)!}{\zeta(3-2k)\cdot 2^{k-1}}\sum_{j=1}^{\dim S_k}\frac{D_{g_j}(2k-2)}{\pi^{3k-3}\parallel g_j\parallel^2}$$
$$\equiv (2k-4)!\sum_{j=1}^{d_k'} p^\alpha\frac{L(2k-3,f_j)}{\pi^{2k-3}\parallel\Phi_j\parallel^2}\, a^{\mathbb{W}\Phi_j}(1) \pmod{p^\alpha},$$

which leads to the desired result. □

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