GENERALIZED L-FUNCTIONS RELATED TO THE RIEMANN ZETA FUNCTION

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ABSTRACT. In this paper, we construct generalized L-functions associated to meromorphic modular forms of weight $\frac{1}{2}$ for the theta group with a single simple pole in the fundamental domain. We then consider their behaviour towards $i\infty$ and relate this to the Riemann zeta function.

1. INTRODUCTION AND STATEMENT OF RESULTS

Arithmetic information $c(n)$ for $n \in \mathbb{N}$ can be naturally encoded in a so-called Dirichlet series, defined for $\mathrm{Re}(s)$ sufficiently large,

$$L(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}.$$

One calls the function L a (classical) *L-function* if it satisfies the following three properties:

(1) It has a meromorphic continuation to the whole complex plane.
(2) It has an *Euler product*, for $\mathrm{Re}(s)$ sufficiently large,

$$L(s) = \prod_p \frac{1}{1 - f_p\left(p^{-s}\right) p^{-s}}.$$

where the product runs over all primes and f_p is a polynomial.
(3) There is some archimedean information $L_\infty(s)$ such that the function

$$\Lambda(s) := L_\infty(s) L(s)$$

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satisfies, for some k and $\varepsilon = \pm 1$, a *functional equation*

$$\Lambda(k-s) = \varepsilon \Lambda(s).$$

In [1], the first two authors constructed functions L_z via regularized Mellin transforms of weight two meromorphic modular forms on $\mathrm{SL}_2(\mathbb{Z})$ with a single simple pole at one point z in the fundamental domain (not on the imaginary axis). These functions satisfy a functional equation and their behaviour as $z \to i\infty$ was shown to be related to a classical L-function (see [1, Theorem 1.1 and Theorem 4.3]), so they were named generalized L-functions.

Slightly more formally, for $k \in \frac{1}{2}\mathbb{Z}$ we call a collection of functions $s \mapsto \mathcal{L}_z(s)$ defined for $s \in \mathbb{C}$ for almost all $z \in \mathbb{H}$ *generalized L-functions of weight k* if they satisfy the following properties:

(1) We have the functional equations

$$\mathcal{L}_z(k-s) = \pm \mathcal{L}_z(s).$$

(2) There exist "simple" functions $C_{\ell,s}$ and $D_{\ell,s}$ such that, for almost all $x \in [0,1]$,

$$\lim_{y\to\infty} \left(\mathcal{L}_z(s) - \sum_{\ell \geq 0} C_{\ell,s}(x) y^{s-\ell} - \sum_{\ell\geq 0} D_{\ell,s}(x) y^{k-s-\ell} \right) \tag{1.1}$$

is a classical completed L-function.

We call the classical L-function whose completion one obtains from the limit (1.1), the *L-function associated to $\mathcal{L}_z(s)$.*

Letting j_ϑ be the Hauptmodul for Γ_ϑ, explicitly given in (3.3), we (formally) define

$$F_z(s) := s\left(\frac{1}{2} - s\right) j_\vartheta(z) \int_0^\infty \frac{\vartheta(it)}{j_\vartheta(it) - j_\vartheta(z)} t^{s-1} dt. \tag{1.2}$$

The main result in this paper is the fact that the functions F_z give a collection of generalized L-functions related to the Riemann zeta function in the sense that (1.1) is the Riemann ξ-function

$$\xi(s) := \frac{s(s-1)}{2\pi^{\frac{s}{2}}} \Gamma\left(\frac{s}{2}\right) \zeta(s),$$

which satisfies the functional equation

$$\xi(1-s) = \xi(s). \tag{1.3}$$

Here $\Gamma(s)$ is the Γ-function, defined by $\Gamma(s) := \int_0^\infty t^{s-1} e^{-t} dt$ for $\mathrm{Re}(s) > 1$ and extended meromorphically to the entire complex plane.

Theorem 1.1. *The functions F_z are a collection of generalized L-functions related to the Riemann zeta function in the following sense:*

(1) *If z is not Γ_ϑ-equivalent to a point on $i\mathbb{R}^+$, then the integral in* (1.2) *converges for all $s \in \mathbb{C}$.*
(2) *If z is not Γ_ϑ-equivalent to a point on $i\mathbb{R}^+$, then we have*
$$F_z\left(\frac{1}{2} - s\right) = F_z(s).$$
(3) *With $C_{\ell,s}(x)$ and $D_{\ell,s}(x)$ defined in* (4.12) *and* (4.13)*, respectively, the limit* (1.1) *is $\xi(2s)$.*

The paper is organized as follows. In Section 2, we recall the properties of the theta function and the zeta function. In Section 3, we construct the Hauptmodul j_ϑ for Γ_ϑ. In Section 4, we show that F_z converges absolutely for all $s \in \mathbb{C}$ under a mild assumption on z and then finally, for $z = x + iy$ with $x \notin \mathbb{Z}$ fixed, take the limit $y \to \infty$ to prove Theorem 1.1.

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2. Preliminaries

2.1. **The theta function.** Define the *theta function* ($q := e^{2\pi i\tau}$)
$$\vartheta(\tau) := \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}}.$$
It is well-known that ϑ is a modular form of weight $\frac{1}{2}$ for the *theta group*
$$\Gamma_\vartheta := \left\langle S, T^2 \right\rangle = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \pmod 2 \text{ and } c \equiv b \pmod 2 \right\},$$
where $S := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. The theta function has the following growth behavior.

Lemma 2.1. *Let $\tau = u + iv$.*

(1) *As* $v \to \infty$*, we have*

$$\vartheta(\tau) = 1 + O\left(e^{-\pi v}\right).$$

(2) *As* $v \to 0^+$*, we have*

$$\vartheta(iv) = \frac{1}{\sqrt{v}} + O\left(\frac{e^{-\frac{\pi}{v}}}{\sqrt{v}}\right).$$

Proof.
(1) The claim follows directly from the Fourier expansion of ϑ.
(2) It is well-known that we have

$$\vartheta(\tau) = (-i\tau)^{-\frac{1}{2}}\vartheta\left(-\frac{1}{\tau}\right). \tag{2.1}$$

Taking $\tau = it$ and plugging in the Fourier expansion for $\vartheta(\frac{i}{t})$ yields the claim. □

2.2. **A regularized Mellin transform of the theta function.** We recall the following well-known relation between the theta function and the Riemann zeta function that goes back to Riemann.[1]

Lemma 2.2. *For any* $t_0 > 0$*, we have*

$$\frac{2}{s(2s-1)}\xi(2s) = \int_{t_0}^{\infty} (\vartheta(it) - 1)t^{s-1}dt + \int_0^{t_0} \left(\vartheta(it) - \frac{1}{\sqrt{t}}\right) t^{s-1}dt - \frac{t_0^s}{s} + \frac{t_0^{s-\frac{1}{2}}}{s-\frac{1}{2}}.$$

2.3. **Asymptotics for special functions.** For $y > 0$ and $s \in \mathbb{C}$ we define the *incomplete gamma function* by

$$\Gamma(s, y) := \int_y^{\infty} t^{s-1}e^{-t}dt.$$

Denoting, for $\ell \in \mathbb{N}$, the *rising factorial* by $(a)_\ell := \prod_{j=0}^{\ell-1}(a + j)$, we require the following asymptotic behavior for $\Gamma(s, y)$, which may be found in [2, 8.11.2].

Lemma 2.3. *For* $s \in \mathbb{C}$ *and* $N \in \mathbb{N}$ *we have, as* $y \to \infty$*,*

$$\Gamma(s, y) = y^{s-1}e^{-y}\left(\sum_{j=0}^{N-1} \frac{(-1)^j(1-s)_j}{y^j} + O\left(y^{-N}\right)\right).$$

[1] See [3, Subsection 2.6] for a modern reference.

We also require bounds for the confluent hypergeometric function ${}_1F_1$. Assuming that $s \notin \mathbb{Z}$, [2, 13.7.1] implies that, as $y \to \infty$, for any $N \in \mathbb{N}_0$

$$ {}_1F_1(s; s+1; y) \sim \frac{se^y}{y} \left(\sum_{j=0}^{N} \frac{(1-s)_j}{y^j} + O_{s,N}\left(y^{-N-1}\right) \right). $$

3. Construction of meromorphic modular forms

In this section, we construct meromorphic modular forms $\mathcal{H}_z$ whose regularized Mellin transforms give the collection of generalized L-functions that are used to prove Theorem 1.1.

3.1. The Hauptmodul j_ϑ for the theta group. In this subsection, we construct a Hauptmodul for the theta group Γ_ϑ and discuss its properties. We recall that a *Hauptmodul* for a congruence subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is a Γ-invariant meromorphic function j_Γ for which every Γ-invariant meromorphic function may be written as a rational function in j_Γ. For this, we require the modular *lambda function*

$$ \lambda(\tau) := \frac{\theta_2(\tau)^4}{\vartheta(\tau)^4}, $$

where

$$ \theta_2(\tau) := \sum_{n\in\mathbb{Z}} e^{\pi i \left(n+\frac{1}{2}\right)^2 \tau}. $$

It is well-known that λ is a Hauptmodul for $\Gamma(2)$ and satisfies the identities

$$ \lambda\left(-\frac{1}{\tau}\right) = 1 - \lambda(\tau), \qquad \lambda(\tau+1) = \frac{\lambda(\tau)}{\lambda(\tau)-1}, \qquad \lambda\left(\frac{1}{1-\tau}\right) = \frac{1}{1-\lambda(\tau)}. \quad (3.1) $$

We have the following growth towards the cusps.[2] For this, let $\mathbb{H}^* := \mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}$ and $\mathbb{C}^* := \mathbb{C} \cup \{i\infty\}$.

Lemma 3.1.
(1) *As $v \to \infty$, we have*

$$ \lambda(\tau) = 16e^{\pi i\tau} + O\left(e^{-2\pi v}\right), \qquad \frac{1}{\lambda(\tau)} = \frac{e^{-\pi i \tau}}{16} + O(1). $$

[2] There are three cusps of $\Gamma(2)$, namely, 0, 1, and $i\infty$, each with cusp width 2.

(2) *As $v \to \infty$, we have*

$$\lambda\left(-\frac{1}{\tau}\right) = 1 - 16e^{\pi i\tau} + O\left(e^{-2\pi v}\right), \qquad \frac{1}{\lambda\left(-\frac{1}{\tau}\right) - 1} = -\frac{e^{-\pi i\tau}}{16} + O(1).$$

(3) *As $v \to \infty$, we have*

$$\lambda\left(\frac{\tau}{\tau+1}\right) = \frac{e^{-\pi i\tau}}{16} + O(1).$$

(4) *The function λ is a bijection from $\Gamma(2) \setminus \mathbb{H}^*$ to $\mathbb{C}^*$.*

Proof.
(1) The statements follow immediately from the Fourier expansion of λ.
(2) The first claim follows from (3.1) and part (1) and the second follows by subtracting 1 from both sides and dividing.
(3) Similarly, we combine the first and second equations of (3.1) to obtain

$$\lambda\left(\frac{\tau}{\tau+1}\right) = \frac{1}{\lambda(\tau)}. \tag{3.2}$$

The claim now follows immediately from part (1).
(4) This follows since a Γ-invariant meromorphic function is a Hauptmodul for Γ if and only if it is a bijection from $\Gamma \setminus \mathbb{H}^*$ to $\mathbb{C}^*$. □

Define

$$j_\vartheta(\tau) := \frac{1}{\lambda(\tau)\left(1 - \lambda(\tau)\right)}. \tag{3.3}$$

We now show that j_ϑ is a Hauptmodul for Γ_ϑ, and give its growth towards the cusps.

Lemma 3.2.
(1) *The function j_ϑ is Γ_ϑ-invariant.*
(2) *As $v \to \infty$, we have*

$$j_\vartheta\left(-\frac{1}{\tau}\right) = j_\vartheta(\tau) = \frac{e^{-\pi i\tau}}{16} + O(1).$$

In particular, as $v \to 0^+$,

$$j_\vartheta(iv) = e^{\frac{\pi}{v}} + O(1).$$

(3) *As* $v \to \infty$, *we have*

$$j_\vartheta\left(\frac{\tau}{\tau+1}\right) = -256e^{2\pi i\tau} + O\left(e^{-3\pi v}\right).$$

(4) *The function* j_ϑ *is a bijection from* $\Gamma_\vartheta \setminus \mathbb{H}^*$ *to* $\mathbb{C}^*$. *Every meromorphic modular function on* $\Gamma_\vartheta \setminus \mathbb{H}$ *can be written as a rational function in* j_ϑ.

Proof.
(1) It is enough to show that j_ϑ is invariant under the generators of Γ_θ, S, and T^2, which follows from the first equation in (3.1) by a direct calculation.
(2) The first identity follows immediately from (1). For the second identity, we first note that, by (3.1),

$$\frac{1}{1-\lambda(\tau)} = 1 + O\left(e^{\pi i\tau}\right). \tag{3.4}$$

Combining (3.4) with (3.2) and Lemma 3.1 (3), we conclude that, as $v \to \infty$,

$$j_\vartheta(\tau) = \frac{1}{\lambda(\tau)}\frac{1}{1-\lambda(\tau)} = \left(\frac{e^{-\pi i\tau}}{16} + O(1)\right)\left(1 + O\left(e^{\pi i\tau}\right)\right) = \frac{e^{-\pi i\tau}}{16} + O(1).$$

(3) A short calculation using (3.2) and (3.4) shows that

$$j_\vartheta\left(\frac{\tau}{\tau+1}\right) = -256e^{2\pi i\tau} + O\left(e^{-3\pi v}\right).$$

(4) To show surjectivity, let $c \in \mathbb{C}^* \setminus \{0\}$. Then we have

$$j_\vartheta(\tau) = c \Leftrightarrow \lambda(\tau)^2 - \lambda(\tau) + \frac{1}{c} = 0.$$

By Lemma 3.1 (4), for any root α of the polynomial $x^2 - x + \frac{1}{c}$ there exists $\tau \in \mathbb{H}$ such that $\lambda(\tau) = \alpha$. For $c = 0$, letting $\tau \to i\infty$ in part (3) implies that $j_\vartheta(1) = 0$. So j_ϑ is surjective.

To show that j_ϑ is injective, suppose for contradiction that $j_\vartheta(\tau_1) = j_\vartheta(\tau_2)$ with τ_1 and τ_2 not Γ_ϑ-equivalent. Then

$$\lambda(\tau_1)(1-\lambda(\tau_1)) = \lambda(\tau_2)(1-\lambda(\tau_2)).$$

Setting $\alpha_1 := \lambda(\tau_1)$ and $\alpha_2 := \lambda(\tau_2)$ and rearranging, we have

$$\alpha_2^2 - \alpha_2 + \alpha_1 - \alpha_1^2 = 0. \tag{3.5}$$

Since we have a quadratic equation in α_2, this has exactly two solutions (counting multiplicity) in $\mathbb{C}$. One directly checks that $\alpha_2 = \alpha_1$ and $\alpha_2 = 1 - \alpha_1$ are both solutions to (3.5). If $\alpha_1 \neq \frac{1}{2}$, then these are distinct solutions, while for $\alpha_1 = \frac{1}{2}$ we see that (3.5) becomes $\alpha_2^2 - \alpha_2 + \frac{1}{4} = 0$, in which case $\alpha_2 = \frac{1}{2} = \alpha_1 = 1 - \alpha_1$ is a double root. We see that in both cases the two solutions (counting multiplicity) to (3.5) are $\alpha_2 = \alpha_1$ and $\alpha_2 = 1 - \alpha_1$.

If $\alpha_2 = \alpha_1$, then by Lemma 3.1 (4) we conclude that τ_2 is $\Gamma(2)$-equivalent to τ_1. Since $\Gamma(2) \subseteq \Gamma_\vartheta$, this contradicts the assumption that τ_1 and τ_2 are not Γ_ϑ-equivalent. Hence $\alpha_2 = 1 - \alpha_1$. From the first equation of (3.1), we conclude that $\lambda(\tau_2) = \lambda(-\frac{1}{\tau_1})$. By Lemma 3.1 (4), there exists $\gamma \in \Gamma(2)$ such that

$$\tau_2 = \gamma \frac{-1}{\tau_1} = \gamma \circ S\tau_1.$$

Since $\gamma \in \Gamma(2) \subset \Gamma_\vartheta$ and $S \in \Gamma_\vartheta$, we have $\gamma \circ S \in \Gamma_\vartheta$, which contradicts the assumption that τ_1 and τ_2 are not Γ_ϑ-equivalent. We therefore conclude that j_ϑ is a Γ_ϑ-invariant meromorphic function and is a bijection from $\Gamma_\vartheta \setminus \mathbb{H}^*$ to $\mathbb{C}^*$. Hence it is a Hauptmodul for Γ_ϑ by the equivalence noted in the proof of Lemma 3.1 (4). □

3.2. **Definition of the meromorphic modular forms.** For $z \in \mathbb{H}$, define

$$\mathcal{H}_z(\tau) := \frac{j_\vartheta(z)\vartheta(\tau)}{j_\vartheta(\tau) - j_\vartheta(z)}.$$

A direct calculation yields the following properties of $\mathcal{H}_z$.

Lemma 3.3.

(1) *The function $\mathcal{H}_z$ is modular for Γ_ϑ of weight $\frac{1}{2}$. Moreover, it has a pole at $\tau = \mathfrak{z}$ if and only if $\mathfrak{z}$ is Γ_ϑ-equivalent to z.*
(2) *The function $z \mapsto \mathcal{H}_z(\tau)$ is Γ_ϑ-invariant.*
(3) *For $\tau \in \mathbb{H}$ fixed, we have*

$$\lim_{z \to i\infty} \mathcal{H}_z(\tau) = \vartheta(\tau).$$

4. Regularized Mellin transforms and the proof of Theorem 1.1

In this section, we investigate the properties of F_z defined in (1.2) and prove Theorem 1.1.

4.1. **Convergence and functional equation.** We first show that, under a mild assumption on z, the function F_z converges for all $s \in \mathbb{C}$.

Proposition 4.1. *The function F_z is well-defined for $z \in \mathbb{H}$ that is not Γ_ϑ-equivalent to any point on $i\mathbb{R}^+$. Moreover, we have*

$$F_z\left(\frac{1}{2} - s\right) = F_z(s). \tag{4.1}$$

Proof. We first show that F_z is well-defined. Since j_ϑ is Γ_ϑ-invariant, we may assume that z lies in the fundamental domain $\mathcal{F}_\vartheta := \mathcal{F} \cup T^{-1}\mathcal{F} \cup (-T^{-1}S\mathcal{F})$ for Γ_ϑ, where $\mathcal{F}$ is the standard fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$. Since $z \neq it$ by assumption, Lemma 3.2 (4) moreover implies that

$$j_\vartheta(it) - j_\vartheta(z) \neq 0.$$

Hence the integrand is finite for every $t > 0$ and therefore, for any $0 < t_1 < t_2 < \infty$, the integral

$$\int_{t_1}^{t_2} \frac{\vartheta(it)}{j_\vartheta(it) - j_\vartheta(z)} t^{s-1} dt$$

is well-defined and finite. Taking $t_1 = \frac{1}{2(y+y^{-1})}$ and $t_2 = 2(y + y^{-1})$, it remains to show that

$$\int_0^{\frac{1}{2(y+y^{-1})}} \frac{\vartheta(it)}{j_\vartheta(it) - j_\vartheta(z)} t^{s-1} dt < \infty \qquad \text{and} \qquad \int_{2(y+y^{-1})}^{\infty} \frac{\vartheta(it)}{j_\vartheta(it) - j_\vartheta(z)} t^{s-1} dt < \infty.$$

For $0 < t < \frac{1}{2(y+y^{-1})}$, Lemma 3.2 (2) implies that

$$\left| \frac{1}{j_\vartheta(it) - j_\vartheta(z)} \right| \ll e^{-\frac{\pi}{t}}.$$

Combining this with Lemma 2.1 (2) hence yields, that for $t \to 0^+$,

$$\left| \frac{\vartheta(it)}{j_\vartheta(it) - j_\vartheta(z)} t^{s-1} \right| \ll t^{\sigma - \frac{3}{2}} e^{-\frac{\pi}{t}},$$

where here and throughout $\sigma := \mathrm{Re}(s)$. Therefore

$$\int_0^{\frac{1}{2(y+y^{-1})}} \left| \frac{\vartheta(it)}{j_\vartheta(it) - j_\vartheta(z)} t^{s-1} \right| dt \ll \int_0^{\frac{1}{2(y+y^{-1})}} t^{\sigma - \frac{3}{2}} e^{-\frac{\pi}{t}} dt < \infty.$$

Similarly, for $t > 2(y+y^{-1})$, Lemma 3.2 (2) implies that, as $t \to \infty$,

$$\left|\frac{1}{j_\vartheta(it)-j_\vartheta(z)}\right| \ll \frac{1}{\left|e^{\pi t}-e^{\pi(y+y^{-1})}\right|+O(1)} \ll \frac{1}{e^{\pi t}-e^{\frac{\pi t}{2}}} \ll e^{-\pi t}.$$

Combining this with Lemma 2.1 (1) hence yields that for $t \to \infty$

$$\left|\frac{\vartheta(it)}{j_\vartheta(it)-j_\vartheta(z)}t^{s-1}\right| \ll t^{\sigma-1}e^{-\pi t}.$$

Therefore

$$\int_{2y}^{\infty}\left|\frac{\vartheta(it)}{j_\vartheta(it)-j_\vartheta(z)}t^{s-1}\right|dt \ll \int_{2y}^{\infty} t^{\sigma-\frac{3}{2}}e^{-\pi t}dt < \infty.$$

The functional equation (4.1) now follows by the change of variables $t \mapsto \frac{1}{t}$, (2.1), and the invariance of j_ϑ under inversion. □

4.2. **Proof of Theorem 1.1.** To show Theorem 1.1, we require the following.

Proposition 4.2. *For every $x \notin \mathbb{Z}$, we have, with $z = x+iy$,*

$$\lim_{y\to\infty}\Bigg(F_z(s)+\left(\frac{1}{2}-s\right)y^s+sy^{\frac{1}{2}-s}-s\left(\frac{1}{2}-s\right)\sum_{\ell=1}^{\lfloor\sigma\rfloor}\frac{(1-s)_{\ell-1}}{\pi^\ell}\operatorname{Li}_\ell\left(e^{\pi ix}\right)y^{s-\ell}$$
$$+s\left(\frac{1}{2}-s\right)\sum_{\ell=1}^{\lfloor\sigma\rfloor}\frac{(s+1-\ell)_{\ell-1}}{\pi^\ell}\operatorname{Li}_\ell\left(e^{-\pi ix}\right)y^{s-\ell}$$
$$-s\left(\frac{1}{2}-s\right)\sum_{\ell=1}^{\left\lfloor\frac{1}{2}-\sigma\right\rfloor}\frac{\left(s+\frac{1}{2}\right)_{\ell-1}}{\pi^\ell}\operatorname{Li}_\ell\left(e^{\pi ix}\right)y^{\frac{1}{2}-s-\ell}$$
$$+s\left(\frac{1}{2}-s\right)\sum_{\ell=1}^{\left\lfloor\frac{1}{2}-\sigma\right\rfloor}\frac{\left(\frac{3}{2}-s-\ell\right)_{\ell-1}}{\pi^\ell}\operatorname{Li}_\ell\left(e^{-\pi ix}\right)y^{\frac{1}{2}-s-\ell}\Bigg)=\xi(2s).$$

Proof. We split, for some $t_0 > 0$,

$$F_z(s) = s\left(\frac{1}{2}-s\right)\left(F_{z,0,\frac{1}{y}}(s)+F_{z,\frac{1}{y},t_0}(s)+F_{z,t_0,y}(s)+F_{z,y,\infty}(s)\right),$$

where for $0 \le y_1 \le y_2 \le \infty$ we define

$$F_{z,y_1,y_2}(s) := j_\vartheta(z) \int_{y_1}^{y_2} \frac{\vartheta(\tau)}{j_\vartheta(\tau) - j_\vartheta(z)} t^{s-1} dt.$$

Since both sides of the claim are invariant under $s \mapsto \frac{1}{2} - s$ by Proposition 4.1 and (1.3), we may assume without loss of generality that $\sigma \ge \frac{1}{4}$.

We claim that in this case, we have

$$F_{z,0,\frac{1}{y}}(s) = \sum_{\ell=1}^{\lfloor \frac{1}{2}-\sigma \rfloor} \frac{\left(\frac{3}{2} - s - \ell\right)_{\ell-1}}{\pi^\ell} \operatorname{Li}_\ell\left(e^{-\pi i x}\right) y^{\frac{1}{2}-s-\ell} + o_{x,s}(1), \tag{4.2}$$

$$F_{z,\frac{1}{y},t_0}(s) = -\int_{\frac{1}{y}}^{t_0} \left(\vartheta(it) - \frac{1}{\sqrt{t}}\right) t^{s-1} dt + \frac{y^{\frac{1}{2}-s}}{s-\frac{1}{2}} - \frac{t_0^{s-\frac{1}{2}}}{s-\frac{1}{2}} - \frac{\operatorname{Li}_1\left(e^{\pi i x}\right)}{\pi y^{s+\frac{1}{2}}} + o_{x,s}(1), \tag{4.3}$$

$$F_{z,t_0,y}(s) = -\int_{t_0}^{y} (\vartheta(it) - 1) t^{s-1} dt + \frac{t_0^s}{s} - \frac{y^s}{s} - \sum_{\ell=1}^{\lfloor\sigma\rfloor} \frac{(1-s)_{\ell-1}}{\pi^\ell} \operatorname{Li}_\ell\left(e^{\pi i x}\right) y^{s-\ell} + o_{x,s}(1),$$

$$F_{z,y,\infty}(s) = \sum_{\ell=1}^{\lfloor\sigma\rfloor} \frac{(s+1-\ell)_{\ell-1}}{\pi^\ell} \operatorname{Li}_\ell\left(e^{-\pi i x}\right) y^{s-\ell} + o_{x,s}(1). \tag{4.4}$$

The proofs of (4.2)–(4.4) are all similar and also analogous to the proofs of [1, (4.5)–(4.8)]. We hence only show (4.3) (we choose the case (4.3) instead of (4.2) to demonstrate how the Mellin transform of ϑ naturally appears), leaving the other cases to the interested reader. We split

$$F_{z,\frac{1}{y},t_0}(s) = \int_{\frac{1}{y}}^{t_0} \frac{j_\vartheta(z)\left(\vartheta(it) - \frac{1}{\sqrt{t}}\right)}{j_\vartheta(it) - j_\vartheta(z)} t^{s-1} dt + \int_{\frac{1}{y}}^{t_0} \frac{j_\vartheta(z)}{j_\vartheta(it) - j_\vartheta(z)} t^{s-\frac{3}{2}} dt, \tag{4.5}$$

$$F_{z,t_0,y}(s) = \int_{t_0}^{y} \frac{j_\vartheta(z)(\vartheta(it) - 1)}{j_\vartheta(it) - j_\vartheta(z)} t^{s-1} dt + \int_{t_0}^{y} \frac{j_\vartheta(z)}{j_\vartheta(it) - j_\vartheta(z)} t^{s-1} dt.$$

Rewriting

$$\frac{j_\vartheta(z)}{j_\vartheta(it) - j_\vartheta(z)} = -1 + \frac{j_\vartheta(it)}{j_\vartheta(it) - j_\vartheta(z)}$$

in (4.5) yields

$$F_{z,\frac{1}{y},t_0}(s) = -\int_{\frac{1}{y}}^{t_0} \left(\vartheta(it) - \frac{1}{\sqrt{t}}\right) t^{s-1}dt + \int_{\frac{1}{y}}^{t_0} \frac{j_\vartheta(it)\left(\vartheta(it) - \frac{1}{\sqrt{t}}\right)}{j_\vartheta(it) - j_\vartheta(z)} t^{s-1}dt - \int_{\frac{1}{y}}^{t_0} t^{s-\frac{3}{2}}dt + \int_{\frac{1}{y}}^{t_0} \frac{j_\vartheta(it)}{j_\vartheta(it) - j_\vartheta(z)} t^{s-\frac{3}{2}}dt. \quad (4.6)$$

The first and the third term contribute the first three terms in (4.3).

We next claim that

$$\lim_{y\to\infty} \int_{\frac{1}{y}}^{t_0} \frac{j_\vartheta(it)\left(\vartheta(it) - \frac{1}{\sqrt{t}}\right)}{j_\vartheta(it) - j_\vartheta(z)} t^{s-1}dt = 0. \quad (4.7)$$

We split the integral into two parts, one from $\frac{1}{y}$ to $\frac{1}{\sqrt{y}}$ and one from $\frac{1}{\sqrt{y}}$ to t_0, and then use Lemma 2.1 (2) and Lemma 3.2 (2). For the integral from $\frac{1}{\sqrt{y}}$ to t_0, Lemma 3.2 (2) implies that

$$\left|\frac{j_\vartheta(it)}{j_\vartheta(it) - j_\vartheta(z)}\right| \ll \frac{e^{\pi\sqrt{y}}}{e^{\pi y} - e^{\pi\sqrt{y}}} \ll e^{-\pi y + \pi\sqrt{y}}.$$

Combining this with Lemma 2.1 (2) yields, as $y \to \infty$,

$$\left|\int_{\frac{1}{\sqrt{y}}}^{t_0} \frac{j_\vartheta(it)\left(\vartheta(it) - \frac{1}{\sqrt{t}}\right)}{j_\vartheta(it) - j_\vartheta(z)} t^{s-1}dt\right| \ll_{s,t_0} e^{-\pi y + \pi\sqrt{y}} \to 0. \quad (4.8)$$

Lemma 3.2 (2) implies that for $\frac{1}{y} < t < t_0$

$$\frac{j_\vartheta(it)}{j_\vartheta(it) - j_\vartheta(z)} = -\frac{e^{\pi\left(\frac{1}{t}+iz\right)}}{1 - e^{\pi\left(\frac{1}{t}+iz\right)}} \left(1 + O\left(e^{-\frac{\pi}{t}}\right)\right). \quad (4.9)$$

Lemma 2.1 (2) and (4.9) then gives that

$$\frac{j_\vartheta(it)\left(\vartheta(it) - \frac{1}{\sqrt{t}}\right)}{j_\vartheta(it) - j_\vartheta(z)} t^{s-1} = O\left(\frac{e^{\pi\left(\frac{1}{t}+iz\right)}}{1 - e^{\pi\left(\frac{1}{t}+iz\right)}} \left(1 + O\left(e^{-\frac{\pi}{t}}\right)\right) t^{\sigma-\frac{3}{2}} e^{-\frac{\pi}{t}}\right). \quad (4.10)$$

We then note that, for $t > \frac{1}{y}$ and $x \notin \mathbb{Z}$, we have

$$\frac{1}{1-e^{\pi\left(\frac{1}{t}+iz\right)}} \ll_x 1,$$

Therefore (4.10) becomes $O_x(e^{-\pi y}t^{\sigma-\frac{3}{2}})$. Hence, as

$$\int_{\frac{1}{y}}^{\frac{1}{\sqrt{y}}} \frac{j_\vartheta(it)\left(\vartheta(it)-\frac{1}{\sqrt{t}}\right)}{j_\vartheta(it)-j_\vartheta(z)} t^{s-1}dt \ll_x e^{-\pi y}\int_{\frac{1}{y}}^{\frac{1}{\sqrt{y}}} t^{\sigma-\frac{3}{2}}dt \to 0,$$

where we use the fact that $\int_{\frac{1}{y}}^{\frac{1}{\sqrt{y}}} t^{\sigma-\frac{3}{2}}dt$ grows at most polynomially in y. Combining this with (4.8) establishes (4.7).

To evaluate the final term in (4.6), we plug in (4.9) to rewrite it as

$$-\lim_{\varepsilon\to 0^+}\int_{\frac{1}{(1-\varepsilon)y}}^{t_0} \frac{e^{\pi\left(\frac{1}{t}+iz\right)}}{1-e^{\pi\left(\frac{1}{t}+iz\right)}}\left(1+O\left(e^{-\frac{\pi}{t}}\right)\right)t^{s-\frac{3}{2}}dt.$$

Making the change of variables $t\mapsto \frac{1}{2t}$, this becomes

$$-2^{\frac{1}{2}-s}\lim_{\varepsilon\to 0^+}\int_{\frac{1}{2t_0}}^{(1-\varepsilon)\frac{y}{2}} \frac{e^{2\pi\left(t+\frac{iz}{2}\right)}}{1-e^{2\pi\left(t+\frac{iz}{2}\right)}}\left(1+O\left(e^{-2\pi t}\right)\right)t^{-s-\frac{1}{2}}dt. \tag{4.11}$$

Up to the factor $-2^{\frac{1}{2}-s}$ in front, this is precisely [1, (4.13)] with $s\mapsto s+\frac{3}{2}$, $z\mapsto\frac{z}{2}$, and $t_0\mapsto 2t_0$. If $\sigma+\frac{3}{2}\geq 1$ (i.e., $\sigma\geq -\frac{1}{2}$), then we can follow [1, (4.14)–(4.17)] and use [1, (2.4) with $x\mapsto -x$] to obtain that the integral in (4.11) equals the second and third terms in [1, (4.6)] with $s\mapsto s+\frac{3}{2}$ and $z\mapsto\frac{z}{2}$, giving that for $\sigma\geq -\frac{1}{2}$

$$\lim_{\varepsilon\to 0^+}\int_{\frac{1}{2t_0}}^{(1-\varepsilon)\frac{y}{2}} \frac{e^{2\pi\left(t+\frac{iz}{2}\right)}}{1-e^{2\pi\left(t+\frac{iz}{2}\right)}}\left(1+O\left(e^{-2\pi t}\right)\right)t^{-s-\frac{1}{2}}dt = \frac{1}{2\pi}\operatorname{Li}_1\left(e^{\pi ix}\right)\left(\frac{y}{2}\right)^{-s-\frac{1}{2}}+o_{x,s}(1).$$

This completes the proof of (4.3). ☐

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Parts (1) and (2) are given in Proposition 4.1. Taking $k=\frac{1}{2}$ and

$$C_{\ell,s}(x) := \begin{cases}\frac{1}{s} & \text{if } \ell=0,\\ \frac{1}{\pi}\left((s+1-\ell)_{\ell-1}\operatorname{Li}_\ell\left(e^{-\pi ix}\right)-(1-s)_{\ell-1}\operatorname{Li}_\ell\left(e^{-\pi ix}\right)\right) & \text{if } \ell\in\mathbb{N},\end{cases} \tag{4.12}$$

$$D_{\ell,s}(x) := C_{\ell,\frac{1}{2}-s}(x), \tag{4.13}$$

the statement of part (3) is precisely the statement of Proposition 4.2. □

References

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