MOCK THETA FUNCTIONS, WEAK MAASS FORMS, AND APPLICATIONS

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1. Introduction

The goal of this article is to provide an overview on mock theta functions and their connection to weak Maass forms.

The theory of modular forms has important applications to many areas of mathematics, e.g. quadratic forms, elliptic curves, partitions as well as other areas throughout mathematics. Let me explain this with the example of partitions. If $p(n)$ denotes the number of partitions of an integer n, then by Euler, we have

$$P(q) := \sum_{n=0}^{\infty} p(n)\, q^{24n-1} = \frac{1}{\eta(24z)},$$

where $\eta(z)$ is Dedekind's η-functions, a weight $\frac{1}{2}$ cusp form ($q = e^{2\pi iz}$ throughout). The theory of modular forms can be employed to show many important properties of $p(n)$. For example Rademacher used the circle method to prove that if n is a positive integer, then

$$p(n) = \frac{2\pi}{(24n-1)^{3/4}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} \cdot I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right). \tag{1.1}$$

Here $I_s(x)$ is the usual I-Bessel function of order s. Furthermore, if $k \geq 1$ and n are integers and $e(x) := e^{2\pi ix}$, then define

$$A_k(n) := \sum_{h \pmod{k}^*} \omega_{h,k}\, e^{-\frac{2\pi ihn}{k}},$$

where h runs through all primitive elements modulo k, and where

$$\omega_{h,k} := \exp\left(\pi is(h,k)\right).$$

Here

$$s(h,k) := \sum_{\mu \pmod{k}} \left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{h\mu}{k}\right)\right)$$

with

$$((x)) := \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

Moreover $p(n)$ satisfies some nice congruence properties. The most famous ones are the Ramanujan congruences:

$$p(5n+4) \equiv 0 \pmod{5}, \tag{1.2}$$
$$p(7n+5) \equiv 0 \pmod{7}, \tag{1.3}$$
$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.4}$$

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In a celebrated paper Ono [32] treated these kinds of congruences systematically. Combining Shimura's theory of modular forms of half-integral weight with results of Serre on modular forms modulo ℓ he showed that for any prime $\ell \geq 5$ there exist infinitely many non-nested arithmetic progressions of the form $An + B$ such that

$$p(An + B) \equiv 0 \pmod{\ell}.$$

Moreover partitions are related to Eulerian series. For example we have

$$\sum_{n=0}^{\infty} p(n) q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)^2(1-q^2)^2 \cdots (1-q^n)^2}. \tag{1.5}$$

Other examples that relate Eulerian series to modular forms are the Rogers-Ramanujan identities

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} &= \frac{1}{\prod_{n=1}^{\infty}(1-q^{5n-1})(1-q^{5n-4})}, \\ 1 + \sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} &= \frac{1}{\prod_{n=1}^{\infty}(1-q^{5n-2})(1-q^{5n-3})}. \end{aligned} \tag{1.6}$$

Mock theta functions, which can also be defined as Eulerian series, stand out of this context. For example the mock theta function $f(q)$, defined by Ramanujan [34] in his last letter to Hardy, is given by

$$f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2}. \tag{1.7}$$

Even if (1.5) and (1.7) have a similar shape $P(q)$ is modular whereas $f(q)$ is not. The mock theta functions were mysterious objects for a long time, for example there was even a discussion how to rigorously define them. Despite those problems they have applications in vast areas of mathematics (e.g. [2, 3, 4, 18, 20, 28, 39] just to mention a few). In this context one has to understand Dyson's quote given 1987 at the Ramanujan's Centary Conference:

"*The mock theta-functions give us tantalizing hints of a grand synthesis still to be discovered. Somehow it should be possible to build them into a coherent group-theoretical structure, analogous to the structure of modular forms which Hecke built around the old theta-functions of Jacobi. This remains a challenge for the future.*"

Zwegers [39, 40] made a first step towards solving this challenge. He observed that Ramanujan's mock theta functions can be interpreted as part of a real analytic vector valued modular form. The author and Ono build on those results to relate functions like $f(q)$ to weak Maass forms (see Section 2 for the definition of a weak Maass form). It turns out that $f(q)$ is the "holomorphic part" of a weak Maass form, the "non-holomorphic part" is a Mordell type integral involving weight $\frac{3}{2}$ theta functions. This is the special case of an infinite family of weak Maass forms that arise from Dyson's rank generating functions (see Section 3). This new theory has a wide range of applications. For example we obtain exact formulas for Ramanujan's mock theta function $f(q)$ (see Section 4), congruences for Dyson's ranks (see Section 5), asymptotics and inequalities for ranks (see Section 6) and identities for rank differences that involve modular forms (see Section 8). In Section 7 we show furthermore a correspondence between weight $\frac{3}{2}$ weak Maass forms and weight $\frac{1}{2}$ theta functions.

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2. General facts on weak Maass forms

Here we recall basic facts on weak Maass forms, first studied by Bruinier and Funke [17]. For $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ and $z = x + iy$ with $x, y \in \mathbb{R}$, the weight k hyperbolic Laplacian is given by

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

If v is odd, then define ϵ_v by

$$\epsilon_v := \begin{cases} 1 & \text{if } v \equiv 1 \pmod 4, \\ i & \text{if } v \equiv 3 \pmod 4. \end{cases}$$

A *(harmonic) weak Maass form of weight k on a subgroup* $\Gamma \subset \Gamma_0(4)$ is any smooth function $f : \mathbb{H} \to \mathbb{C}$ satisfying the following:

(1) For all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and all $z \in \mathbb{H}$, we have

$$f(Az) = \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} (cz + d)^k \ f(z).$$

(2) We have that $\Delta_k f = 0$.

(3) The function $f(z)$ has at most linear exponential growth at all the cusps of Γ.

In a similar manner one defines weak Maass forms on $\Gamma_0(4N)$ (N a positive integer) with Nebentypus χ (a Dirichlet character) by requiring

$$f(Az) = \chi(d) \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} (cz + d)^k \ f(z)$$

instead of (1). Harmonic weak Maass forms have Fourier expansions of the form

$$f(z) = \sum_{n=n_0}^{\infty} \gamma_y(n) q^{-n} + \sum_{n=n_1}^{\infty} a(n) q^n,$$

with $n_0, n_1 \in \mathbb{Z}$. The $\gamma_y(n)$ are functions in y, the imaginary part of z. We refer to $\sum_{n=n_0}^{\infty} \gamma_y(n) q^{-n}$ as the *non-holomorphic part* of $f(z)$, and we refer to $\sum_{n=n_1}^{\infty} a(n) q^n$ as its *holomorphic part.*

Moreover we need the anti-linear differential operator ξ_k defined by

$$\xi_k(g)(z) := 2iy^k \overline{\tfrac{\partial}{\partial \bar{z}} g(z)}.$$

If g is a harmonic weak Maass form of weight k for the group Γ, then $\xi_k(g)$ is a weakly holomorphic modular form (i.e, a modular form with poles at most at the cusps of Γ) of weight $2 - k$ on Γ. Furthermore, ξ_k has the property that its kernel consists of those weight k weak Maass forms which are weakly holomorphic modular forms.

3. Dyson's ranks and weak Maass forms

In order to explain the Ramanujan congruences Dyson introduced the so-called rank of a partition [23]. The *rank* of a partition is defined to be its largest part minus the number of its parts. In his famous paper Dyson conjectured that ranks could be used to "explain" the congruences (1.2) and (1.3) with modulus 5 and 7. More precisely, he conjectured that the partitions of $5n+4$ (resp. $7n+5$) form 5 (resp. 7) groups of equal size when sorted by their ranks modulo 5 (resp. 7). In 1954, Atkin and Swinnerton-Dyer proved Dyson's rank conjecture [7].

To study ranks, it is natural to investigate a generating function. If $N(m,n)$ denotes the number of partitions of n with rank m, then it is well known that

$$R(w;q) := 1 + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} N(m,n) w^m q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(wq;q)_n (w^{-1}q;q)_n},$$

where

$$(a;q)_n = (a)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

If we let $w=1$ we recover $P(q)$ in its Eulerian form (1.5), i.e., (up to a q-power) a weight $-\frac{1}{2}$ modular form. Moreover $R(-1;q)$ is the generating function for the number of partitions with even rank minus the number of partitions with odd rank and equals

$$R(-1;q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2} = f(q)$$

with $f(q)$ as in (1.7). The author and Ono [12] showed that the functions $R(w;q)$ for $w \neq 1$ a root of unity are the holomorphic parts of weak Maass forms. To make this statement more precise suppose that $0 < a < c$ are integers, and let $\zeta_c := e^{\frac{2\pi i}{c}}$. If $f_c := \frac{2c}{\gcd(c,6)}$, then define for $\tau \in \mathbb{H}$ the weight $\frac{3}{2}$ cuspidal theta function $\Theta\left(\frac{a}{c};\tau\right)$ by

$$\Theta\left(\frac{a}{c};\tau\right) := \sum_{m \pmod{f_c}} (-1)^m \sin\left(\frac{a\pi(6m+1)}{c}\right) \cdot \theta\left(6m+1, 6f_c; \frac{\tau}{24}\right),$$

where

$$\theta(\alpha,\beta;\tau) := \sum_{n\equiv\alpha \pmod{\beta}} n e^{2\pi i \tau n^2}.$$

Moreover let $\ell_c := \operatorname{lcm}(2c^2, 24)$ and define

$$D\left(\frac{a}{c};q\right) = D\left(\frac{a}{c};z\right) := -S_1\left(\frac{a}{c};z\right) + q^{-\frac{\ell_c}{24}} R(\zeta_c^a; q^{\ell_c}), \tag{3.1}$$

where the period integral $S_1\left(\frac{a}{c};z\right)$ is given by

$$S_1\left(\frac{a}{c};z\right) := \frac{-i\sin\left(\frac{\pi a}{c}\right)\ell_c^{\frac{1}{2}}}{\sqrt{3}} \int_{-\bar{z}}^{i\infty} \frac{\Theta\left(\frac{a}{c};\ell_c\tau\right)}{\sqrt{-i(\tau+z)}}\, d\tau. \tag{3.2}$$

Theorem 3.1. (1) *If $0<a<c$, then $D\left(\frac{a}{c};z\right)$ is a weak Maass form of weight $\frac{1}{2}$ on $\Gamma_c := \left\langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ \ell_c^2 & 1 \end{smallmatrix}\right) \right\rangle$.*

(2) *If c is odd, then $D\left(\frac{a}{c};z\right)$ is a weak Maass form of weight $\frac{1}{2}$ on $\Gamma_1\left(6f_c^2\ell_c\right)$.*

If $\frac{a}{c}=\frac{1}{2}$, it turns out, using results of Zwegers [39], that $D\left(\frac{1}{2};z\right)$ is a weak Maass form on $\Gamma_0(144)$ with Nebentypus $\chi_{12}(\cdot) := \left(\frac{12}{\cdot}\right)$.

A similar phenomenon as in Theorem 3.1 occurs in the case of overpartitions. Recall that an *overpartition* is a partition, where the first occurance of a summand may be overlined. For a non-negative integer n we denote by $\overline{p}(n)$ the number of overpartions of n. We have the generating function [21]

$$\overline{P}(q) := \sum_{n\geq 0} \overline{p}(n)\, q^n = \frac{\eta(2z)}{\eta(z)^2}, \tag{3.3}$$

which is a weight $-\frac{1}{2}$ modular form. Moreover the generating function for $\overline{N}(m,n)$, the number of overpartitions of n with rank m, is given by

$$\mathcal{O}(w;q) := 1+\sum_{n=1}^{\infty}\overline{N}(m,n)w^mq^n = \sum_{n=0}^{\infty}\frac{(-1)_n q^{\frac{1}{2}n(n+1)}}{(wq;q)_n\left(\frac{q}{w};q\right)_n}.$$

In particular the case $w=1$ gives by (3.3) a modular form. Moreover it turns out [10] that for $w\notin\{-1,1\}$ a root of unity $\mathcal{O}(w;q)$ is the holomorphic part of a weight $\frac{1}{2}$ weak Maass form. In contrast to the case of usual partitions one obtains in the case $w=-1$ the holomorphic part of a weight $\frac{3}{2}$ weak Maass form.

Sketch of proof of Theorem 3.1. We first determine the transformation law of $R\left(\zeta_c^a;q\right)$. For this we define certain related functions. For $q:=e^{2\pi iz}$ let

$$M\left(\frac{a}{c};q\right) := \frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n+\frac{a}{c}}}{1-q^{n+\frac{a}{c}}}\cdot q^{\frac{3}{2}n(n+1)},$$

$$M_1\left(\frac{a}{c};q\right) := \frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}q^{n+\frac{a}{c}}}{1+q^{n+\frac{a}{c}}}\cdot q^{\frac{3}{2}n(n+1)},$$

$$N\left(\frac{a}{c};q\right) := \frac{(1-\zeta_c^a)}{(q;q)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n\,q^{\frac{n}{2}(3n+1)}}{1-\zeta_c^a\,q^n},$$

$$N_1\left(\frac{a}{c};q\right) := \sum_{n\in\mathbb{Z}}\frac{(-1)^n\,q^{\frac{3n}{2}(n+1)}}{1-\zeta_c^a\,q^{n+\frac{1}{2}}}.$$

As an abuse of notation we also write $M\left(\frac{a}{c};z\right)$ instead of $M\left(\frac{a}{c};q\right)$ and in the same way we treat the other functions. One can show [27] that

$$R\left(\zeta_c^a;q\right) = N\left(\frac{a}{c};q\right).$$

Moreover for $0\le b<c$, define $M(a,b,c;z)$ by

$$M(a,b,c;q) := \frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n+\frac{a}{c}}}{1-\zeta_c^bq^{n+\frac{a}{c}}}\cdot q^{\frac{3}{2}n(n+1)}.$$

In addition, if $\frac{b}{c}\notin\left\{0,\frac{1}{2},\frac{1}{6},\frac{5}{6}\right\}$, then define the integer $k(b,c)$ by

$$k(b,c) := \begin{cases} 0 & \text{if } 0<\frac{b}{c}<\frac{1}{6},\\ 1 & \text{if } \frac{1}{6}<\frac{b}{c}<\frac{1}{2},\\ 2 & \text{if } \frac{1}{2}<\frac{b}{c}<\frac{5}{6},\\ 3 & \text{if } \frac{5}{6}<\frac{b}{c}<1,\end{cases} \tag{3.4}$$

and let

$$N(a,b,c;q) := -\frac{i\zeta_{2c}^a\,q^{-\frac{b}{2c}}}{2\,(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{\frac{n}{2}(3n+1)-k(b,c)n}}{1-\zeta_c^aq^{n-\frac{b}{c}}}.$$

Remark. The above defined functions can also be rewritten in terms of the functions

$$T_k(x;q) := \frac{1}{(q;q)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n\,q^{\frac{n}{2}(3n+2k+1)}}{1-xq^n}$$

with $k \in \mathbb{Z}$. These functions can all be expressed in terms of T_0:

$$T_m(x;q) - xT_{m+1}(x;q) = (-1)^m \chi_3(1-m) q^{-\frac{m}{6}(m+1)}$$

with

$$\chi_3(m) := \begin{cases} 1 & \text{if } m \equiv 1 \pmod 3, \\ -1 & \text{if } m \equiv -1 \pmod 3, \\ 0 & \text{if } m \equiv 0 \pmod 3. \end{cases}$$

Also

$$T_m\left(x^{-1};q\right) = -xT_{-m}(x;q).$$

Moreover we need the following Mordell type integrals.

$$J\left(\frac{a}{c};\alpha\right) := \int_0^\infty e^{-\frac{3}{2}\alpha x^2} \frac{\cosh\left(\left(\frac{3a}{c}-2\right)\alpha x\right) + \cosh\left(\left(\frac{3a}{c}-1\right)\alpha x\right)}{\cosh(3\alpha x/2)}\, dx,$$

$$J_1\left(\frac{a}{c};\alpha\right) := \int_0^\infty e^{-\frac{3}{2}\alpha x^2} \frac{\sinh\left(\left(\frac{3a}{c}-2\right)\alpha x\right) - \sinh\left(\left(\frac{3a}{c}-1\right)\alpha x\right)}{\sinh(3\alpha x/2)}\, dx,$$

$$J(a,b,c;\alpha) := \int_{-\infty}^\infty e^{-\frac{3}{2}\alpha x^2 + 3\alpha x\frac{a}{c}} \frac{\left(\zeta_c^b e^{-\alpha x} + \zeta_c^{2b} e^{-2\alpha x}\right)}{\cosh\left(3\alpha x/2 - 3\pi i\frac{b}{c}\right)}\, dx.$$

Modifying an argument of Watson [38] one can show using contour integration.

Lemma 3.2. *Suppose that $0<a<c$ are coprime integers, and that α and β have the property that $\alpha\beta = \pi^2$. If $q := e^{-\alpha}$ and $q_1 := e^{-\beta}$, then we have*

$$q^{\frac{3a}{2c}\left(1-\frac{a}{c}\right)-\frac{1}{24}} \cdot M\left(\frac{a}{c};q\right) = \sqrt{\frac{\pi}{2\alpha}} \csc\left(\frac{a\pi}{c}\right) q_1^{-\frac{1}{6}} \cdot N\left(\frac{a}{c};q_1^4\right) - \sqrt{\frac{3\alpha}{2\pi}} \cdot J\left(\frac{a}{c};\alpha\right),$$

$$q^{\frac{3a}{2c}\left(1-\frac{a}{c}\right)-\frac{1}{24}} \cdot M_1\left(\frac{a}{c};q\right) = -\sqrt{\frac{2\pi}{\alpha}} q_1^{\frac{4}{3}} \cdot N_1\left(\frac{a}{c};q_1^2\right) - \sqrt{\frac{3\alpha}{2\pi}} \cdot J_1\left(\frac{a}{c};\alpha\right).$$

If moreover $\frac{b}{c} \notin \left\{\frac{1}{2},\frac{1}{6},\frac{5}{6}\right\}$, then

$$q^{\frac{3a}{2c}\left(1-\frac{a}{c}\right)-\frac{1}{24}} \cdot M(a,b,c;q) =$$
$$\sqrt{\frac{8\pi}{\alpha}} e^{-2\pi i\frac{a}{c}k(b,c) + 3\pi i\frac{b}{c}\left(\frac{2a}{c}-1\right)} \zeta_c^{-b} q_1^{\frac{4b}{c}k(b,c) - \frac{6b^2}{c^2} - \frac{1}{6}} \cdot N(a,b,c;q_1^4) - \sqrt{\frac{3\alpha}{8\pi}} \zeta_{2c}^{-5b} \cdot J(a,b,c;\alpha).$$

Two remarks.
1) The case $b=0$ is contained in [27].
2) It is nowadays more common to write modular transformation laws in terms of τ and $-\frac{1}{\tau}$ than in q and q_1.

The above transformation laws allow us to construct an infinite family of a vector valued weight $\frac{1}{2}$ weak Maass forms (see [12] for the definition of vector valued weak Maass form). For simplicity we assume for the remainder of this section that c is odd. Using the functions

$$\mathcal{N}\left(\frac{a}{c};q\right) = \mathcal{N}\left(\frac{a}{c};z\right) := \csc\left(\frac{a\pi}{c}\right) \cdot q^{-\frac{1}{24}} \cdot N\left(\frac{a}{c};q\right),$$

$$\mathcal{M}\left(\frac{a}{c};q\right) = \mathcal{M}\left(\frac{a}{c};z\right) := 2q^{\frac{3a}{2c}\cdot\left(1-\frac{a}{c}\right)-\frac{1}{24}} \cdot M\left(\frac{a}{c};q\right),$$

we define the vector valued (holomorphic) function $F\left(\frac{a}{c};z\right)$ by

$$F\left(\frac{a}{c};z\right) := \left(F_1\left(\frac{a}{c};z\right), F_2\left(\frac{a}{c};z\right)\right)^T = \left(\sin\left(\frac{\pi a}{c}\right)\mathcal{N}\left(\frac{a}{c};\ell_c z\right), \sin\left(\frac{\pi a}{c}\right)\mathcal{M}\left(\frac{a}{c};\ell_c z\right)\right)^T.$$

Similarly, define the vector valued (non-holomorphic) function $G\left(\frac{a}{c};z\right)$ by

$$G\left(\frac{a}{c};z\right)=\left(G_1\left(\frac{a}{c};z\right),G_2\left(\frac{a}{c};z\right)\right)^T$$
$$:=\left(2\sqrt{3}\sin\left(\frac{\pi a}{c}\right)\sqrt{-i\ell_c z}\cdot J\left(\frac{a}{c};-2\pi i\ell_c z\right),\frac{2\sqrt{3}\sin\left(\frac{\pi a}{c}\right)}{i\ell_c z}\cdot J\left(\frac{a}{c};\frac{2\pi i}{\ell_c z}\right)\right)^T.$$

Following a method of Zwegers [39], which uses the Mittag-Leffler partial fraction decomposition, we can realize the function $G\left(\frac{a}{c};z\right)$ as a vector valued theta integral.

Lemma 3.3. *For $z\in\mathbb{H}$, we have*

$$G\left(\frac{a}{c};z\right)=\frac{i\ell_c^{\frac{1}{2}}\sin\left(\frac{\pi a}{c}\right)}{\sqrt{3}}\int_0^{i\infty}\frac{\left((-i\ell_c\tau)^{-\frac{3}{2}}\Theta\left(\frac{a}{c};-\frac{1}{\ell_c\tau}\right),\Theta\left(\frac{a}{c};\ell_c\tau\right)\right)^T}{\sqrt{-i(\tau+z)}}\,d\tau.$$

We next determine the necessary modular transformation properties of the vector

$$S\left(\frac{a}{c};z\right)=\left(S_1\left(\frac{a}{c};z\right),S_2\left(\frac{a}{c};z\right)\right)$$
$$:=\frac{-i\sin\left(\frac{\pi a}{c}\right)\ell_c^{\frac{1}{2}}}{\sqrt{3}}\int_{-\bar{z}}^{i\infty}\frac{\left(\Theta\left(\frac{a}{c};\ell_c\tau\right),(-i\ell_c\tau)^{-\frac{3}{2}}\Theta\left(\frac{a}{c};-\frac{1}{\ell_c\tau}\right)\right)^T}{\sqrt{-i(\tau+z)}}\,d\tau.$$

Lemma 3.4. *We have*

$$\begin{aligned}S\left(\frac{a}{c};z+1\right)&=S\left(\frac{a}{c};z\right),\\ \frac{1}{\sqrt{-i\ell_c z}}\cdot S\left(\frac{a}{c};-\frac{1}{\ell_c{}^2 z}\right)&=\begin{pmatrix}0&1\\1&0\end{pmatrix}\cdot S\left(\frac{a}{c};z\right)+G\left(\frac{a}{c};z\right).\end{aligned}$$

Combining the above and using the transformation law for $\Theta\left(\frac{a}{c};\tau\right)$ [35], one can now conclude that $D\left(\frac{a}{c};z\right)$ satisfies the correct transformation law under the stated group. To see that $D\left(\frac{a}{c};z\right)$ is annihilated by $\Delta_{\frac{1}{2}}$, we write

$$\Delta_{\frac{1}{2}}=-4y^{\frac{3}{2}}\frac{\partial}{\partial z}\sqrt{y}\frac{\partial}{\partial\bar{z}}. \tag{3.5}$$

Since $q^{-\frac{\ell_c}{24}}R(\zeta_b^a;q^{\ell_c})$ is a holomorphic function in z, it is thus clearly annihilated by $\Delta_{\frac{1}{2}}$. Moreover

$$\frac{\partial}{\partial\bar{z}}\left(S_1\left(\frac{a}{c};z\right)\right)=-\frac{\sin\left(\frac{\pi a}{c}\right)\,\ell_c^{\frac{1}{2}}}{\sqrt{6y}}\cdot\Theta\left(\frac{a}{c};-\ell_c\bar{z}\right).$$

Hence, we find that $\sqrt{y}\frac{\partial}{\partial\bar{z}}\left(D\left(\frac{a}{c};z\right)\right)$ is anti-holomorphic, and therefore by (3.5) annihilated by $\Delta_{\frac{1}{2}}$. Using that $\Theta\left(\frac{a}{c};\tau\right)$ is a weight $\frac{3}{2}$ cusp form it is not hard to conclude that $D\left(\frac{a}{c};z\right)$ has at most linear exponential growth at the cusps. □

4. The Andrews-Dragonette-Conjecture

One can use the theory of weak Maass form to obtain exact formulas for the coefficients of the mock theta function $f(q)$ which we denote by $\alpha(n)$ [11]. Recall that

$$f(q)=R(-1;q)=1+\sum_{n=1}^{\infty}\left(N_e(n)-N_o(n)\right)q^n=1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2},$$

where $N_e(n)$ (resp. $N_o(n)$) denotes the number of partions of n with even (resp. odd) rank. It is a classical problem to find exact formulas for $N_e(n)$ and $N_o(n)$. Since by (1.1) we have an exact formula for the partition function $p(n)$ this is equivalent to the problem of determining exact formulas for $\alpha(n)$. Ramanujan's last letter to Hardy includes the claim that

$$\alpha(n) = (-1)^{n-1}\frac{\exp\left(\pi\sqrt{\frac{n}{6}-\frac{1}{144}}\right)}{2\sqrt{n-\frac{1}{24}}} + O\left(\frac{\exp\left(\frac{1}{2}\pi\sqrt{\frac{n}{6}-\frac{1}{144}}\right)}{\sqrt{n-\frac{1}{24}}}\right). \tag{4.1}$$

Typical of his writings, Ramanujan did not give a proof. Dragonette finally showed (4.1) in her 1951 Ph.D. thesis [22] written under the direction of Rademacher. In his 1964 Ph.D. thesis, also written under Rademacher, Andrews [2] improved upon Dragonette's work, and showed that

$$\alpha(n) = \pi(24n-1)^{-\frac{1}{4}}\sum_{k=1}^{[\sqrt{n}]}\frac{(-1)^{\lfloor\frac{k+1}{2}\rfloor}A_{2k}\left(n-\frac{k(1+(-1)^k)}{4}\right)}{k}\cdot I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12k}\right) + O(n^{\epsilon}).$$

Moreover they made the

Conjecture. (Andrews-Dragonette)
If n is a positive integer, then

$$\alpha(n) = \pi(24n-1)^{-\frac{1}{4}}\sum_{k=1}^{\infty}\frac{(-1)^{\lfloor\frac{k+1}{2}\rfloor}A_{2k}\left(n-\frac{k(1+(-1)^k)}{4}\right)}{k}\cdot I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12k}\right). \tag{4.2}$$

The author and Ono [11] used the theory of weak Maass forms to prove this conjecture.

Theorem 4.1. *The Andrews-Dragonette Conjecture is true.*

Sketch of Proof. From Section 3 we know that $D\left(\frac{1}{2};z\right)$ is a weight $\frac{1}{2}$ weak Maass form on $\Gamma_0(144)$ with Nebentypus character χ_{12}. We will construct a Maass-Poincaré series which we will show equals $D\left(\frac{1}{2};z\right)$. The Andrews-Dragonette Conjecture can be concluded by computing the coefficients of the Poincaré series. For $s\in\mathbb{C}, k\in\frac{1}{2}+\mathbb{Z}$, and $y\in\mathbb{R}\setminus\{0\}$, let

$$\begin{aligned}\mathcal{M}_s(y) &:= |y|^{-\frac{k}{2}}M_{\frac{k}{2}\,\mathrm{sgn}(y),\, s-\frac{1}{2}}(|y|),\\ \mathcal{W}_s(y) &:= |y|^{-\frac{1}{4}}W_{\frac{1}{4}\,\mathrm{sgn}(y),\, s-\frac{1}{2}}(|y|),\end{aligned}$$

where $M_{\nu,\mu}(z)$ and $W_{\nu,\mu}(z)$ are the standard Whittaker function. Furthermore, let

$$\varphi_{s,k}(z) := \mathcal{M}_s\left(-\frac{\pi y}{6}\right)e\left(-\frac{x}{24}\right).$$

It is straightforward to confirm that $\varphi_{s,k}(z)$ is an eigenfunction of Δ_k. Moreover for matrices $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_0(2)$, with $c\geq 0$, let

$$\chi\left(\begin{pmatrix}a&b\\c&d\end{pmatrix}\right) := \begin{cases} e\left(-\frac{b}{24}\right) & \text{if } c=0,\\ i^{-1/2}(-1)^{\frac{1}{2}(c+ad+1)}e\left(-\frac{a+d}{24c}-\frac{a}{4}+\frac{3dc}{8}\right)\cdot\omega_{-d,c}^{-1} & \text{if } c>0.\end{cases}$$

Define the Poincaré series $P_k(s;z)$ by

$$P_k(s;z) := \frac{2}{\sqrt{\pi}}\sum_{M\in\Gamma_\infty\setminus\Gamma_0(2)}\chi(M)^{-1}(cz+d)^{-k}\varphi_{s,k}(Mz), \tag{4.3}$$

were $\Gamma_\infty := \{\pm \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \ : \ n \in \mathbb{Z}\}$. The series $P_{\frac{1}{2}}\left(1-\frac{k}{2}; z\right)$ is absolute convergent for $k < \frac{1}{2}$ and annihilated by $\Delta_{\frac{k}{2}}$. The function $P_{\frac{1}{2}}\left(\frac{3}{4}; z\right)$ can be analytically continued by its Fourier expansion, which bases on a modification of an argument of Hooley involving the interplay between solutions of quadratic congruences and the representation of integers by quadratic forms. This calculation is lengthy, and is carried out in detail in [11]. We compute the Fourier expansion of $P_{\frac{1}{2}}\left(\frac{3}{4}; z\right)$ as

$$P_{\frac{1}{2}}\left(\frac{3}{4}; z\right) = \left(1 - \pi^{-\frac{1}{2}} \cdot \Gamma\left(\frac{1}{2}, \frac{\pi y}{6}\right)\right) \cdot q^{-\frac{1}{24}} + \sum_{n=-\infty}^{0} \gamma_y(n) q^{n-\frac{1}{24}} + \sum_{n=1}^{\infty} \beta(n) q^{n-\frac{1}{24}},$$

where for positive integers n we have

$$\beta(n) = \pi(24n-1)^{-\frac{1}{4}} \sum_{k=1}^{\infty} \frac{(-1)^{\lfloor \frac{k+1}{2} \rfloor} A_{2k}\left(n - \frac{k(1+(-1)^k)}{4}\right)}{k} \cdot I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12k}\right),$$

and for non-positive integers n we have

$$\begin{aligned} \gamma_y(n) = \pi^{\frac{1}{2}} |24n-1|^{-\frac{1}{4}} \cdot \Gamma\left(\frac{1}{2}, \frac{\pi|24n-1| \cdot y}{6}\right) & \\ \times \sum_{k=1}^{\infty} \frac{(-1)^{\lfloor \frac{k+1}{2} \rfloor} A_{2k}\left(n - \frac{k\left(1+(-1)^k\right)}{4}\right)}{k} \cdot J_{\frac{1}{2}}\left(\frac{\pi\sqrt{|24n-1|}}{12k}\right). & \end{aligned}$$

Here the incomplete gamma function $\Gamma(a; x)$ is defined by

$$\Gamma(a; x) := \int_x^\infty e^{-t} t^{a-1} \, dt. \tag{4.4}$$

To finish the proof, we have to show that $\alpha(n) = \beta(n)$. For this we let

$$\begin{aligned} P(z) &:= P_{\frac{1}{2}}\left(\frac{3}{4}; 24z\right) = P_{nh}(z) + P_h(z), \\ M(z) &:= D\left(\frac{1}{2}; z\right) = M_{nh}(z) + M_h(z) \end{aligned}$$

canonically decomposed into a non-holomorphic and a holomorphic part. In particular

$$\begin{aligned} P_h(z) &= q^{-1} + \sum_{n=1}^{\infty} \beta(n)\, q^{24n-1}, \\ M_h(z) &= q^{-1} f(q^{24}) = q^{-1} + \sum_{n=1}^{\infty} \alpha(n)\, q^{24n-1}. \end{aligned}$$

The function $P(z)$ and $M(z)$ are weak Maass forms of weight $\frac{1}{2}$ for $\Gamma_0(144)$ with Nebentypus χ_{12}. We first prove that $P_{nh}(z) = M_{nh}(z)$. For this compute that $\xi_{\frac{1}{2}}\left(P(z)\right)$ and $\xi_{\frac{1}{2}}\left(M(z)\right)$ are holomorphic modular forms of weight $\frac{3}{2}$ with Nebentypus χ_{12} with the property that their non-zero Fourier coefficients are supported on arithmetic progression congruent to $1 \pmod{24}$. Choose a constant c such that the coefficients up to q^{24} of $\xi_{\frac{1}{2}}\left(P(z)\right)$ and $c\xi_{\frac{1}{2}}(M(z))$ agree and since $\dim_{\mathbb{C}}\left(M_{\frac{3}{2}}\left(\Gamma_0(144), \chi_{12}\right)\right) = 24$, $\xi_{\frac{1}{2}}\left(P(z)\right) = c\xi_{\frac{1}{2}}(M(z))$, which implies that $P_{nh}(z) = cM_{nh}(z)$. Thus the function $H(z) := P(z) - cM(z)$ is a weakly holomorphic modular form. We have to show that $c = 1$. For this we apply the inversion $z \mapsto -\frac{1}{z}$. By work of Zwegers [39] this produces a nonholomorphic part unless $c = 1$. Since $H(z)$ is weakly holomorphic we conclude that $c = 1$. To be more precise, the Poincaré series considered here is a component of a vector valued weak Maass form whose transformation law is known by work of Zwegers (see also [26], where such a vector valued Poincaré series is construced). Estimating the coefficients of $P(z)$ and $M(z)$ against $n^{\frac{3}{4}+\epsilon}$, one obtains that $H(z)$ is a holomorphic modular form of weight $\frac{1}{2}$ on $\Gamma_0(144)$ with Nebentypus χ_{12}. Since this space is trivial we obtain $H(z) = 0$ which employs the claim. $\square$

5. Congruences for Dyson's rank generating functions

In this section we prove an infinite family of congruences for Dyson's ranks which generalizes partitions congruences [12].

Theorem 5.1. *Let t be a positive odd integer, and let $\mathcal{Q} \nmid 6t$ be prime. If j is a positive integer, then there are infinitely many non-nested arithmetic progressions $An + B$ such that for every $0 \leq r < t$ we have*

$$N(r, t; An + B) \equiv 0 \pmod{\mathcal{Q}^j}.$$

Two remarks.
1) The congruences in Theorem 5.1 may be viewed as a combinatorial decomposition of the partition function congruence

$$p(An + B) \equiv 0 \pmod{\mathcal{Q}^j}.$$

2) Congruences for $t = \mathcal{Q}^j$ were shown in [9].

Sketch of proof. First observe that

$$\sum_{n=0}^{\infty} N(r, t; n)q^n = \frac{1}{t} \sum_{n=0}^{\infty} p(n)q^n + \frac{1}{t} \sum_{j=1}^{t-1} \zeta_t^{-rj} \cdot R(\zeta_t^j; q). \tag{5.1}$$

Using the results from Section 3 we can conclude that

$$\sum_{n=0}^{\infty} \left(N(r, t; n) - \frac{p(n)}{t} \right) q^{\ell_t n - \frac{\ell_t}{24}}$$

is the holomorphic part of a weak Maass form of weight $\frac{1}{2}$ on $\Gamma_1\left(6 f_t^2 \ell_t\right)$.

We wish to apply certain quatratic twists which "kill" the non-holomorphic part of $D\left(\frac{a}{c}; q\right)$. For this we compute on which arithmetic progressions it is supported. This will enable us to use results on congruences for half integer weight modular forms. We obtain

$$\begin{aligned} D\left(\frac{a}{c}; z\right) &= q^{-\frac{\ell_c}{24}} + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} N(m, n) \zeta_c^{am} q^{\ell_c n - \frac{\ell_c}{24}} \\ &- \frac{2 \sin\left(\frac{\pi a}{c}\right)}{\sqrt{\pi}} \sum_{m \pmod{f_c}} (-1)^m \sin\left(\frac{a\pi(6m+1)}{c}\right) \sum_{n \equiv 6m+1 \pmod{6f_c}} \Gamma\left(\frac{1}{2}; \frac{\ell_c n^2 y}{6}\right) q^{-\frac{\ell_c n^2}{24}}. \end{aligned}$$

In particular the non-holomorphic part of $D\left(\frac{a}{c}; z\right)$ is supported on certain fixed arithmetic progression. Generalizing the theory of twists of modular forms to twists of weak Maass forms, one can show.

Proposition 5.2. *If $0 \leq r < t$ are integers, where t is odd, and $\mathcal{P} \nmid 6t$ is prime, then*

$$\sum_{\substack{n \geq 1 \\ \left(\frac{24\ell_t n - \ell_t}{\mathcal{P}}\right) = -\left(\frac{-\ell_t}{\mathcal{P}}\right)}} \left(N(r, t; n) - \frac{p(n)}{t} \right) q^{\ell_t n - \frac{\ell_t}{24}}$$

is a weight $\frac{1}{2}$ weakly holomorphic modular form on $\Gamma_1\left(6 f_t^2 \ell_t \mathcal{P}^4\right)$.

To prove Theorem 5.1, we shall employ a recent general result of Treneer [36]. We use the following fact, which generalized Serre's results on p-adic modular forms.

Proposition 5.3. *Suppose that $f_1(z), f_2(z), \ldots, f_s(z)$ are half-integral weight cusp forms where*

$$f_i(z) \in S_{\lambda_i+\frac{1}{2}}(\Gamma_1(4N_i)) \cap \mathcal{O}_K[[q]],$$

and where $\mathcal{O}_K$ is the ring of integers of a fixed number field K. If $\mathcal{Q}$ is prime and $j \geq 1$ is an integer, then the set of primes L for which

$$f_i(z) \mid T_{\lambda_i}(L^2) \equiv 0 \pmod{\mathcal{Q}^j},$$

for each $1 \leq i \leq s$, has positive Frobenius density. Here $T_{\lambda_i}(L^2)$ denotes the usual L^2 index Hecke operator of weight $\lambda_i + \frac{1}{2}$.

Now suppose that $\mathcal{P} \nmid 6t\mathcal{Q}$ is prime. By Proposition 5.2, for every $0 \leq r < t$

$$F(r,t,\mathcal{P};z) = \sum_{n=1}^{\infty} a(r,t,\mathcal{P};n)q^n := \sum_{\left(\frac{24\ell_t n-\ell_t}{\mathcal{P}}\right)=-\left(\frac{-\ell_t}{\mathcal{P}}\right)} \left(N(r,t;n) - \frac{p(n)}{t}\right) q^{\ell_t n-\frac{\ell_t}{24}} \tag{5.2}$$

is a weakly holomorphic modular form of weight $\frac{1}{2}$ on $\Gamma_1\left(6f_t^2\ell_t\mathcal{P}^4\right)$. Furthermore, by the work of Ahlgren and Ono [1], it is known that

$$P(t,\mathcal{P};z) = \sum_{n=1}^{\infty} p(t,\mathcal{P};n)q^n := \sum_{\left(\frac{24\ell_t n-\ell_t}{\mathcal{P}}\right)=-\left(\frac{-\ell_t}{\mathcal{P}}\right)} p(n) q^{\ell_t n-\frac{\ell_t}{24}} \tag{5.3}$$

is a weakly holomorphic modular form of weight $-\frac{1}{2}$ on $\Gamma_1\left(24\ell_t\mathcal{P}^4\right)$.

Now since $\mathcal{Q} \nmid 24f_t^2\ell_t\mathcal{P}^4$, a generalization of a result of Treneer (see Theorem 3.1 of [36]), implies that there is a sufficiently large integer m for which

$$\sum_{\mathcal{Q}\nmid n} a(r,t,\mathcal{P};\mathcal{Q}^m n)q^n,$$

for all $0 \leq r < t$, and

$$\sum_{\mathcal{Q}\nmid n} p(t,\mathcal{P};\mathcal{Q}^m n)q^n$$

are all congruent modulo $\mathcal{Q}^j$ to forms in the graded ring of half-integral weight cusp forms with algebraic integer coefficients on $\Gamma_1\left(24f_t^2\ell_t\right)$. Applying Proposition 5.3 to these $t+1$ forms gives that a positive proportion of primes L have the property that these $t+1$ half-integral weight cusp forms modulo $\mathcal{Q}^j$ are annihilated by the index L^2 half-integral weight Hecke operators. Theorem 5.1 now follows *mutatis mutandis* as in the proof of Theorem 1 of [32]. $\square$

6. Asymptotics for Dyson's rank partition functions

We obtain asymptotic formulas for Dyson's rank generating functions [8]. As an application, we solve a conjecture of Andrews and Lewis on inequalities between ranks. We write

$$R(\zeta_c^a;q) =: 1 + \sum_{n=1}^{\infty} A\left(\frac{a}{c};n\right)q^n.$$

Let k and h be coprime integers, h' defined by $hh' \equiv -1 \pmod{k}$ if k is odd and by $hh' \equiv -1 \pmod{2k}$ if k is even, $k_1 := \frac{k}{\gcd(k,c)}$, $c_1 := \frac{c}{\gcd(k,c)}$, and $0 < l < c_1$ is defined by the congruence $l \equiv ak_1 \pmod{c_1}$. Furthermore we define, for $n, m \in \mathbb{Z}$, the following sums of Kloosterman type

$$\begin{aligned} D_{a,c,k}(n,m) &:= (-1)^{ak+l} \sum_{h \pmod{k}^*} \omega_{h,k} \cdot e^{\frac{2\pi i}{k}(nh+mh')}, \\ B_{a,c,k}(n,m) &:= (-1)^{ak+1} \sin\left(\frac{\pi a}{c}\right) \sum_{h \pmod{k}^*} \frac{\omega_{h,k}}{\sin\left(\frac{\pi a h'}{c}\right)} \cdot e^{-\frac{3\pi i a^2 k_1 h'}{c}} \cdot e^{\frac{2\pi i}{k}(nh+mh')}, \end{aligned}$$

where for $B_{a,c,k}(n,m)$ we require that $c|k$. Moreover, for $c \nmid k$, let

$$\delta_{c,k,r} := \begin{cases} -\left(\frac{1}{2}+r\right)\frac{l}{c_1} + \frac{3}{2}\left(\frac{l}{c_1}\right)^2 + \frac{1}{24} & \text{if } 0 < \frac{l}{c_1} < \frac{1}{6}, \\ -\frac{5l}{2c_1} + \frac{3}{2}\left(\frac{l}{c_1}\right)^2 + \frac{25}{24} - r\left(1 - \frac{l}{c_1}\right) & \text{if } \frac{5}{6} < \frac{l}{c_1} < 1, \\ 0 & \text{otherwise}, \end{cases} \tag{6.1}$$

and for $0 < \frac{l}{c_1} < \frac{1}{6}$ or $\frac{5}{6} < \frac{l}{c_1} < 1$

$$m_{a,c,k,r} := \begin{cases} \frac{1}{2c_1^2}\left(-3a^2k_1^2 + 6lak_1 - ak_1c_1 - 3l^2 + lc_1 - 2ark_1c_1 + 2lc_1 r\right) & \text{if } 0 < \frac{l}{c_1} < \frac{1}{6}, \\ \frac{1}{2c_1^2}\left(-6ak_1c_1 - 3a^2k_1^2 + 6lak_1 + ak_1c_1 + 6lc_1 \right. & \text{if } \frac{5}{6} < \frac{l}{c_1} < 1. \\ \qquad \left. -3l^2 - 2c_1^2 - lc_1 + 2ark_1c_1 + 2c_1(c_1 - l)r\right) & \end{cases}$$

Using the Circle Method, we obtain [8] the following asymptotic formulas for the coefficients $A\left(\frac{a}{c}; n\right)$.

Theorem 6.1. *If $0 < a < c$ are coprime integers and c is odd, then for positive integers n we have that*

$$\begin{aligned} A\left(\frac{a}{c}; n\right) = \frac{4\sqrt{3}i}{\sqrt{24n-1}} \sum_{\substack{1 \le k \le \sqrt{n} \\ c|k}} \frac{B_{a,c,k}(-n,0)}{\sqrt{k}} \cdot \sinh\left(\frac{\pi\sqrt{24n-1}}{6k}\right) \\ + \frac{8\sqrt{3} \cdot \sin\left(\frac{\pi a}{c}\right)}{\sqrt{24n-1}} \sum_{\substack{1 \le k \le \sqrt{n} \\ c \nmid k \\ r \ge 0 \\ \delta_{c,k,r} > 0}} \frac{D_{a,c,k}(-n, m_{a,c,k,r})}{\sqrt{k}} \cdot \sinh\left(\frac{\pi\sqrt{2\delta_{c,k,r}(24n-1)}}{\sqrt{3}k}\right) + O_c\left(n^{\epsilon}\right). \end{aligned}$$

Using (5.1), one can conclude asymptotics for $N(a,c;n)$ from Theorem 6.1.

Corollary 6.2. *For integers $0 \le a < c$, where c is an odd integer, we have*

$$\begin{aligned} N(a,c;n) = \frac{2\pi}{c \cdot \sqrt{24n-1}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} \cdot I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right) \\ + \frac{1}{c} \sum_{j=1}^{c-1} \zeta_c^{-aj} \left(\frac{4\sqrt{3}i}{\sqrt{24n-1}} \sum_{c|k} \frac{B_{j,c,k}(-n,0)}{\sqrt{k}} \sinh\left(\frac{\pi}{6k}\sqrt{24n-1}\right) \right. \\ \left. + \frac{8\sqrt{3}\sin\left(\frac{\pi j}{c}\right)}{\sqrt{24n-1}} \sum_{\substack{k,r \\ c \nmid k \\ \delta_{c,k,r} > 0}} \frac{D_{j,c,k}(-n, m_{j,c,k,r})}{\sqrt{k}} \sinh\left(\sqrt{\frac{2\delta_{c,k,r}(24n-1)}{3}} \frac{\pi}{k}\right) \right) + O_c\left(n^{\epsilon}\right). \end{aligned}$$

This corollary implies a conjecture of Andrews and Lewis. In [6, 31] they showed

$$\begin{aligned} N(0,2;2n) &< N(1,2;2n) && \text{if } n \geq 1, \\ N(0,4;n) &> N(2,4;n) && \text{if } 26 < n \equiv 0,1 \pmod 4, \\ N(0,4;n) &< N(2,4;n) && \text{if } 26 < n \equiv 2,3 \pmod 4. \end{aligned}$$

Moreover, they conjectured (see Conjecture 1 of [6]).

Conjecture. (Andrews and Lewis)
For all $n > 0$, we have

$$\begin{array}{ll} N(0,3;n) < N(1,3;n) & \textit{if } n \equiv 0 \textit{ or } 2 \pmod 3, \\ N(0,3;n) > N(1,3;n) & \textit{if } n \equiv 1 \pmod 3, \end{array} \tag{6.2}$$

A careful analysis of the asymptotics in Corollary 6.2 gives the following theorem.

Theorem 6.3. *The Andrews-Lewis Conjecture is true for all $n \notin \{3, 9, 21\}$ in which case we have equality in (6.2).*

Sketch of proof of Theorem 6.1. We use the Hardy Littlewood method. By Cauchy's Theorem we have for $n > 0$

$$A\left(\frac{a}{c};n\right) = \frac{1}{2\pi i}\int_C \frac{N\left(\frac{a}{c};q\right)}{q^{n+1}}\,dq,$$

where C is an arbitrary path inside the unit circle surrounding 0 counterclockwise. Now let

$$\vartheta'_{h,k} := \frac{1}{k(k_1+k)}, \quad \vartheta''_{h,k} := \frac{1}{k(k_2+k)},$$

where $\frac{h_1}{k_1} < \frac{h}{k} < \frac{h_2}{k_2}$ are adjacent Farey fractions in the Farey sequence of order $N := \lfloor n^{1/2} \rfloor$. We make the substitution $q = e^{-\frac{2\pi}{n}+2\pi i t}$ $(0 \leq t \leq 1)$ and then decompose the path of integration into paths along the Farey arcs $-\vartheta'_{h,k} \leq \Phi \leq \vartheta''_{h,k}$, where $\Phi = t - \frac{h}{k}$ and $0 \leq h \leq k \leq N$ with $(h,k) = 1$. One obtains

$$A\left(\frac{a}{c};n\right) = \sum_{h,k} e^{-\frac{2\pi i h n}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} N\left(\frac{a}{c}; e^{\frac{2\pi i}{k}(h+iz)}\right)\cdot e^{\frac{2\pi n z}{k}}\,d\Phi,$$

where $z = \frac{k}{n} - k\Phi i$. One can conclude from the transformation law of $N\left(\frac{a}{c};q\right)$ (in a modified version of Lemma 3.2) that

$$A\left(\frac{a}{c};n\right) = \sum_1 + \sum_2 + \sum_3,$$

where

$$\sum_1 := i\sin\left(\frac{\pi a}{c}\right)\sum_{\substack{h,k\\ c|k}}\omega_{h,k}\,\frac{(-1)^{ak+1}}{\sin\left(\frac{\pi a h'}{c}\right)}\cdot e^{-\frac{3\pi i a^2k_1h'}{c}-\frac{2\pi i h n}{k}}\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{-\frac{1}{2}}\cdot e^{\frac{2\pi z}{k}\left(n-\frac{1}{24}\right)+\frac{\pi}{12kz}}\, N\left(\frac{ah'}{c};q_1\right)d\Phi,$$

$$\sum_2 := -4i\sin\left(\frac{\pi a}{c}\right)\sum_{\substack{h,k\\ c\nmid k}}\omega_{h,k}\,(-1)^{ak+l}\,e^{-\frac{2\pi i h' s a}{c}-\frac{3\pi i h' a^2 k_1}{cc_1}+\frac{6\pi i h' l a}{cc_1}-\frac{2\pi i h n}{k}}$$

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{-\frac{1}{2}}\cdot e^{\frac{2\pi z}{k}\left(n-\frac{1}{24}\right)+\frac{\pi}{12kz}}\cdot q_1^{\frac{sl}{c_1}-\frac{3l^2}{2c_1^2}}\cdot N\left(ah',\frac{lc}{c_1},c;q_1\right)d\Phi,$$

$$\sum_3 := 2\sin^2\left(\frac{\pi a}{c}\right)\sum_{h,k}\frac{\omega_{h,k}}{k}\cdot e^{-\frac{2\pi i h n}{k}}\sum_{\nu \pmod k}(-1)^{\nu}\,e^{-\frac{3\pi i h'\nu^2}{k}+\frac{\pi i h'\nu}{k}}$$

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{2\pi z}{k}\left(n-\frac{1}{24}\right)}\cdot z^{\frac{1}{2}}\cdot I_{a,c,k,\nu}(z)d\Phi.$$

Here $q_1 := e^{\frac{2\pi i}{k}\left(h'+\frac{i}{\tau}\right)}$ and

$$I_{a,c,k,\nu}(z) := \int_{\mathbb{R}} e^{-\frac{3\pi z x^2}{k}}\cdot H_{a,c}\left(\frac{\pi i\nu}{k}-\frac{\pi i}{6k}-\frac{\pi z x}{k}\right)dx$$

with

$$H_{a,c}(x) := \frac{\cosh(x)}{\sinh\left(x+\frac{\pi i a}{c}\right)\cdot\sinh\left(x-\frac{\pi i a}{c}\right)}.$$

Two important steps are the estimation of $I_{a,c,k,\nu}(z)$ and certain Kloosterman sums.

Lemma 6.4. *We have*

$$z^{\frac{1}{2}}\cdot I_{a,c,k,\nu}(z)\ll k\cdot n^{\frac{1}{4}}\cdot g_{a,c,k,\nu},$$

where $g_{a,c,k,\nu} := \left(\min\left(6kc\left\{\frac{\nu}{k}-\frac{1}{6k}+\frac{a}{c}\right\},6kc\left\{\frac{\nu}{k}-\frac{1}{6k}-\frac{a}{c}\right\}\right)\right)^{-1}$, *with* $\{x\} := x-\lfloor x\rfloor$ *for* $x\in\mathbb{R}$.

Lemma 6.5. *For* $n,m\in\mathbb{Z}$, $0\le\sigma_1<\sigma_2\le k$, $D\in\mathbb{Z}$ *with* $(D,k)=1$, *we have*

$$\sum_{\substack{h \pmod k^*\\ \sigma_1\le Dh'\le\sigma_2}}\omega_{h,k}\cdot e^{\frac{2\pi i}{k}(hn+h'm)} \ll \gcd(24n+1,k)^{\frac{1}{2}}\cdot k^{\frac{1}{2}+\epsilon}.$$

If $c|k$, *then we have*

$$(-1)^{ak+1}\sin\left(\frac{\pi a}{c}\right)\sum_{\substack{h \pmod k^*\\ \sigma_1\le Dh'\le\sigma_2}}\frac{\omega_{h,k}}{\sin\left(\frac{\pi a h'}{c}\right)}\cdot e^{-\frac{3\pi i a^2k_1h'}{c}}\cdot e^{\frac{2\pi i}{k}(hn+h'm)} \ll \gcd(24n+1,k)^{\frac{1}{2}}\cdot k^{\frac{1}{2}+\epsilon}.$$

To estimate $\sum_1$, we write

$$N\left(\frac{ah'}{c};q_1\right) =: 1+\sum_{r\in\mathbb{N}}a(r)\cdot e^{\frac{2\pi i m_r h'}{k}}\cdot e^{-\frac{2\pi r}{kz}},$$

where m_r is a sequence in $\mathbb{Z}$ and the coefficients $a(r)$ are independent of a,c,k, and h. We treat the constant term and the term coming from from $r\ge 1$ seperately since they contribute to the main term and to the error term, respectively. We denote the associated sums by S_1 and S_2, respectively and first estimate S_2. Throughout we need the easily verified fact that $\mathrm{Re}(z) = \frac{k}{n}$, $\mathrm{Re}\left(\frac{1}{z}\right) > \frac{k}{2}$, $|z|^{-\frac{1}{2}} \leq n^{\frac{1}{2}} \cdot k^{-\frac{1}{2}}$, and $\vartheta'_{h,k} + \vartheta''_{h,k} \leq \frac{2}{k(N+1)}$. Decompose

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} = \int_{-\frac{1}{k(N+k)}}^{\frac{1}{k(N+k)}} + \int_{-\frac{1}{k(k_1+k)}}^{-\frac{1}{k(N+k)}} + \int_{\frac{1}{k(N+k)}}^{\frac{1}{k(k_2+k)}} \tag{6.3}$$

and denote the associated sums by S_{21}, S_{22}, and S_{23}, respectively. Furthermore in S_{22} (and similarly S_{23}) we write

$$\int_{-\frac{1}{k(k+k_1)}}^{-\frac{1}{k(N+k)}} = \sum_{l=k_1+k}^{N+k-1} \int_{-\frac{1}{kl}}^{-\frac{1}{k(l+1)}}.$$

We have that

$$k_1 \equiv -h' \pmod{k}, \quad k_2 \equiv h' \pmod{k},$$
$$N - k < k_1 \leq N, \quad N - k < k_2 \leq N.$$

Using Lemma 6.5, we can show that $\sum_1$ equals

$$i \sin\left(\frac{\pi a}{c}\right) \sum_{\substack{h,k \\ c|k}} \omega_{h,k} \cdot \frac{(-1)^{ak+1}}{\sin\left(\frac{\pi a h'}{c}\right)} \cdot e^{-\frac{3\pi i a^2 k_1 h'}{c} - \frac{2\pi i h n}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{-\frac{1}{2}} \cdot e^{\frac{2\pi z}{k}\left(n - \frac{1}{24}\right) + \frac{\pi}{12kz}}\, d\Phi + O\left(n^{\epsilon}\right).$$

In a similar (but more complicated) manner we prove that $\sum_2$ equals

$$2 \sin\left(\frac{\pi a}{c}\right) \sum_{\substack{k,r \\ c\nmid k \\ \delta_{c,k,r}>0}} (-1)^{ak+l} \sum_{h} \omega_{h,k} e^{\frac{2\pi i}{k}\left(-nh + m_{a,c,k,r} h'\right)} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{-\frac{1}{2}} e^{\frac{2\pi z}{k}\left(n - \frac{1}{24}\right) + \frac{2\pi}{kz}\delta_{c,k,r}} d\Phi + O\left(n^{\epsilon}\right).$$

Using Lemma 6.4 we estimate $\sum_3$ in a similar way. In $\sum_1$ and $\sum_2$ we next write

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} = \int_{-\frac{1}{kN}}^{\frac{1}{kN}} - \int_{-\frac{1}{kN}}^{-\frac{1}{k(k+k_1)}} - \int_{\frac{1}{k(k+k_2)}}^{\frac{1}{kN}}$$

and denote the associated sums by S_{11}, S_{12}, and S_{13}, respectively. The sums S_{12} and S_{13} contribute to the error terms which can be bounded as before. To finish the proof, we have to estimate integrals of the shape

$$I_{k,r} := \int_{-\frac{1}{kN}}^{\frac{1}{kN}} z^{-\frac{1}{2}} \cdot e^{\frac{2\pi}{k}\left(z\left(n - \frac{1}{24}\right) + \frac{r}{z}\right)} d\Phi.$$

One can show that

$$I_{k,r} = \frac{1}{ki} \int_{\Gamma} z^{-\frac{1}{2}} \cdot e^{\frac{2\pi}{k}\left(z\left(n - \frac{1}{24}\right) + \frac{r}{z}\right)}\, dz + O\left(n^{-\frac{3}{4}}\right),$$

where Γ denotes the circle through $\frac{k}{n} \pm \frac{i}{N}$ and tangent to the imaginary axis at 0. Making the substitution $t = \frac{2\pi r}{kz}$ and using the Hankel formula, we obtain

$$I_{k,r} = \frac{4\sqrt{3}}{\sqrt{k\left(24n-1\right)}} \sinh\left(\sqrt{\frac{2r(24n-1)}{3}}\, \frac{\pi}{k}\right) + O\left(n^{-\frac{3}{4}}\right)$$

from which we can easily conclude the theorem. $\square$

7. Correspondences for weight $\frac{3}{2}$ weak Maass forms

In [13] we classify those weak Maass forms whose holomorphic part arise from basic hypergeometric series, and we obtain a one-to-one correspondence

$$\{\Theta(\chi;z)\} \quad \longleftrightarrow \quad \{M_\chi(z)\}$$

between holomorphic weight $\frac{1}{2}$ theta functions $\Theta(\chi;z)$ and certain weight $\frac{3}{2}$ weak Maass forms $M_\chi(z)$. To state our result, define the basic hypergeometric-type series

$$F(\alpha,\beta,\gamma,\delta,\epsilon,\zeta) := \sum_{n=0}^{\infty} \frac{(\alpha;\zeta)_n \delta^{n^2} \epsilon^n}{(\beta;\zeta)_n(\gamma;\zeta)_n}$$

and by differentiation of a special case of the Rogers and Fine identity [25]

$$F_{RF}(\alpha,\beta) := \frac{\alpha}{2} \cdot \frac{\partial}{\partial \alpha}\left(\frac{1}{1+\alpha} F(\alpha,-\alpha\beta,0,1,\alpha,\beta)\right) = \sum_{n=1}^{\infty} (-1)^n n \alpha^{2n} \beta^{n^2}.$$

Moreover let $\Theta_0(z) := \sum_{n\in\mathbb{Z}} q^{n^2}$ be the classical Jacobi theta function. For a non-trivial even Dirichlet character χ with conductor b define

$$\Theta(\chi;z) := \sum_{n\in\mathbb{Z}} \chi(n) q^{n^2},$$

which is a weight $\frac{1}{2}$ modular form on $\Gamma_0(4b^2)$. Moreover let the non-holomorphic theta integral $N_\chi(z)$ be given by

$$N_\chi(z) := -\frac{ib^2}{\pi} \int_{-\bar{z}}^{i\infty} \frac{\Theta(\chi;\tau)}{(-i(\tau+z))^{\frac{3}{2}}}\, d\tau.$$

Lastly, define the function Q_χ by

$$Q_\chi(z) := \frac{4b\sqrt{2}}{\Theta_0(b^2 z)} \sum_{a \pmod{b}} \chi(a) \sum_{j \pmod{b}} \zeta_b^{ja} L\left(q^{2a}\zeta_b^j, 2b; q\right), \tag{7.1}$$

where

$$L(w,d;q) := \sum_{n\in\mathbb{Z}} n \frac{q^{n^2} w^n}{1-q^{dn}} = \sum_{k\geq 0}\left(F_{RF}\left(w^{\frac{1}{2}} q^{\frac{kd}{2}}, -q\right) + F_{RF}\left(w^{-\frac{1}{2}} q^{\frac{(k+1)d}{2}}, -q\right)\right). \tag{7.2}$$

In [13], we show:

Theorem 7.1. *If χ is a non-trivial even Dirichlet character with conductor b, then*

$$M_\chi(z) := Q_\chi(z) - N_\chi(z)$$

is a weight $\frac{3}{2}$ weak Maass form on $\Gamma_0(4b^2)$ with Nebentypus χ.

Corollary 7.2. *We have that*

$$y^{3/2} \cdot \overline{\tfrac{\partial}{\partial \bar{z}} M_\chi(z)} = -\frac{ib^2}{2\sqrt{2}\pi} \cdot \Theta(\chi;z).$$

Since the Serre-Stark Basis Theorem asserts that the spaces of holomorphic weight $\frac{1}{2}$ modular forms have explicit bases of theta functions, Corollary 7.2 gives a bijection between weight $\frac{3}{2}$ weak Maass forms and weight $\frac{1}{2}$ modular forms.

Sketch of proof of Theorem 7.1. We only give a sketch of proof here; details can be found in [13]. For $0 < a < b$, we let

$$\psi_{a,b}(z) := 2\sum_{m\geq 1}\sum_{\substack{-bm\leq d<0\\ d\equiv \pm a \pmod b}} (d+bm)\, e^{2\pi i\left(b^2m^2-d^2\right)z}.$$

Note for fixed b, the functions $\psi_{a,b}$ are determined by representatives $a \in \mathbb{Z}/b\mathbb{Z}$. These q-series will be associated to the period integral

$$\mathcal{N}(a,b;z) := -\frac{ib^2}{\pi}\int_{-\overline{z}}^{i\infty} \frac{\Theta_{a,b}(\tau)}{\left(-i(z+\tau)\right)^{\frac{3}{2}}}\, d\tau,$$

where

$$\Theta_{a,b}(z) := \sum_{\substack{n\in\mathbb{Z}\\ n\equiv a \pmod b}} e^{2\pi i n^2 z}.$$

More precisely, we define the functions $\mathcal{M}(a,b;z)$ by

$$\mathcal{M}(a,b;z) := 2^{\frac{3}{2}} b^2 \frac{\psi_{a,b}(z)}{\Theta_0(b^2z)} - \mathcal{N}(a,b;z).$$

We first show the following theorem.

Theorem 7.3. *If $0 < a < b$ are coprime positive integers, where $b \geq 4$ is even, then $\mathcal{M}(a,b;z)$ is a weight $\frac{3}{2}$ weak Maass form on $\Gamma(4b^2)$.*

In order to prove Theorem 7.3, we define functions that are "dual" to $\psi_{a,b}$ under the involution $z \mapsto -\frac{1}{z}$:

$$\phi_{a,b}(z) := 2\sum_{n=1}^{\infty} n\cdot e^{\pi i n^2 z}\, \frac{\left(1-e^{4\pi i n z}\right)}{\left(1-2\cos\left(\frac{2\pi a}{b}\right)e^{2\pi i n z}+e^{4\pi i n z}\right)}.$$

As with $\psi_{a,b}$, the function $\phi_{a,b}$ depends only on the residue class of $a \pmod b$. We show.

Lemma 7.4. *Assuming the hypotheses above, we have*

$$\frac{\phi_{a,b}\left(z\right)}{\Theta_0\left(\frac{z}{2}\right)} = -\frac{1}{b}\cdot(-iz)^{-3/2}\cdot\frac{\psi_{a,b}\left(-\frac{1}{2b^2z}\right)}{\Theta_0\left(-\frac{1}{2z}\right)} + 2\sqrt{-iz}\int_{\mathbb{R}} \frac{u\, e^{\pi i u^2 z}}{1-\zeta_c^a e^{\pi i u^2 z}}\, du.$$

Next define the Mordell-type integral

$$\mathcal{I}_{a,b}(z) := 2\sqrt{-iz}\int_{\mathbb{R}} \frac{u\, e^{\pi i u^2 z}}{1-\zeta_c^a e^{\pi i u^2 z}}\, du.$$

This integral can be rewritten as a theta integral.

Lemma 7.5. *We have*

$$\mathcal{I}_{a,b}(z) = \frac{1}{2\pi}\int_0^{\infty} \frac{\Theta_{a,b}\left(-\frac{1}{2iub^2}\right)\cdot u^{-\frac{1}{2}}}{\left(-i(iu+z)\right)^{\frac{3}{2}}}\, du.$$

To prove Theorem 7.3, we first show that $\mathcal{M}(a,b;z)$ is annihilated by the weight $\frac{3}{2}$ Laplace operator. Now we show that $\mathcal{M}(a,b;z)$ obeys the weight $\frac{3}{2}$ transformation laws with respect to $\Gamma(4b^2)$. To see this, we use the fact that by work of Shimura $\Theta_{a,b}(\tau)$ is a weight $\frac{1}{2}$ modular form on $\Gamma(4b^2)$. By definition, the period integral $\mathcal{N}(a,b;z)$ inherits the transformation properties of $\Theta_{a,b}(\tau)$ with a shift in weight. To finish the proof of Theorem 7.1, observe that

$$\begin{aligned}\psi_{a,b}(z) &= \frac{2}{b} \sum_{j \pmod b} \zeta_b^{ja} L\left(\zeta_b^j q^{2a}, 2b; q\right)\\ &= \frac{2}{b} \sum_{j \pmod b} \zeta_b^{ja} \sum_{k\geq 0} \left(F_{RF}\left(\zeta_b^{\frac{j}{2}} q^{kb+a}, -q\right) + F_{RF}\left(\zeta_b^{-\frac{j}{2}} q^{b-a+kb}, -q\right)\right).\end{aligned}$$

This yields

$$Q_\chi(z) = \frac{2^{\frac{3}{2}} b^2}{\Theta_0(b^2 z)} \sum_{a \pmod b} \chi(a) \psi_{a,b}(z).$$

Now the claim follows using that

$$\mathcal{N}_\chi(z) = \sum_{a=1}^{b-1} \chi(a) \mathcal{N}(a,b;z).$$

□

8. Identities for rank differences

There are a lot of identities that relate modular forms and Eulerian series, e.g. the Roger Ramanujan identities (1.6). Let $f_0(q), f_1(q), \Phi(q)$, and $\Psi(q)$ denote the mock theta functions

$$\begin{aligned} f_0(q) &:= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q)_n}, \qquad & \Phi(q) &:= -1 + \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q;q^5)_{n+1}(q^4;q^5)_n},\\ f_1(q) &:= \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q)_n}, \qquad & \Psi(q) &:= -1 + \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q^2;q^5)_{n+1}(q^3;q^5)_n}.\end{aligned}$$

The mock theta conjectures of Ramanujan are a list of ten identities involving these functions. Andrews and Garvan [5] proved that these conjectures are equivalent to the following pair of identities that essentially express two weight $\frac{1}{2}$ modular forms as linear combinations of Eulerian series:

$$\frac{(q^5;q^5)_\infty (q^5;q^{10})_\infty}{(q;q^5)_\infty (q^4;q^5)_\infty} = f_0(q) + 2\Phi(q^2), \tag{8.1}$$

$$\frac{(q^5;q^5)_\infty (q^5;q^{10})_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty} = f_1(q) + 2q^{-1}\Psi(q^2). \tag{8.2}$$

These were shown by Hickerson in 1988 [28]. He proved later on several more identities of this type [29], and, more recently, Choi and Yesilyurt have obtained even further such identities (for example, see [18, 19, 37]) using methods similar to those of Hickerson. The difficulty in proving the mock theta identities lies in the fact that mock theta functions are not modular forms. Moreover identities for ranks are known, e.g.

$$N(r,t;n) = N(r-t,t;n)$$

or more complicated identities [7]

$$N(1,7;7n+1) = N(2,7;7n+1) = N(3,7;7n+1).$$

Furthermore Atkin and Swinnerton-Dyer [7] proved some very surprising identities such as

$$-\frac{(q;q^7)_\infty^2(q^6;q^7)_\infty^2(q^7;q^7)_\infty^2}{(q;q)_\infty} = \sum_{n=0}^{\infty}\left(N(0,7;7n+6)-N(1,7;7n+6)\right)q^n.$$

This identity expresses a weight $\frac{1}{2}$ modular form as a linear combination of Eulerian series. From the new perspective described in Section 3 that the rank generating functions are the holomorphic parts of weak Maass forms, the mock theta conjectures arise naturally in the theory of Maass forms and arise from linear relations between the non-holomorphic parts of independent Maass forms [16].

Theorem 8.1. *Suppose that $t \geq 5$ is prime, $0 \leq r_1, r_2 < t$ and $0 \leq d < t$. Then the following are true:*

(1) *If $\left(\frac{1-24d}{t}\right) = -1$, then*

$$\sum_{n=0}^{\infty}\left(N(r_1,t;tn+d)-N(r_2,t;tn+d)\right)q^{24(tn+d)-1}$$

is a weight $\frac{1}{2}$ weakly holomorphic modular form on $\Gamma_1(576t^{10})$.

(2) *Suppose that $\left(\frac{1-24d}{t}\right) = 1$. If $r_1, r_2 \not\equiv \pm\frac{1}{2}(1+\alpha) \pmod{t}$, for any $0 \leq \alpha < 2t$ satisfying $1-24d \equiv \alpha^2 \pmod{2t}$, then*

$$\sum_{n=0}^{\infty}\left(N(r_1,t;tn+d)-N(r_2,t;tn+d)\right)q^{24(tn+d)-1}$$

is a weight $\frac{1}{2}$ weakly holomorphic modular form on $\Gamma_1\left(576t^6\right)$.

Theorem 8.1 is optimal in a way that for all other pairs r_1 and r_2 (apart from trivial cases) that

$$\sum_{n=0}^{\infty}\left(N(r_1,t;tn+d)-N(r_2,t,tn+d)\right)q^{24(tn+d)-1}$$

is the holomorphic part of a weak Maass form which has a non-vanishing non-holomorphic part.

Theorem 8.2. *Suppose that $t > 1$ is an odd integer. If $0 \leq r_1, r_2 < t$ are integers, and $\mathcal{P} \nmid 6t$ is prime, then*

$$\sum_{\substack{n\geq 1\\ \left(\frac{24l_tn-l_t}{\mathcal{P}}\right)=-\left(\frac{-l_t}{\mathcal{P}}\right)}}\left(N(r_1,t;n)-N(r_2,t;n)\right)q^{l_tn-\frac{l_t}{24}}$$

is a weight $\frac{1}{2}$ weakly holomorphic modular form on $\Gamma_1(6f_t^2l_t\mathcal{P}^4)$.

Since Theorem 8.2 follows easily from Section 3, we only consider Theorem 8.1

Proof of Theorem 8.1. Using the results from Sections 3 and 5, one can reduce the claim to the identity

$$\sum_{j=1}^{t-1}\left(\zeta_t^{-r_1j}-\zeta_t^{-r_2j}\right)\sin\left(\frac{\pi j}{t}\right)\sin\left(\frac{\pi j\alpha}{t}\right) = 0$$

which can be easily verified using that $\sin(x) = \frac{1}{2i}\left(e^{ix}-e^{-ix}\right)$. $\square$

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