UNIMODAL SEQUENCES AND "STRANGE" FUNCTIONS: A FAMILY OF QUANTUM MODULAR FORMS

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Abstract. In this paper, we construct an infinite family of quantum modular forms from combinatorial rank "moment" generating functions for strongly unimodal sequences. The first member of this family is Kontsevich's "strange" function studied by Zagier. These results rely upon the theory of mock Jacobi forms. As a corollary, we exploit the quantum and mock modular properties of these combinatorial functions in order to obtain asymptotic expansions.

1. Introduction and Statement of Results

A sequence of integers $\{a_j\}_{j=1}^s$ is called a *strongly unimodal sequence of size* n if there exists an integer k such that

$$0 < a_1 < a_2 < \cdots < a_k > a_{k+1} > \cdots > a_s > 0 \tag{1.1}$$

and $a_1 + \cdots + a_s = n$. A number of familiar sequences are strongly unimodal, for example, the sequence of binomial coefficients $\{\binom{n}{j-1}\}_{j=1}^{n+1}$ with n even. Attached to strongly unimodal sequences is a notion of rank, analogous to the well known notion of the rank of an integer partition. For more on partition ranks, see for example original works by Atkin and Swinnerton-Dyer [9], Dyson [19], and Ramanujan [27], and the more recent joint work of the first author and Ono [15] related to mock modular forms. The *rank* of a strongly unimodal sequence is equal to $s-2k+1$, the number of terms after the maximal term minus the number of terms that precede it. For example, there are six strongly unimodal sequences of size 5: $\{5\}$, $\{1,4\}$, $\{4,1\}$, $\{1,3,1\}$, $\{2,3\}$, $\{3,2\}$. Their respective ranks are $0,-1,1,0,-1,1$. By letting w (resp. w^{-1}) keep track of the terms after (resp. before) a maximal term, we have that $u(m,n)$, the number of size n and rank m sequences, satisfies

$$U(w;q) := \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} u(m,n)(-w)^m q^n = \sum_{n=0}^{\infty} (wq;q)_n (w^{-1}q;q)_n q^{n+1}, \tag{1.2}$$

where for $n \in \mathbb{N}_0$, $(w;q)_n := \prod_{j=0}^{n-1}(1-wq^j)$.

Recently, Bryson, Ono, Pitman, and the third author [16] studied this function in the special case $w=1$, namely [1]

$$U(1;q) = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} (-1)^m u(m,n) q^n = \sum_{n=1}^{\infty} \left(u_e(n) - u_o(n)\right) q^n,$$

[1]Note. The function $U(w;q)$, given in (1.2), is equal to the function $U(-w;q)$ as defined in [16].

where $u_e(n)$ (resp. $u_o(n)$) denotes the number of unimodal sequences of size n with even (resp. odd) rank. They showed that for every root of unity ζ,

$$U(1;\zeta) = F\left(\zeta^{-1}\right),$$

where Kontsevich's "strange" function is defined by

$$F(q) := \sum_{n=0}^{\infty} (q;q)_n.$$

Prior, Zagier [31] proved that this function satisfies the "identity"

$$F(q) = -\frac{1}{2}\sum_{n=1}^{\infty} n \left(\frac{12}{n}\right) q^{\frac{n^2-1}{24}}, \tag{1.3}$$

where $\left(\frac{\cdot}{\cdot}\right)$ is the Kronecker symbol. Neither side of the identity (1.3) makes sense simultaneously. Indeed, the right hand side of (1.3) converges in the unit disk $|q| < 1$, but nowhere on the unit circle. The identity (1.3) means that at roots of unity ζ, $F(\zeta)$ (which is clearly a finite sum) agrees with the limit as q approaches ζ radially within the unit disk of the function on the right hand side of (1.3). Moreover, Zagier proved that for $x \in \mathbb{Q} \setminus \{0\}$

$$\phi(x) + (-ix)^{-\frac{3}{2}} \phi\left(-\frac{1}{x}\right) = \frac{\sqrt{3i}}{2\pi} \int_0^{i\infty} (w+x)^{-\frac{3}{2}} \eta(w) dw, \tag{1.4}$$

where $\phi(x) := e^{-\frac{\pi i x}{12}} F(e^{-2\pi i x})$ and $\eta(w) := e^{\frac{\pi i w}{12}} \prod_{n=1}^{\infty}(1-e^{2\pi i n w})$ is the Dedekind eta-function. Note that the constant $\sqrt{3i}/2\pi$ in (1.4) is given explicitly in [16]. There, the authors also gave a new proof of (1.4), using the fact that $U(1;q)$ is a (weak) mixed mock modular form for $|q| < 1$. Here, we slightly modify the definition of "mixed mock modular form" given in [18] to mean functions that lie in the tensor product of the general spaces of mock modular forms and weakly holomorphic modular forms (up to possible rational multiples of q powers). In particular, we do not require (as in [18]) these functions to be holomorphic at the cusps. Weak mixed mock modular forms in this sense occur in a variety of areas including combinatorics [3], algebraic geometry [29], Lie theory [23], Joyce invariants [25], and quantum black holes [18, 24].

The similarity between (1.4) and the usual modular transformation formula of a modular form in part motivated Zagier [32] to introduce the notion of a quantum modular form. A *weight* $k \in \frac{1}{2}\mathbb{Z}$ *quantum modular form* is a complex-valued function f on $\mathbb{Q}$, such that for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, the function $h_\gamma : \mathbb{Q} \setminus \gamma^{-1}(\infty) \to \mathbb{C}$ defined by

$$h_\gamma(x) := f(x) - \varepsilon(\gamma)(cx+d)^{-k} f\left(\frac{ax+b}{cx+d}\right) \tag{1.5}$$

satisfies a "suitable" property of continuity or analyticity. The $\varepsilon(\gamma)$ in (1.5) are suitable complex numbers, such as those in the theory of half-integral weight modular forms when $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$.

This paper gives an infinite family of quantum modular forms from the "moments" of the unimodal rank statistic. In general, such moment functions are of both number theoretic and combinatorial interest. For example, in their celebrated work [8], Atkin and Garvan discovered a partial differential equation relating the bivariate generating functions for the partition statistics rank and crank, leading to exact linear relations between rank and crank moments. In [4], Andrews provided a beautiful combinatorial interpretation of partition rank moments in terms of "k-marked Durfee symbols". Andrews also discovered a relationship between partition rank moments and the "smallest parts" partition statistic in [5], which has led to further work by Garvan [22], for example. In addition to intrinsic combinatorial interest, moment functions have been shown to satisfy modular properties. For example, works including [2, 12, 13] exhibit relationships to weak Maass forms and mock theta functions.

To state our results, we define for $r \in \mathbb{N}_0$ the following "weighted" moment functions, where throughout $q := e^{2\pi i\tau}$

$$\phi_r(\tau) := (\pi i)^{2r+1} \sum_{n=1}^{\infty} \sum_{m\in\mathbb{Z}} (-1)^m u(m,n) Q_r\left(m^2, n - \tfrac{1}{24}\right) q^{n-\frac{1}{24}}, \tag{1.6}$$

and $Q_r(X,Y) \in \mathbb{Q}[X,Y]$ is the polynomial

$$Q_r(X,Y) := \sum_{\substack{0\le\mu\le r\\ 0\le\ell\le r-\mu}} c_r(\mu,\ell) X^\ell Y^\mu \in \mathbb{Q}[X,Y], \tag{1.7}$$

with rational coefficients $c_r(\mu,\ell)$ defined in (1.9). For example, the first few polynomials (normalized, with $Y \to Y - \frac{1}{24}$) are given by

$$\begin{aligned}
Q_0\left(X, Y - \frac{1}{24}\right) &= -2,\\
Q_1\left(X, Y - \frac{1}{24}\right) &= -4(X+2Y),\\
Q_2\left(X, Y - \frac{1}{24}\right) &= -\frac{4}{105}\Big(10X + 35X^2 + 6Y + 180XY + 108Y^2\Big),\\
Q_3\left(X, Y - \frac{1}{24}\right) &= -\frac{4}{3465}\Big(7X + 140X^2 + 154X^3 + 2Y + 420XY\\
&\qquad + 1260X^2Y + 120Y^2 + 2520XY^2 + 720Y^3\Big).
\end{aligned}$$

Note that in particular the first member of the family $\phi_r(\tau)$ is (up to a constant) the "strange" function studied by Zagier and Kontsevich discussed above. That is, $\phi_0(\tau) = -2\pi i q^{-\frac{1}{24}} U(1;q) = -2\pi i \phi(\tau)$. It is not difficult to see that the functions $\phi_r(\tau)$ may also be written in terms of the "twisted" unimodal moment functions u_r, defined for integers $r \ge 0$ by

$$u_r(q) := \sum_{n=1}^{\infty} \sum_{m\in\mathbb{Z}} (-1)^m u(m,n) m^r q^n.$$

The moments $\sum_m u(m,n)m^r$ of the unimodal rank statistic are analogous with the rank and crank partition moments, functions which have drawn wide combinatorial interest since Atkin and Garvan famously introduced them [8]. There is a vast literature on such objects, including asymptotic questions and congruence properties. While the unimodal rank moments are exponentially large for even r [14], it is surprising that the twisted moments $\sum_m (-1)^m u(m,n)m^r$, as a consequence of our results, are only polynomially large in n. We have chosen to handle the more complicated expressions $\sum_m (-1)^m u(m,n) Q_r(m^2, n - 1/24)$ because the generating functions for these numbers have a fixed weight as modular objects as seen in Theorem 1.1, while the generating function for the twisted moments will have a mixed weight. To relate these generating functions $\phi_r(\tau)$ to the twisted unimodal moments $u_r(\tau)$, by symmetry, we note that $u_{2r+1}(q)=0$ for integers $r\geq 0$. In particular, using (1.6), we find that

$$\phi_r(\tau)=(\pi i)^{2r+1}\sum_{\substack{0\leq\mu\leq r\\ 0\leq\ell\leq r-\mu}}\frac{c_r\left(\mu,\ell\right)}{(2\pi i)^{\mu}}\cdot\frac{\partial^{\mu}}{\partial\tau^{\mu}}\left(u_{2\ell}(q)q^{-\frac{1}{24}}\right), \tag{1.8}$$

where we define

$$c_r(\mu,\ell):=\frac{-2^{2\ell+1}6^{\mu}\Gamma\left(\frac{1}{2}+2r-\mu\right)}{\Gamma\left(\frac{1}{2}+2r\right)\mu!(2\ell)!(2r-2\mu-2\ell+1)!}\in\mathbb{Q}. \tag{1.9}$$

The coefficients $c_r(\mu,\ell)$ are indeed in $\mathbb{Q}$, as it is well known for integers $k\in\mathbb{N}_0$, that $\Gamma\left(\frac{1}{2}+k\right)\in\sqrt{\pi}\cdot\mathbb{Q}$. The twisted moment functions also naturally extend the unimodal function $U(1;q)$ discussed above; namely, $u_0(q)=U(1;q)=-q^{\frac{1}{24}}(2\pi i)^{-1}\phi_0(\tau)$.

To state our first result, we define another polynomial

$$P_r\left(X,Y\right):=\sum_{\substack{0\leq N\leq r\\ 0\leq M\leq 3r}}b_r\left(N,M\right)X^{2N+1}Y^{M}, \tag{1.10}$$

where the coefficients $b_r(N,M)$ are given explicitly in (3.14). Our first theorem establishes that the unimodal moment functions ϕ_r are quantum modular forms on $\mathbb{Q}\backslash\{0\}$, and that their transformation law also extends to $\mathbb{H}$. The function $\mathcal{H}_r$ below is defined in (3.15).

Theorem 1.1. *Let $r\in\mathbb{N}_0$. If $\tau\in\mathbb{H}\cup\mathbb{Q}\backslash\{0\}$, we have that*

$$\phi_r(\tau)-(-i\tau)^{-\frac{3}{2}-2r}\phi_r\left(-\frac{1}{\tau}\right)=\int_{\mathbb{R}}P_r\left(w,(-i\tau)^{-1}\right)e^{\frac{\pi i\tau w^2}{3}}\frac{\sinh\left(\frac{2\pi w}{3}\right)}{\cosh(\pi w)}dw+\mathcal{H}_r(\tau), \tag{1.11}$$

where $\mathcal{H}_r(\tau)=0$ for $\tau\in\mathbb{Q}\setminus\{0\}$. In particular, the functions ϕ_r are quantum modular forms.

Remarks.
1) The transformation law given in (1.11) in the case $\tau\in\mathbb{H}$ essentially establishes the mock modular properties of the unimodal rank moment functions $\phi_r(\tau)$.

2) In the course of proving (1.11) in the case $\tau\in\mathbb{Q}\setminus\{0\}$, we show that for each integer $r\geq 0$, the function ϕ_r is defined for $\tau\in\mathbb{Q}$. Moreover, in Theorem 5.1 of §5, we pay special attention to the case $r=1$, and establish an explicit finite value for $\phi_1\left(\frac{h}{k}\right)$ $(h,k\in\mathbb{Z})$ as the value of a polynomial in the root of unity $e^{2\pi ih/k}$.

3) Our functions naturally arise from mock Jacobi forms. It would be interesting to investigate whether a theory of quantum Jacobi forms could be developed that contains functions arising in this paper as special cases.

Our next theorem exploits the automorphic properties given in Theorem 1.1, and establishes the asymptotic behavior of the moment functions u_r. While such properties are of independent interest, we also point out that these functions are related to the quantum moment functions ϕ_r by (1.8). To describe their asymptotic behavior, we use the Bernoulli polynomials $B_k(x)$ and Euler polynomials $E_k(x)$, defined by the generating functions

$$\frac{ze^{xz}}{e^z-1}=\sum_{k=0}^{\infty}B_k(x)\frac{z^k}{k!}, \tag{1.12}$$

$$\frac{2e^{xz}}{e^z+1}=\sum_{k=0}^{\infty}E_k(x)\frac{z^k}{k!}, \tag{1.13}$$

respectively.

Theorem 1.2. *For non-negative integers r, as $t\to 0^+$, we have that*

$$e^{\frac{\pi t}{12}}u_{2r}\left(e^{-2\pi t}\right)=\frac{3^{2r+1}}{(2r+1)}\sum_{k=0}^{\infty}\frac{(3\pi t)^k}{k!}\sum_{0\le n\le r}\binom{2r+1}{2n}3^{-2n}B_{2n}\left(\frac{1}{2}\right)E_{2r+1+2k-2n}\left(\frac{5}{6}\right),$$

In particular, we have that

$$e^{\frac{\pi t}{12}}u_{2r}\left(e^{-2\pi t}\right)\sim\frac{2\cdot 6^{2r}}{2r+1}\left(B_{2r+1}\left(\frac{2}{3}\right)+B_{2r+1}\left(\frac{5}{6}\right)\right).$$

The paper is organized as follows. In Section 2 we provide relevant background information on modular forms, Jacobi forms, and mock Jacobi forms, as well as Bernoulli and Euler polynomials. In Section 3 we prove Theorem 1.1, and in Section 4 we establish Theorem 1.2. In Section 5 we pay special consideration to the moment function ϕ_1.

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2. Preliminaries

Here, we provide preliminary information on automorphic forms in §2.1, and Bernoulli and Euler polynomials in §2.2.

2.1. Automorphic forms.

In this section, we recall some fundamental properties of certain modular and (mock) Jacobi forms. We start with the well known transformation law for the Dedekind η-function.

Lemma 2.1. *For $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$, we have that*

$$\eta\left(\gamma\tau\right)=\chi\left(\gamma\right)\left(c\tau+d\right)^{\frac{1}{2}}\eta(\tau), \tag{2.1}$$

where $\chi\left(\gamma\right)$ is a 24th root of unity, which can be given explicitly in terms of Dedekind sums [26]*. In particular, we have that*

$$\eta\left(-\frac{1}{\tau}\right)=\sqrt{-i\tau}\,\eta(\tau).$$

Here and throughout the square root is defined by the principal branch of the logarithm. Moreover, we require the usual Jacobi theta function, defined for $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$, by

$$\vartheta(z;\tau) := \sum_{\nu \in \frac{1}{2}+\mathbb{Z}} e^{\pi i \nu^2 \tau + 2\pi i \nu\left(z+\frac{1}{2}\right)}. \tag{2.2}$$

This function is well known to satisfy the following transformation law [26, (80.31) and (80.8)].

Lemma 2.2. *For $\lambda, \mu \in \mathbb{Z}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we have that*

$$\begin{aligned} \vartheta(z+\lambda\tau+\mu;\tau) &= (-1)^{\lambda+\mu} q^{-\frac{\lambda^2}{2}} e^{-2\pi i \lambda z} \vartheta(z;\tau), \\ \vartheta\left(\frac{z}{c\tau+d};\gamma\tau\right) &= \chi^3(\gamma)(c\tau+d)^{\frac{1}{2}} e^{\frac{\pi i c z^2}{c\tau+d}} \vartheta(z;\tau). \end{aligned}$$

In particular, we have that

$$\vartheta\left(\frac{z}{\tau}; -\frac{1}{\tau}\right) = -i\sqrt{-i\tau}\, e^{\frac{\pi i z^2}{\tau}} \vartheta(z;\tau).$$

The Jacobi theta function also satisfies the well known triple product identity ($w = e^{2\pi i z}$)

$$\vartheta(z;\tau) = -iq^{\frac{1}{8}} w^{-\frac{1}{2}} \prod_{n=1}^{\infty} (1-q^n)\left(1-wq^{n-1}\right)\left(1-w^{-1}q^n\right).$$

Additionally, we require the following classical Taylor expansion (see for example [30]):

$$\vartheta(z;\tau) = -2\pi z \cdot \eta^3(\tau) \exp\left(-2\sum_{k=1}^{\infty} G_{2k}(\tau) \frac{(2\pi i z)^{2k}}{(2k)!}\right). \tag{2.3}$$

Here for even integers $k \geq 2$, the Eisenstein series are defined by

$$G_k(\tau) := -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where $\sigma_\ell(n) := \sum_{d|n} d^\ell$ and B_k denotes the kth Bernoulli number.

We also make use of Zwegers's functions $A_\ell(z_1, z_2; \tau)$ [34] (see also [7] and [10]), defined for $\ell \in \mathbb{N}$, $\tau \in \mathbb{H}$, $z_2 \in \mathbb{C}$, and $z_1 \in \mathbb{C} \setminus (\mathbb{Z}\tau + \mathbb{Z})$, by

$$A_\ell(z_1, z_2; \tau) := e^{\ell \pi i z_1} \sum_{n \in \mathbb{Z}} \frac{(-1)^{\ell n} q^{\frac{\ell n(n+1)}{2}} e^{2\pi i n z_2}}{1 - q^n e^{2\pi i z_1}}. \tag{2.4}$$

These functions may be "completed" into non-holomorphic Jacobi forms by setting

$$\widehat{A}_\ell(z_1, z_2; \tau) := A_\ell(z_1, z_2; \tau) + R_\ell(z_1, z_2; \tau).$$

The non-holomorphic completions of these higher level Appell functions are defined by

$$R_\ell(z_1, z_2; \tau) := \frac{i}{2} \sum_{k=0}^{\ell-1} e(kz_1) \vartheta\left(z_2 + k\tau + \frac{\ell-1}{2}; \ell\tau\right) R\left(\ell z_1 - z_2 - k\tau - \frac{\ell-1}{2}; \ell\tau\right),$$

where $e(x) := e^{2\pi i x}$ and where ($\tau = u + iv$)

$$R(z;\tau) := \sum_{n \in \frac{1}{2}+\mathbb{Z}} \left(\mathrm{sgn}(n) - E\left(\left(n + \frac{\mathrm{Im}(z)}{v}\right)\sqrt{2v}\right)\right)(-1)^{n-\frac{1}{2}} q^{-\frac{n^2}{2}} e^{-2\pi i n z},$$

with $E(z) := 2\int_0^z e^{-\pi t^2}dt$. Proposition 2.3 below shows that the so-called "error to modularity" of the function $R(z;\tau)$ is the Mordell integral, defined for $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$ by

$$h(z;\tau) := \int_{\mathbb{R}} \frac{e^{\pi i\tau w^2 - 2\pi zw}}{\cosh(\pi w)}dw. \tag{2.5}$$

Proposition 2.3 (Zwegers, [33])**.** *For $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$, we have that*

$$R(z+1;\tau) = -R(z;\tau),$$
$$R\left(\frac{z}{\tau};-\frac{1}{\tau}\right) = \sqrt{-i\tau}e^{-\frac{\pi i z^2}{\tau}}\left(-R(z;\tau)+h(z;\tau)\right).$$

Moreover, the completed higher level Appell functions $A_\ell(z_1,z_2;\tau)$ transform as follows.

Proposition 2.4 (Zwegers, [34])**.** *For $n_1,n_2,m_1,m_2 \in \mathbb{Z}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we have*

$$\widehat{A}_\ell(z_1+n_1\tau+m_1, z_2+n_2\tau+m_2;\tau) = (-1)^{\ell(n_1+m_1)}e(z_1(\ell n_1 - n_2) - n_1 z_2)q^{\frac{\ell n_1^2}{2}-n_1n_2}\widehat{A}_\ell(z_1,z_2;\tau),$$
$$\widehat{A}_\ell\left(\frac{z_1}{c\tau+d},\frac{z_2}{c\tau+d};\gamma\tau\right) = (c\tau+d)e\left(\frac{c(-\ell z_1^2+2z_1z_2)}{2(c\tau+d)}\right)\widehat{A}_\ell(z_1,z_2;\tau).$$

We further require "dissection properties" of the functions ϑ and R (see [11, 28, 34]).

Lemma 2.5. *With notation as above, we have for $n \in \mathbb{N}$*

$$\vartheta\left(z;\frac{\tau}{n}\right) = \sum_{\ell=0}^{n-1} q^{\frac{\left(\ell-\frac{n-1}{2}\right)^2}{2n}} e^{2\pi i\left(\ell-\frac{n-1}{2}\right)\left(z+\frac12\right)}\vartheta\left(nz+\left(\ell-\frac{n-1}{2}\right)\tau+\frac{n-1}{2};n\tau\right),$$
$$R\left(z;\frac{\tau}{n}\right) = \sum_{\ell=0}^{n-1} q^{-\frac{\left(\ell-\frac{n-1}{2}\right)^2}{2n}} e^{-2\pi i\left(\ell-\frac{n-1}{2}\right)\left(z+\frac12\right)}R\left(nz+\left(\ell-\frac{n-1}{2}\right)\tau+\frac{n-1}{2};n\tau\right).$$

2.2. **Bernoulli and Euler polynomials.** In this section, we recall certain properties of the Bernoulli polynomials $B_k(x)$ and Euler polynomials $E_k(x)$, defined in (1.12) and (1.13), respectively, as well as their special values

$$B_k := B_k(0), \quad E_k := 2^k E_k\left(\frac12\right).$$

One property we make use of is a "dissection" property of the Bernoulli polynomials. Namely, for $m \in 2\mathbb{N}_0+1$, it is well known that (see Chapter 23 of [1])

$$B_k(mx) = m^{k-1}\sum_{a=0}^{m-1} B_k\left(x+\frac{a}{m}\right). \tag{2.6}$$

Another "splitting" property that we use is the following:

$$2^k B_k\left(\frac{x+y}{2}\right) = \sum_{j=0}^{k}\binom{k}{j}B_j(x)E_{k-j}(y), \tag{2.7}$$

which follows easily from the definition of the Euler and Bernoulli polynomials, using the fact that

$$\frac{2z\cdot e^{(x+y)z}}{e^{2z}-1} = \frac{ze^{xz}}{e^z-1}\cdot\frac{2e^{yz}}{e^z+1}.$$

Here and throughout, we let $\zeta_N := e^{2\pi i/N}$ for $N \in \mathbb{N}$. The next lemma expresses derivatives of secant in terms of Euler polynomials.

Lemma 2.6. *With notation as above, we have that for $c \in \mathbb{N}_0$*

$$\sec^{(2c+1)}\left(\frac{\pi}{3}\right) = (-1)^c\sqrt{3}\cdot 6^{2c+1}E_{2c+1}\left(\frac{5}{6}\right).$$

Proof. This follows quickly from Theorem 2 of [17]. Namely, using the facts that $E_{2c-1}\left(\frac{1}{6}\right) = -E_{2c-1}\left(\frac{5}{6}\right)$, and $E_{2c-1}\left(\frac{1}{2}\right) = 0$, gives the claim. □

A fourth property that we use expresses the Euler numbers as integrals. Namely, it is known (see (18) on p.42 of [21] for example) that for $k \in \mathbb{N}_0$,

$$\int_{\mathbb{R}} \frac{w^{2k}}{\cosh(\pi w)}dw = (2i)^{-2k}E_{2k}. \tag{2.8}$$

Note that $E_{2k-1} = 0$ for $k \in \mathbb{N}$.

3. Proof of Theorem 1.1

Here, we ultimately conclude Theorem 1.1 from Proposition 3.6, Proposition 3.7, and Proposition 3.8 below. In §3.1, we establish properties of mock Jacobi forms related to the unimodal rank generating function, and in §3.2, we construct mock modular forms from its Taylor coefficients. In §3.3, we establish quantum modularity and prove Theorem 1.1. Until otherwise indicated, throughout this section, we take $\tau \in \mathbb{H}$.

3.1. **Mock Jacobi forms and unimodal ranks.** Here we establish properties of mock Jacobi forms associated to the unimodal rank generating function. We begin by writing $U(w;q)$ in terms of the Appell functions $A_\ell(u,v;\tau)$ defined in (2.4). Throughout, for $w_1, w_2 \in \mathbb{C}$, we let

$$\mathcal{U}(w_1;w_2) := U(e(w_1);e(w_2)).$$

Lemma 3.1. *Let $w = e(z)$. With notation as above, we have that*

$$\mathcal{U}(z;\tau) = \frac{1}{\left(w^{\frac{1}{2}} - w^{-\frac{1}{2}}\right)(q;q)_\infty}\left(A_1(z,-z;\tau) - w^{-1}A_3(z,-\tau;\tau)\right).$$

Proof. Entry 3.4.7 of "Ramanujan's lost notebook" (see p.67 of [6]) gives with $a = -w, b = -w^{-1}$ that $\mathcal{U}(z;\tau)$ equals

$$\frac{-1}{(1-w)\left(1-w^{-1}\right)}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(wq;q)_n\left(w^{-1}q;q\right)_n} + \frac{1}{(1-w^{-1})(q;q)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{\frac{n(n+1)}{2}}w^{-n}}{1-wq^n}. \tag{3.1}$$

We note that second sum on the right hand side of (3.1) is easily seen to be equal to

$$\frac{1}{\left(w^{\frac{1}{2}} - w^{-\frac{1}{2}}\right)(q;q)_\infty}A_1(z,-z;\tau).$$

Using these facts, the result follows after applying the identity [9]

$$\frac{-1}{(1-w^{-1})(1-w)}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(wq;q)_n(w^{-1}q;q)_n} = \frac{-1}{\left(w^{\frac{1}{2}} - w^{-\frac{1}{2}}\right)}\frac{1}{(q;q)_\infty}A_3(z,-\tau;\tau). \qquad \square$$

Next we define a normalization of the function $\mathcal{U}(z;\tau)$

$$Y^+(z;\tau) := -\left(w^{\frac{1}{2}} - w^{-\frac{1}{2}}\right) q^{-\frac{1}{24}} \cdot \mathcal{U}(z;\tau) = \eta^{-1}(\tau)\left(w^{-1}A_3(z,-\tau;\tau) - A_1(z,-z;\tau)\right), \tag{3.2}$$

where the second equality follows from Lemma 3.1. Using Proposition 3.3, we now establish a transformation law for Y^+, which is a key step in showing quantum modularity of the functions ϕ_r. To state this, we define

$$H(z;\tau) := \frac{i}{2}\frac{\vartheta(z;\tau)}{\eta(\tau)} h(2z;\tau) - g(z;\tau),$$

where $h(z;\tau)$ is given in (2.5), and

$$g(z;\tau) := \frac{i}{\sqrt{3}}\int_{\mathbb{R}} e^{\frac{\pi i\tau w^2}{3} - 2\pi wz} \frac{\sinh\left(\frac{2\pi w}{3}\right)}{\cosh(\pi w)} dw.$$

Proposition 3.2. *With notation as above, we have that*

$$-ie^{\frac{3\pi iz^2}{\tau}} Y^+\left(\frac{z}{\tau}; -\frac{1}{\tau}\right)\frac{1}{\sqrt{-i\tau}} - Y^+(z;\tau) = H(z;\tau).$$

To prove Proposition 3.2 we rather work with a second normalization of the function $\mathcal{U}(z;\tau)$, namely

$$X^+(z;\tau) := -e^{-\frac{3\pi z^2}{2v}}\left(w^{\frac{1}{2}} - w^{-\frac{1}{2}}\right)(q;q)_\infty \mathcal{U}(z;\tau) = \left(w^{-1}A_3(z,-\tau;\tau) - A_1(z,-z;\tau)\right)e^{-\frac{3\pi z^2}{2v}}.$$

Moreover we need the completed function

$$\widehat{X}(z;\tau) := \left(w^{-1}\widehat{A}_3(z,-\tau;\tau) - \widehat{A}_1(z,-z;\tau)\right)e^{-\frac{3\pi z^2}{2v}} = \left(\widehat{A}_3(z,0;\tau) - \widehat{A}_1(z,-z;\tau)\right)e^{-\frac{3\pi z^2}{2v}}, \tag{3.3}$$

where the second equality in (3.3) follows from the first transformation in Proposition 2.4. Using Proposition 2.4, it is not difficult to establish the following modularity result for $\widehat{X}(z;\tau)$.

Proposition 3.3. *With notation as above, for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we have that*

$$\widehat{X}\left(\frac{z}{c\tau+d}; \gamma\tau\right) = (c\tau+d)\widehat{X}(z;\tau).$$

From Proposition 3.3, we can establish the following transformation property of $X^+(z;\tau)$.

Proposition 3.4. *With notation as above, we have that*

$$X^+\!\left(\frac{z}{\tau}; -\frac{1}{\tau}\right)\tau^{-1} - X^+(z;\tau) = \left(\frac{i}{2}\vartheta(z;\tau)\, h(2z;\tau) + \frac{i}{2\sqrt{3}}\,\eta(\tau)\sum_{\pm} \pm h\left(z \pm \frac{1}{3}; \frac{\tau}{3}\right)\right)e^{-\frac{3\pi z^2}{2v}}.$$

Proof. Using Proposition 3.3 we obtain that

$$\left(X^+\left(\frac{z}{\tau}; -\frac{1}{\tau}\right)\tau^{-1} - X^+(z;\tau)\right)2i = f_1(z;\tau) + f_2(z;\tau),$$

with

$$f_1(z;\tau) := \vartheta\left(-\frac{1}{\tau};-\frac{3}{\tau}\right)e^{-\frac{3\pi z^2\overline{\tau}}{2v\tau}}\tau^{-1}\sum_{\pm}\pm e^{\pm\frac{2\pi iz}{\tau}}R\left(\frac{3z}{\tau}\pm\frac{1}{\tau};-\frac{3}{\tau}\right)$$
$$-\vartheta(\tau;3\tau)e^{-\frac{3\pi z^2}{2v}}\sum_{\pm}\pm e^{\pm 2\pi iz}R(3z\mp\tau;3\tau),$$
$$f_2(z;\tau) := \vartheta\left(\frac{z}{\tau};-\frac{1}{\tau}\right)R\left(\frac{2z}{\tau};-\frac{1}{\tau}\right)e^{-\frac{3\pi z^2\overline{\tau}}{2v\tau}}\tau^{-1}-\vartheta(z;\tau)R(2z;\tau)e^{-\frac{3\pi z^2}{2v}}.$$

We next simplify f_1 and f_2. Firstly, using Lemma 2.2 and Proposition 2.3, we obtain that

$$f_2(z;\tau) = -\vartheta(z;\tau)h(2z;\tau)e^{-\frac{3\pi z^2}{2v}}. \tag{3.4}$$

Next Lemma 2.2 and Proposition 2.3 yield that

$$\vartheta\left(-\frac{1}{\tau};-\frac{3}{\tau}\right)e^{-\frac{3\pi z^2\overline{\tau}}{2v\tau}}\tau^{-1}\sum_{\pm}\pm e^{\pm\frac{2\pi iz}{\tau}}R\left(\frac{3z}{\tau}\pm\frac{1}{\tau};-\frac{3}{\tau}\right)$$
$$= -\frac{1}{3}e^{-\frac{3\pi z^2}{2v}}\vartheta\left(-\frac{1}{3};\frac{\tau}{3}\right)\sum_{\pm}\pm\left(-R\left(z\pm\frac{1}{3};\frac{\tau}{3}\right)+h\left(z\pm\frac{1}{3};\frac{\tau}{3}\right)\right).$$

Now Lemma 2.5, the fact that $\vartheta(0;\tau)=0$, and Proposition 2.3, give that

$$\vartheta\left(-\frac{1}{3};\frac{\tau}{3}\right) = 2i\sin\left(\frac{\pi}{3}\right)q^{\frac{1}{6}}\vartheta(\tau;3\tau),$$
$$R\left(z\pm\frac{1}{3};\frac{\tau}{3}\right) = -q^{-\frac{1}{6}}e^{2\pi i\left(z\pm\frac{1}{3}\right)}R(3z-\tau;3\tau)+R(3z;3\tau)-q^{-\frac{1}{6}}e^{-2\pi i\left(z\pm\frac{1}{3}\right)}R(3z+\tau;3\tau).$$

Thus

$$\sum_{\pm}\mp R\left(z\pm\frac{1}{3};\frac{\tau}{3}\right) = 2i\sin\left(\frac{2\pi}{3}\right)q^{-\frac{1}{6}}\sum_{\pm}\pm e^{\pm 2\pi iz}R(3z\mp\tau;3\tau),$$

and hence

$$f_1(z;\tau) = -\frac{i}{\sqrt{3}}q^{\frac{1}{6}}\vartheta(\tau;3\tau)\sum_{\pm}\pm h\left(z\pm\frac{1}{3};\frac{\tau}{3}\right)e^{-\frac{3\pi z^2}{2v}}. \tag{3.5}$$

Combining (3.4), (3.5), and the fact that $\vartheta(\tau;3\tau) = -iq^{-\frac{1}{6}}\eta(\tau)$ gives the claim. □

Proof of Proposition 3.2. First note that

$$\sum_{\pm}\pm h\left(z\pm\frac{1}{3};\frac{\tau}{3}\right) = 2i\sqrt{3}\cdot g(z;\tau).$$

The result now follows immediately from Proposition 3.4 and Lemma 2.1, using the fact that

$$Y^+(z;\tau) = \frac{e^{\frac{3\pi z^2}{2v}}}{\eta(\tau)}X^+(z;\tau).$$

□

3.2. **Taylor coefficients and unimodal ranks.** Next, using the results from §3.1, we construct mock modular forms from the Taylor coefficients of the unimodal rank generating function. The fuctions $H(z;\tau)$ and $Y^+(z;\tau)$ are holomorphic in z, and it is not difficult to see that they are both odd functions in z, so we may write

$$Y^+(z;\tau) = \sum_{r=0}^{\infty} a_{2r}(\tau) z^{2r+1}, \tag{3.6}$$

$$H(z;\tau) = \sum_{r=0}^{\infty} h_{2r}(\tau) z^{2r+1}. \tag{3.7}$$

The next lemma describes the modularity properties of the Taylor coefficients $a_{2r}(\tau)$ of $Y^+(z;\tau)$.

Lemma 3.5. *With notation as above, we have that*

$$a_{2r}\left(-\frac{1}{\tau}\right)(-i\tau)^{-\frac{3}{2}-2r} = \sum_{0\leq j\leq r} \frac{(3\pi)^{r-j}}{(r-j)!}(-1)^{j+1}(-i\tau)^{j-r}\left(a_{2j}(\tau)+h_{2j}(\tau)\right).$$

Proof. Proposition 3.2 directly yields

$$Y^+\left(\frac{z}{\tau};-\frac{1}{\tau}\right) = ie^{-\frac{3\pi i z^2}{\tau}}\sqrt{-i\tau}\left(Y^+(z;\tau)+H(z;\tau)\right).$$

Inserting the Taylor expansions (3.6) and (3.7), and Taylor expanding the exponential function, gives that

$$\begin{aligned}\sum_{r=0}^{\infty} a_{2r}\left(-\frac{1}{\tau}\right)\left(\frac{z}{\tau}\right)^{2r+1} &= i\sqrt{-i\tau}\sum_{\ell=0}^{\infty}\frac{\left(\frac{-3\pi i z^2}{\tau}\right)^{\ell}}{\ell!}\sum_{j=0}^{\infty}\left(a_{2j}(\tau)+h_{2j}(\tau)\right)z^{2j+1}\\ &= i\sqrt{-i\tau}\sum_{r=0}^{\infty} z^{2r+1}\sum_{0\leq j\leq r}\frac{(3\pi)^{r-j}}{(r-j)!}(-1)^{r+j}(-i\tau)^{j-r}\left(a_{2j}(\tau)+h_{2j}(\tau)\right).\end{aligned}$$

Equating the coefficients of z^{2r+1} gives the claim. $\square$

To prove the transformation law for the functions ϕ_r, we define for $r\in\mathbb{N}_0$

$$b_{2r}(\tau) := \sum_{0\leq\mu\leq r}\frac{(3\pi i)^{\mu}\Gamma\left(\frac{1}{2}+2r-\mu\right)}{\Gamma(\frac{1}{2}+2r)\mu!}a_{2r-2\mu}^{(\mu)}(\tau). \tag{3.8}$$

We will later show that $\phi_r(\tau)=b_{2r}(\tau)$. The functions $b_{2r}(\tau)$ transform as described in the following proposition, a fact which follows as in [20], using Lemma 3.5.

Proposition 3.6. *With notation as above, for $r\in\mathbb{N}_0$, we have that*

$$\begin{aligned}b_{2r}\left(-\frac{1}{\tau}\right)(-i\tau)^{-\frac{3}{2}-2r}-b_{2r}(\tau) = -(-i\tau)^{-\frac{3}{2}-2r}\sum_{0\leq\mu\leq r}&\frac{(3\pi i)^{\mu}\Gamma\left(\frac{1}{2}+2r-\mu\right)}{\Gamma(\frac{1}{2}+2r)\mu!}\\ &\times\sum_{0\leq j\leq r-\mu}\frac{(3\pi)^{r-\mu-j}(-1)^j}{(r-\mu-j)!}\frac{\partial^{\mu}}{\partial\tau^{\mu}}\left((-i\tau)^{j+r-\mu+\frac{3}{2}}h_{2j}(\tau)\right).\end{aligned}$$

Our next proposition shows that the "errors to modularity" h_{2r} are C^∞, a fact we use in the course of establishing the quantum modularity of the unimodal rank functions ϕ_r in Theorem 1.1. In doing so, we split the Taylor expansion of $H(z;\tau)$ into two pieces

$$H(z;\tau) = H_1(z;\tau) + H_2(z;\tau), \tag{3.9}$$

with

$$\begin{aligned} H_1(z;\tau) &= \sum_{r=0}^{\infty} h_{1,2r}(\tau) z^{2r+1} := \frac{i}{2}\frac{\vartheta(z;\tau)}{\eta(\tau)} h(2z;\tau), \\ H_2(z;\tau) &= \sum_{r=0}^{\infty} h_{2,2r}(\tau) z^{2r+1} := -g(z;\tau). \end{aligned}$$

Proposition 3.7. *The functions h_{2r} are C^∞ on $\mathbb{R}$. To be more precise, $h_{1,2r}(\tau)$ vanishes to infinite order for $\tau \in \mathbb{Q}$, and we extend this function to equal* 0 *on all of $\mathbb{R}$. Moreover, for $\tau \in \mathbb{H} \cup \mathbb{Q}$, the function $h_{2,2r}$ satisfies*

$$h_{2,2r}(\tau) = \frac{i}{\sqrt{3}} \frac{(2\pi)^{2r+1}}{(2r+1)!} \int_{\mathbb{R}} e^{\frac{\pi i \tau w^2}{3}} w^{2r+1} \frac{\sinh\left(\frac{2\pi w}{3}\right)}{\sinh(\pi w)} dw.$$

Proof. Firstly, we have that

$$\begin{aligned} H_1(z;\tau) &= \frac{i}{2\eta(\tau)} \sum_{r=0}^{\infty} \frac{\partial^r}{\partial z^r} \left[\vartheta(z;\tau) h(2z;\tau)\right]_{z=0} \frac{z^r}{r!} \\ &= \frac{i}{2\eta(\tau)} \sum_{r=0}^{\infty} \frac{z^{2r+1}}{(2r+1)!} \sum_{\ell=0}^{r} \binom{2r+1}{2\ell+1} \frac{\partial^{2\ell+1}}{\partial z^{2\ell+1}} \left[\vartheta(z;\tau)\right]_{z=0} \frac{\partial^{2r-2\ell}}{\partial z^{2r-2\ell}} \left[h(2z;\tau)\right]_{z=0}, \end{aligned}$$

so that

$$h_{1,2r}(\tau) = \frac{i}{2\eta(\tau)} \sum_{\ell=0}^{r} \frac{1}{(2\ell+1)!(2r-2\ell)!} \frac{\partial^{2\ell+1}}{\partial z^{2\ell+1}} \left[\vartheta(z;\tau)\right]_{z=0} \frac{\partial^{2r-2\ell}}{\partial z^{2r-2\ell}} \left[h(2z;\tau)\right]_{z=0}.$$

It is not hard to see that $h(2z;\tau)$ is C^∞ as a function of τ near $z=0$. Moreover by (2.3), we see that

$$\frac{i}{2\eta(\tau)} \frac{\partial^{2\ell+1}}{\partial z^{2\ell+1}} \left[\vartheta(z;\tau)\right]_{z=0}$$

gives a linear combination of Eisenstein series multiplied by $\eta^2(\tau)$. It is well known that the Eisenstein series satisfy

$$G_k\left(-\frac{1}{\tau}\right) = \tau^k G_k(\tau) \quad (k>2 \text{ even}), \qquad G_2\left(-\frac{1}{\tau}\right) = \tau^2 G_2(\tau) + \frac{i\tau}{4\pi}.$$

This implies that the function $h_{2r}(\tau)$ and its derivatives vanish exponentially for $\tau \in \mathbb{Q}$. The second claim follows directly by inserting the Taylor expansion of $e^{-2\pi z x}$. $\square$

3.3. **Quantum unimodal ranks.** Building from the results in §3.1 and §3.2, here we prove Theorem 1.1.

Proof of Theorem 1.1. We first relate the Taylor coefficients of $Y^+(z;\tau)$ to the unimodal moments u_{2r}. Using the definition of u_{2r}, it is not difficult to verify that

$$\mathcal{U}\left(z;\tau\right)=\sum_{r=0}^{\infty}u_{2r}(q)\frac{(2\pi iz)^{2r}}{(2r)!}. \tag{3.10}$$

Using the Taylor expansion of $\sin(\pi z)$ we find that

$$Y^+\left(z;\tau\right)=-2iq^{-\frac{1}{24}}\sin(\pi z)\mathcal{U}\left(z;\tau\right)=-(2\pi iz)\sum_{r=0}^{\infty}(2\pi iz)^{2r}\sum_{0\le\ell\le r}\frac{u_{2\ell}(q)q^{-\frac{1}{24}}2^{2\ell-2r}}{(2\ell)!(2r-2\ell+1)!},$$

yielding

$$\frac{a_{2r}(\tau)}{(2\pi i)^{2r+1}}=-\sum_{0\le\ell\le r}\frac{u_{2\ell}(q)q^{-\frac{1}{24}}2^{2\ell-2r}}{(2\ell)!(2r-2\ell+1)!}. \tag{3.11}$$

Using (3.11), the definition of $\phi_r(\tau)$ in (1.6), or its equivalent formulation given in (1.8), as well as the definition of $b_{2r}(\tau)$ in (3.8), it is not difficult to see that for each $r\in\mathbb{N}_0$, $b_{2r}(\tau)=\phi_r(\tau)$. Combining this with the fact that $h_{2j}(\tau)=h_{1,2j}(\tau)+h_{2,2j}(\tau)$, Proposition 3.6 yields

$$\begin{aligned}\phi_r\left(-\frac{1}{\tau}\right)&(-i\tau)^{-\frac{3}{2}-2r}-\phi_r(\tau)\\&=-(-i\tau)^{-\frac{3}{2}-2r}\sum_{\substack{0\le\mu\le r\\0\le j\le r-\mu}}\frac{(3\pi)^{r-j}(-1)^j i^\mu\Gamma\left(\frac12+2r-\mu\right)}{\Gamma\left(\frac12+2r\right)\mu!(r-\mu-j)!}\frac{\partial^\mu}{\partial\tau^\mu}\left((-i\tau)^{j+r-\mu+\frac32}\left(h_{1,2j}(\tau)+h_{2,2j}(\tau)\right)\right).\end{aligned} \tag{3.12}$$

By continuation, (3.12) and what follows hold on $\mathbb{H}\cup\mathbb{Q}\setminus\{0\}$.

We first consider the first summand. We have by Proposition 3.7

$$\begin{aligned}&\frac{\partial^\mu}{\partial\tau^\mu}\left((-i\tau)^{j+r-\mu+\frac32}h_{2,2j}(\tau)\right)\\&\quad=\frac{i}{\sqrt3}\frac{(2\pi)^{2j+1}}{(2j+1)!}\int_{\mathbb{R}}\sum_{\ell=0}^{\mu}\binom{\mu}{\ell}\frac{\partial^\ell}{\partial\tau^\ell}\left((-i\tau)^{j+r-\mu+\frac32}\right)\frac{\partial^{\mu-\ell}}{\partial\tau^{\mu-\ell}}\left(e^{\frac{\pi iw^2\tau}{3}}\right)w^{2j+1}\frac{\sinh\left(\frac{2\pi w}{3}\right)}{\cosh(\pi w)}dw\\&\quad=\sum_{\ell=0}^{\mu}(-1)^\ell i^{\mu+1}\pi^{2j+1+\mu-\ell}2^{2j+1}3^{\ell-\mu-\frac12}\binom{\mu}{\ell}\frac{\Gamma\left(j+r-\mu+\frac52\right)}{(2j+1)!\Gamma\left(j+r-\mu+\frac52-\ell\right)}\\&\qquad\qquad\times(-i\tau)^{j+r+\frac32-\mu-\ell}\int_{\mathbb{R}}w^{2j+2\mu-2\ell+1}e^{\frac{\pi iw^2\tau}{3}}\frac{\sinh\left(\frac{2\pi w}{3}\right)}{\cosh(\pi w)}dw.\end{aligned} \tag{3.13}$$

We now define the numbers

$$b_r(\mu,j,\ell):=\frac{i(-1)^{j+\ell+\mu}2^{2j+1}\pi^{r+j+\mu+1-\ell}3^{r+\ell-\mu-j-\frac12}\Gamma\left(\frac12+2r-\mu\right)\Gamma\left(j+r-\mu+\frac52\right)}{(2j+1)!\ell!(\mu-\ell)!(r-\mu-j)!\Gamma\left(\frac12+2r\right)\Gamma\left(j+r-\mu+\frac52-\ell\right)},$$

and let

$$b_r(N,M) := \sum_{0\leq \mu\leq r} \sum_{\substack{0\leq j\leq r-\mu\\ 0\leq \ell\leq \mu\\ N=j+\mu-\ell\\ M=\mu+\ell+r-j}} b_r(\mu,j,\ell). \tag{3.14}$$

Moreover, we define

(3.15)

$$\mathcal{H}_r(\tau) := (-i\tau)^{-\frac{3}{2}-2r} \sum_{\substack{0\leq\mu\leq r\\ 0\leq j\leq r-\mu}} \frac{(3\pi)^{r-j}(-1)^j i^{\mu}\Gamma\left(\frac{1}{2}+2r-\mu\right)}{\Gamma\left(\frac{1}{2}+2r\right)\mu!(r-\mu-j)!} \frac{\partial^{\mu}}{\partial\tau^{\mu}}\left((-i\tau)^{j+r-\mu+\frac{3}{2}} h_{1,2j}(\tau)\right).$$

Note that $\mathcal{H}_r(\tau)=0$ for $\tau\in\mathbb{Q}\setminus\{0\}$. We have thus shown for $\tau\in\mathbb{H}\cup\mathbb{Q}\setminus\{0\}$,

$$(-i\tau)^{-\frac{3}{2}-2r}\phi_r\left(-\frac{1}{\tau}\right)-\phi_r(\tau) = -\int_{\mathbb{R}} P_r\left(w,(-i\tau)^{-1}\right) e^{\frac{\pi i\tau w^2}{3}} \frac{\sinh\left(\frac{2\pi w}{3}\right)}{\cosh(\pi w)} dw - \mathcal{H}_r(\tau),$$

as claimed in (1.11).

Finally, under translation $\tau\to\tau+1$, it is clear using the definition of $\phi_r(\tau)$ in (1.6) that $\phi_r(\tau+1)=e^{-\frac{\pi i}{12}}\phi_r(\tau)$. With the proof of Proposition 3.8 below, using (1.8), Theorem 1.1 now follows. ☐

We are left to show existence of the moment function and their derivatives.

Proposition 3.8. *For $r,n\in\mathbb{N}_0$, the moment functions*

$$\frac{\partial^n}{\partial\tau^n}\left[q^{-\frac{1}{24}}u_{2r}(q)\right]$$

are defined for every root of unity $q=\zeta$ and lie in $\mathbb{Z}[\zeta]$.

Proof. For ease of notation, we let

$$\begin{aligned} D_\alpha &:= \alpha\frac{\partial}{\partial\alpha},\\ J_m(w;q) &:= (wq;q)_m(w^{-1}q;q)_m. \end{aligned}$$

To finish the proof it is enough to show that for m sufficiently large, and every $n,r\in\mathbb{N}_0$, the function

$$D_q^n\left(D_w^r\left[J_m(w;q)\right]_{w=1}\right) \tag{3.16}$$

vanishes for $q=\zeta$.

It is not difficult to see that for $m\in\mathbb{N}$

$$\frac{D_w\left(J_m(w;q)\right)}{J_m(w;q)} = -\sum_{k=1}^m \frac{wq^k}{1-wq^k} + \sum_{k=1}^m \frac{w^{-1}q^k}{1-w^{-1}q^k} =: R_m(w;q). \tag{3.17}$$

We further relax notation and let $J := J_m(w;q)$, $R := R_m(w;q)$, and $R^{(r)} := D_w^r R$ for $r \in \mathbb{N}_0$. Using (3.17), we find that

$$\begin{aligned}
D_w J &= JR,\\
D_w^2 J &= J\left(R^2 + R^{(1)}\right),\\
D_w^3 J &= J\left(R^3 + 3RR^{(1)} + R^{(2)}\right),\\
D_w^4 J &= J\left(R^4 + 4RR^{(2)} + 3(R^{(1)})^2 + 6R^2R^{(1)} + R^{(3)}\right)\\
&\ \ \vdots
\end{aligned}$$

Note that each $D_w^r J$ can be expressed as J multiplied by a sum over the partitions or r. That is, given a partition $\pi = \ell_1(\pi)\cdot 1 + \ell_2(\pi)\cdot 2 + \cdots + \ell_{r-1}(\pi)\cdot(r-1) + \ell_r(\pi)\cdot r$ of r (where each $\ell_j(\pi) \in \mathbb{N}_0$) we may assign the product

$$\prod_{1\le j\le r} \left(D_w^{j-1} R\right)^{\ell_j(\pi)}.$$

Conversely, every such product appearing as a summand as above for $D_w^r J$ corresponds to a partition of r. In general, we have that

$$D_w^r \left[J_m(w;q)\right]_{w=1} = (q;q)_m^2 \sum_{\pi\vdash r} c(\pi) \prod_{1\le j\le r} \left(D_w^{j-1}\left[R_m(w;q)\right]_{w=1}\right)^{\ell_j(\pi)},$$

where we sum over all partitions π of r. The exponents $\ell_j(\pi)$ correspond to the number of parts of the partition π of r, and the constants $c(\pi) = c_r(\pi)$ also depend on the partition π of r. Now using the definition of $R_m(w;q)$ in (3.17), we may write

$$\sum_{\pi\vdash r} c(\pi) \prod_{1\le j\le r} \left(D_w^{j-1}\left[R_m(w;q)\right]_{w=1}\right)^{\ell_j(\pi)} = \sum_{\vec{k}=(k_1,\ldots,k_c)} \frac{P_{\vec{k},r}(q)}{\prod_{j=1}^{c}\left(1-q^{k_j}\right)^r} =: R_{m,r}(q), \tag{3.18}$$

where $c = c_r \in \mathbb{N}$ depends only on r, and $P_{\vec{k},r} \in \mathbb{Z}[q]$. Next we apply the operator D_q^n to $(q;q)_m^2$ multiplied by $R_{m,r}(q)$ in (3.18) above. Using the product rule, we have that (3.16) equals

$$\sum_{0\le j\le n} \binom{n}{j} D_q^j\left((q;q)_m^2\right) D_q^{n-j}\left(R_{m,r}(q)\right).$$

It is not difficult to see that

$$\frac{D_q\left((q;q)_m^2\right)}{(q;q)_m^2} = -2\sum_{k=1}^{m} \frac{kq^k}{1-q^k} =: T_m(q),$$

and for $l \in \mathbb{N}$, that

$$D_q^{\ell-1}\left(T_m(q)\right) = \sum_{k=1}^{m} \frac{Q_{k,l}(q)}{\left(1-q^k\right)^{\ell}},$$

with $Q_{k,l}(q) \in \mathbb{Z}[q]$. Therefore, we may conclude that (3.16) has the shape

$$(q;q)_m^2 \sum_{\vec{k}=(k_1,\ldots,k_d)} \frac{P_{\vec{k},r,n}(q)}{\prod_{j=1}^{d}\left(1-q^{k_j}\right)^{r+n}},$$

where $d = d_{r,n} \in \mathbb{N}$ depends only on r and n, and $P_{\vec{k},r,n} \in \mathbb{Z}[q]$. Now if $\zeta = \zeta_m$ then $(q;q)_M^2$ ($M \in \mathbb{N}$) vanishes at $q = \zeta$ of order $\geq 2\lfloor \frac{m}{M} \rfloor$. On the other hand each term

$$\frac{P_{\vec{k},r,n}(q)}{\prod_{j=1}^{d} \left(1 - q^{k_j}\right)^{r+n}}$$

vanishes at $q = \zeta$ of order at most $d(r+n)$, which is a constant independent of m. Thus, the claim follows.

$\square$

4. Proof of Theorem 1.2

To prove Theorem 1.2, we recall (3.2). It is not difficult to see from Proposition 3.2 that

$$Y^+(z;it) = -H(z;it) + \sum_{r\geq 0} \beta_r(t) z^r$$

with

$$\beta_r(t) \ll_r e^{-\frac{N}{t}}$$

for some $N > 0$. To find the asymptotic expansion of $H(z;it)$, we split as in (3.9) and bound using (2.3)

$$h_{1,2r}(it) \ll e^{-\frac{M}{t}}$$

for some $M > 0$. Thus we are left to determine the asymptotic expansion of $H_2(z;it)$. For this, we write

$$\begin{aligned} H_2(z;it) &= -\frac{i}{\sqrt{3}} \int_{\mathbb{R}} e^{-\frac{\pi t w^2}{3} - 2\pi w z} \frac{\sinh\left(\frac{2\pi w}{3}\right)}{\cosh\left(\pi w\right)} dw \\ &= -\frac{i}{\sqrt{3}} \sum_{r=0}^{\infty} \frac{\left(-2\pi z\right)^{2r+1}}{(2r+1)!} \sum_{k=0}^{\infty} \frac{\left(-\frac{\pi t}{3}\right)^k}{k!} \int_{\mathbb{R}} \frac{w^{2r+2k+1} \sinh\left(\frac{2\pi w}{3}\right)}{\cosh\left(\pi w\right)} dw, \end{aligned} \tag{4.1}$$

where the identity in (4.1) refers to an asymptotic expansion. Thus, to determine the asymptotic expansion of $H_2(z;it)$, we are left to evaluate explicitly for $a \in \mathbb{N}_0$

$$\mathcal{C}_a := \int_{\mathbb{R}} \frac{w^{2a+1} \sinh\left(\frac{2\pi w}{3}\right)}{\cosh(\pi w)} dw = \frac{1}{2} \int_{\mathbb{R}} \frac{w^{2a+1} \left(e^{\frac{2\pi w}{3}} - e^{-\frac{2\pi w}{3}}\right)}{\cosh(\pi w)} dw = \sum_{r=1}^{\infty} \frac{\left(\frac{2\pi}{3}\right)^{2r-1}}{(2r-1)!} \int_{\mathbb{R}} \frac{w^{2a+2r}}{\cosh(\pi w)} dw.$$

From (2.8), we have that the integral above equals $(2i)^{-2a-2r} E_{2a+2r}$, yielding

$$\mathcal{C}_a = (-2i)^{-2a-1} \sum_{r=1}^{\infty} \frac{\left(\frac{\pi i}{3}\right)^{2r-1}}{(2r-1)!} E_{2a+2r} = (-2i)^{-2a-1} \sum_{r=0}^{\infty} \frac{\left(\frac{\pi i}{3}\right)^{r}}{r!} E_{2a+r+1}.$$

The second equality above holds due to the fact that $E_j = 0$ for j odd.

We are thus left to understand $\sum_{r=0}^{\infty} \frac{v^r}{r!} E_{r+b}$ for positive integers b and $v = \frac{\pi i}{3}$. Set

$$f(v) := \sum_{r=0}^{\infty} \frac{E_r}{r!} v^r = \operatorname{sech}(v),$$

where the second equality above is simply the definition of the Euler numbers. Then

$$f^{(b)}(v) = \sum_{r=0}^{\infty} \frac{E_{r+b}}{r!} v^r.$$

Thus

$$\mathcal{C}_a = (-2i)^{-2a-1} \operatorname{sech}^{(2a+1)}\left(\frac{\pi i}{3}\right) = 2^{-2a-1} \sec^{(2a+1)}\left(\frac{\pi}{3}\right). \tag{4.2}$$

Next we deduce from (1.12) that

$$\frac{i}{2} \frac{1}{\sin(\pi z)} = -\sum_{n=0}^{\infty} \frac{B_{2n}\left(\frac{1}{2}\right)}{(2n)!} (2\pi i z)^{2n-1}.$$

Combining the above, we have established that the asymptotic expansion of $\mathcal{U}(z; it) e^{\frac{\pi t}{12}}$ as $t \to 0^+$ is given by

$$\frac{1}{\sqrt{3}} \sum_{r=0}^{\infty} (2\pi i z)^{2r} (-1)^r \sum_{0 \le n \le r} \frac{B_{2n}\left(\frac{1}{2}\right)}{(2n)!} \frac{(-1)^n}{(2r-2n+1)!} \sum_{k=0}^{\infty} \frac{\left(-\frac{\pi t}{3}\right)^k}{k!} \mathcal{C}_{r-n+k}.$$

Thus, using (4.2), we have the asymptotic expansion as $t \to 0^+$

$$\begin{aligned} e^{\frac{\pi t}{12}} u_{2r}\left(e^{-2\pi t}\right) = \frac{(2r)!(-1)^r 2^{-2r-1}}{\sqrt{3}} \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(-\frac{\pi}{3}\right)^k 2^{-2k} \\ \times \sum_{0 \le n \le r} \frac{(-1)^n B_{2n}\left(\frac{1}{2}\right) 2^{2n}}{(2n)!(2r-2n+1)!} \sec^{(2r-2n+2k+1)}\left(\frac{\pi}{3}\right). \end{aligned} \tag{4.3}$$

Using Lemma 2.6 together with (4.3), we have that

$$e^{\frac{\pi t}{12}} u_{2r}\left(e^{-2\pi t}\right) = \frac{3^{2r+1}}{2r+1} \sum_{k=0}^{\infty} \frac{(3\pi t)^k}{k!} \sum_{0 \le n \le r} \binom{2r+1}{2n} 3^{-2n} B_{2n}\left(\frac{1}{2}\right) E_{2r+2k+1-2n}\left(\frac{5}{6}\right), \tag{4.4}$$

which concludes the proof of the first statement of Theorem 1.2.

Next we prove the claimed asymptotic for the main term. Since $B_{2n+1}\left(\frac{1}{2}\right) = 0$, we may rewrite the $k = 0$ summand of (4.4) as

$$\frac{3^{2r+1}}{2r+1} \sum_{0 \le n \le 2r+1} \binom{2r+1}{n} 3^{-n} B_n\left(\frac{1}{2}\right) E_{2r+1-n}\left(\frac{5}{6}\right). \tag{4.5}$$

Now we use (2.6), which yields that

$$B_n\left(\frac{1}{2}\right) = 3^{n-1} \sum_{a=0}^{2} B_n\left(\frac{1}{6} + \frac{a}{3}\right).$$

Thus, (4.5) equals

$$\frac{3^{2r}}{2r+1} \sum_{a=0}^{2} \sum_{0 \le n \le 2r+1} \binom{2r+1}{n} B_n\left(\frac{1}{6} + \frac{a}{3}\right) E_{2r+1-n}\left(\frac{5}{6}\right). \tag{4.6}$$

Using (2.7), (4.6) reduces to

$$\frac{2\cdot 6^{2r}}{2r+1}\sum_{a=0}^{2} B_{2r+1}\left(\frac{1}{2}+\frac{a}{6}\right).$$

Noting again that $B_{2r+1}\left(\frac{1}{2}\right)=0$, we find that as claimed, as $t\to 0^+$,

$$e^{\frac{\pi t}{12}}u_{2r}\left(e^{-2\pi t}\right)\sim \frac{2\cdot 6^{2r}}{2r+1}\left(B_{2r+1}\left(\frac{2}{3}\right)+B_{2r+1}\left(\frac{5}{6}\right)\right).$$

5. An Example: the moment function $\phi_1(\tau)$

In this section, we give an exact value for the quantum moment function

(5.1)
$$\phi_1(\tau)=4\pi^3 i q^{-\frac{1}{24}}\sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}}(-1)^m u(m,n)\left(m^2+2n\right)q^n = 4\pi^3 i q^{-\frac{1}{24}}\left(u_2(q)-i\pi^{-1}\frac{\partial}{\partial\tau}u_0(q)\right).$$

To describe this, we define for positive integers n the polynomials

$$d_n(q):=n(q;q)_{n-1}^2q^n-2q^{n+2}(q;q)_n\sum_{j=1}^{n} jq^{j-1}\prod_{\substack{k=1\\k\neq j}}^{n}\left(1-q^k\right)\in\mathbb{Z}[q], \tag{5.2}$$

$$b_n(q):=q^{n+1}\sum_{j=1}^{n}q^j\prod_{\substack{k=1\\k\neq j}}^{n}\left(1-q^k\right)^2\in\mathbb{Z}[q]. \tag{5.3}$$

Theorem 5.1. *If* $h,k\in\mathbb{N}, with\, \gcd(h,k)=1,$ *we have that*

$$\phi_1\left(\frac{h}{k}\right)=8\pi^3 i\zeta_{24k}^{-h}\left(\sum_{n=1}^{k}d_n\left(\zeta_k^h\right)-\sum_{n=1}^{2k-1}b_n\left(\zeta_k^h\right)\right).$$

Remark. Theorem 5.1, together with (1.11) in the case $\tau\in\mathbb{Q}\setminus\{0\}$ of Theorem 1.1, gives an exact value for the integral

$$\int_{\mathbb{R}}P_1\left(w,(-i\tau)^{-1}\right)e^{\frac{\pi i\tau w^2}{3}}\frac{\sinh\left(\frac{2\pi w}{3}\right)}{\cosh(\pi w)}dw.$$

To prove Theorem 5.1, we first establish Proposition 5.2 and Proposition 5.3 below. These propositions give alternate expressions for the functions defining $\phi_1(\tau)$ (see (5.1)), which we subsequently evaluate for $q=\zeta$, where ζ is any root of unity.

Proposition 5.2. *With notation as above, we have that*

$$\frac{\partial}{\partial\tau}u_0(q)=2\pi i\sum_{n\geq 1}d_n(q).$$

Moreover, if $gcd(h,k)=1$ *we have that*

$$\frac{\partial}{\partial\tau}\left[u_0(q)\right]_{q=\zeta_k^h}=2\pi i\sum_{n=1}^{k}d_n\left(\zeta_k^h\right).$$

Proof. The first statement follows by straightforward differentiation, using that $u_0(q) = \mathcal{U}(0;\tau)$, definition (1.2), and the fact that $\frac{1}{2\pi i}\frac{\partial}{\partial\tau} = q\frac{d}{dq}$. To prove the second statement, we observe that $d_n(q)$ is of the form $d_n(q) = (q;q)_{n-1}\widetilde{d}_n(q)$, where $\widetilde{d}_n(\zeta_k^h) < \infty$. The statement now follows, observing that for $n \geq k+1$, the factor $(q;q)_{n-1}$ of $d_n(q)$ vanishes when $q = \zeta_k^h$. □

Proposition 5.3. *With notation as above, we have that*

$$(2\pi i)^2 u_2(q) = \frac{\partial^2}{\partial z^2}\left[\mathcal{U}(z;\tau)\right]_{z=0} = -2(2\pi i)^2 \sum_{n\geq 1} b_n(q).$$

Moreover, if $h, k \in \mathbb{N}$, *with* $\gcd(h,k) = 1$, *we have that*

$$(2\pi i)^2 u_2(\zeta_k^h) = -2(2\pi i)^2 \sum_{n=1}^{2k-1} b_n\left(\zeta_k^h\right).$$

Proof of Proposition 5.3. The first statement follows by straightforward differentiation, using definition (1.2), and the fact that $\frac{1}{2\pi i}\frac{\partial}{\partial z} = w\frac{d}{dw}$ for $w = e^{2\pi i z}$. To prove the second statement, using the first statement, we see for $n \geq 2k$, the jth summand defining $b_n(q)$ (for any $j \geq 1$) contains either the factor $(1-q^k)$ or $(1-q^{2k})$ (or both), both of which vanish when $q = \zeta_k^h$. □

Proof of Theorem 5.1. Theorem 5.1 now follows from the definition of $\phi_1(\tau)$ (see (5.1)), Proposition 5.2, and Proposition 5.3. □

References

[1] M. Abramowitz and I. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables,* National Bureau of Standards Applied Mathematics Series, 55, Washington, D.C. (1964).

[2] C. Alfes, K. Bringmann, and J. Lovejoy, *Automorphic properties of generating functions for generalized odd rank moments and odd Durfee symbols*, Math. Proc. Cambridge Philos. Soc. **151** (2011), 385-406.

[3] G. Andrews, *Partitions with short sequences and mock theta functions,* Proc. Natl. Acad. Sci. USA, **102** (2005), 4666-4671.

[4] G. Andrews, *Partitions, Durfee symbols, and the Atkin-Garvan moments of ranks,* Invent. Math. **169** no. 1 (2007), 37-73.

[5] G. Andrews, *The number of smallest parts in the partitions of n,* J. Reine Angew. Math. **624** (2008), 133-142.

[6] G. Andrews and B. Berndt, *Ramanujan's Lost Notebook, Part II.* Springer, New York, 2009.

[7] G.E. Andrews, R.C. Rhoades, and S.P. Zwegers, *Modularity of the concave composition generating function,* Algebra and Number Theory, to appear.

[8] A. Atkin and F. Garvan, *Relations between the ranks and cranks of partitions,* Rankin memorial issues, Ramanujan J. **7** (2003), 343-366.

[9] A. Atkin and H. Swinnerton-Dyer, *Some properties of partitions,* Proc. Lond. Math. Soc. **66** (4) (1954), 84-106.

[10] K. Bringmann, *On the explicit construction of higher deformations of partition statistics,* Duke Math. 144 (2008), 195-233.

[11] K. Bringmann and A. Folsom, *On the asymptotic behavior of Kac-Wakimoto characters,* Proc. Amer. Math. Soc., to appear.

[12] K. Bringmann, F. Garvan, and K. Mahlburg, *Partition statistics and quasiharmonic Maass forms,* Int. Math. Res. Not. (2009), 63-97.

[13] K. Bringmann, J. Lovejoy, and R. Osburn, *Automorphic properties of generating functions for generalized rank moments and Durfee symbols,* Int. Math. Res. Not. (2010), no. 2, 238-260.

[14] K. Bringmann, K. Mahlburg, and R. Rhoades, *Peak positions of strongly unimodal sequences*, in preparation.
[15] K. Bringmann and K. Ono, *Dyson's ranks and Maass forms,* Ann. of Math. **171** (2010), 419-449.
[16] J. Bryson, K. Ono, S. Pitman, and R. C. Rhoades, *Unimodal sequences and quantum and mock modular forms*, Proc. Natl. Acad. Sci. USA 109 (2012), no. 40, 16063-16067.
[17] D. Cvijovic, *Closed-form formulae for the derivatives of trigonometric functions at rational multiples of* π*,* Appl. Math. Lett. **22** (2009), 906-909.
[18] A. Dabholkar, S. Murthy, and D. Zagier *Quantum black holes, wall crossing, and mock modular forms,* submitted for publication.
[19] F. Dyson, *Some guesses in the theory of partitions,* Eureka (Cambridge) **8** (1944), 10-15.
[20] M. Eichler and D. Zagier, *The theory of Jacobi forms,* Progr. Math., 55. Birkhäuser Boston, Boston, MA, (1985).
[21] A. Erdelyi, W. Magnus, F. Oberhettinger, and F. Tricomi, *Higher transcendental functions. Vol. I.* Based, in part, on notes left by Harry Bateman and compiled by the Staff of the Bateman Manuscript Project, reprint of the 1955 original, Robert E. Krieger Publishing, Malabar 1981.
[22] F. Garvan, *Higher order spt-functions,* Adv. Math. **228** (2011), 241-265.
[23] V. Kac and M. Wakimoto, *Integrable highest weight modules over affine superalgebras and Appell's function,* Comm. Math. Phys. **215** (2001), 631-682.
[24] J. Manschot, *Wall-crossing of D*4*-branes using flow trees,* Adv. Theor. Math. Phys. **15** (2011).
[25] A. Mellit and S. Okada, *Joyce invariants for* $K3$ *surfaces and mock theta functions,* Comm. Number Theory Phys. **3** (2009), 655-676.
[26] H. Rademacher, *Topics in analytic number theory*, Die Grundlehren der math. Wiss., Band 169, Springer-Verlag, Berlin, (1973).
[27] S. Ramanujan, *Some properties of* $p(n)$*, number of partitions of* n*,* Proc. Cambridge Phil. Soc. **XIX** (1919), 207-210.
[28] G. Shimura, *On modular forms of half integral weight,* Ann. of Math. (2) **97** (1973), 440-481.
[29] C. Vafa and E. Witten, *A strong coupling test of S-duality,* Nuclear Phys. B **431** (1994), 3-77.
[30] D. Zagier, *Periods of modular forms and Jacobi theta functions,* Invent. Math. **104** (1991), 449-465.
[31] D. Zagier, *Vassiliev invariants and a strange identity related to the Dedekind eta-function,* Topology **40** (2001), 945-960.
[32] D. Zagier, *Quantum modular forms*, Quanta of maths, 659-675, Clay Math. Proc., 11, Amer. Math. Soc., Providence, RI, 2010.
[33] S. Zwegers, *Mock theta functions*, Ph.D. Thesis, Universiteit Utrecht, (2002).
[34] S. Zwegers, *Multivariable Appell functions,* preprint.

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