ON A CONJECTURE OF KOIKE ON IDENTITIES BETWEEN THOMPSON SERIES AND ROGERS-RAMANUJAN FUNCTIONS

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ABSTRACT. One of the many amazing things Ramanujan did in his lifetime was to list 40 identities involving what are now called the Rogers-Ramanujan functions $G(q)$ and $H(q)$ on one side, and products of functions of the form $Q_m = \prod_{n=1}^{\infty}(1 - q^{mn})$ on the other side. The identities are rather complicated and seem too difficult to guess. Recently however, Koike devised a strategy for finding (but not proving) these types of identities by connecting them to Thompson series. He was able to conjecture many new Rogers-Ramanujan type identities between $G(q)$ and $H(q)$, and Thompson series. Here we prove these identities.

1. INTRODUCTION AND STATEMENT OF RESULTS

One of the many amazing things Ramanujan did in his lifetime was to list 40 identities involving the *Rogers-Ramanujan functions*

$$(1.1) \qquad G(q) \quad := \quad \prod_{n=0}^{\infty} \frac{1}{\left(1 - q^{5n+1}\right)\left(1 - q^{5n+4}\right)} \,,$$

$$(1.2) \qquad H(q) \quad := \quad \prod_{n=0}^{\infty} \frac{1}{\left(1 - q^{5n+2}\right)\left(1 - q^{5n+3}\right)} \,.$$

One example of such an identity is the following one:

$$(1.3) \qquad G(q)G(q^4) + qH(q)H(q^4) = q^{1/12}\left(\frac{\eta(2z)^2}{\eta(z)\eta(4z)}\right)^2 \qquad \left(q = e^{2\pi i z}\right),$$

where $\eta(z) := q^{1/24}\prod_{n=1}^{\infty}\left(1 - q^n\right)$ is Dedekind's η-function. Many people have contributed to proving Ramanujan's identities (for example [Bia89],[Dar21],[Rog21], and [Wat33]).

According to Birch [Bia89] these identities seem too complicated to guess even for one with Ramanujan's incredible instinct for formulae. However, Koike recently made an interesting observation which enabled him to conjecture several new identities of the same type as Ramanujan's. Koike observed that certain eta-products which are connected to Thompson series in Conway and Norton's famous paper "Monstrous Moonshine" [CN79] appear on both sides of these identities. His method yields some of the identities in Ramanujan's list as well as new ones (see [Koi04]). Here we prove the new identities conjectured by Koike.

Let M denote the *monster*, the largest sporadic finite simple group. For every $g \in M$, the *Thompson series* T_g is defined by

$$T_g := q^{-1} + 0 + H_1(g)q + H_2(g)q^2 + \cdots,$$

where the $H_i(g)$ are character values for certain representations H_i of M (see [CN79]).

Let N be a positive integer, and let e be a *Hall divisor* of N (written $e \parallel N$), which is a positive divisor of N such that $(e, N/e) = 1$. An *Atkin-Lehner* involution W_e is a matrix of the form

$$W_e := \begin{pmatrix} ae & b \\ cN & de \end{pmatrix},$$

such that $\det W_e = e$.

Consider subgroups of $\mathrm{PSL}_2(\mathbb{R})$ obtained by extending $\Gamma_0(N)$ by certain Atkin-Lehner involutions. In particular, if S is a subset of Hall divisors of N, we write $N + S$ to denote the group generated by $\Gamma_0(N)$ and all Atkin-Lehner involutions W_e for $e \in S$. For shorthand, we write $N+$ when we adjoin all Atkin-Lehner involutions.

For $N + S \notin \{25-, 49+49, 50+50\}$, let X_{N+S} be the Riemann surface obtained by compactifying $\mathcal{H}/(N + S)$. If X_{N+S} has genus zero, then by work of Borcherds, there is an element $g \in M$ such that T_g is the normalized generator of the function field of X_{N+S}. In this case we define

$$T_{N+S} := T_g.$$

Let

(1.4) $$g(z) \quad := \quad q^{-1/60} G(q),$$

(1.5) $$h(z) \quad := \quad q^{11/60} H(q).$$

Moreover, if $r \equiv \pm s \pmod 5$ we define the functions $u_{r,s}(z)$ by

$$u_{r,s}(z) := \begin{cases} g(rz)g(sz) + h(rz)h(sz) & \text{for } r + s \equiv 0 \pmod 5, \\ g(rz)h(sz) - h(rz)g(sz) & \text{for } r - s \equiv 0 \pmod 5. \end{cases}$$

The purpose of this paper is to prove the following conjecture made by Koike (see [Koi04]), which we now state as a theorem.

Theorem 1.1. *Let M be the monster group, and suppose N is among the following set of orders of elements of M*

$$\left\{ \begin{array}{l} 4, 6, 9, 14, 19, 21, 24, 26, 29, 31, 34, 36, 39, \\ 41, 44, 46, 51, 54, 56, 59, 66, 69, 71, 119 \end{array} \right\}.$$

Then there exists an element $g \in M$ of order N and a constant c_N depending on N such that $T_g + c_N$ can be expressed as a product of Rogers-Ramanujan functions $u_{r,s}(z)$. (see Appendix A for a complete list of the identities and the constants c_N)

As an example of Theorem 1.1 we give the first of Koike's identities in Appendix A

(1.6) $$(u_{4,1}(z))^{12} = T_{4+} + 24 = \frac{\eta(2z)^{48}}{\eta(z)^{24}\eta(4z)^{24}},$$

where the second equality comes from Conway and Norton's Monstrous Moonshine [CN79].

Remark. *The identities (A.1), (A.2), (A.4), (A.13), (A.15), (A.16), and (A.21) in Appendix A already follow from the Ramanujan identities [Bir75].*

2. Preliminaries

Before we give the proof of Theorem 1.1, we require some facts about modular forms. See [BBP87, Bia89, Mar96] for more details.

If f is a modular form on a congruence subgroup Γ and ζ is a cusp of Γ, then f has a Fourier expansion at ζ of the form

$$\sum_{m \geq m_0} c(m) q_h^m,$$

where $q_h := e^{2\pi i z/h}$, $m_0 \in \mathbb{Z}$, $c(m_0) \neq 0$, and $h := h(\Gamma, \zeta)$ is the width of Γ at the cusp ζ. We let $\mathrm{Ord}_\Gamma(f, \zeta) := m_0$, and call it the *relative order of vanishing of f with respect to Γ*. Furthermore, we define $\mathrm{ord}(f, \zeta)$, the *invariant order of f at ζ*, to be the smallest power of $q := e^{2\pi i z}$ appearing in the Fourier expansion of f at ζ. Thus the two orders satisfy

$$(2.1) \qquad \mathrm{Ord}_\Gamma(f, \zeta) = h(\Gamma, \zeta) \cdot \mathrm{ord}(f, \zeta).$$

When we consider $z \in \mathcal{H}$, the upper half plane, $\mathrm{ord}(f, z)$ is the usual order of f as an analytic function at z. If a is the order of the stabilizer of z in Γ (considered as a subgroup of $\mathrm{PSL}_2(\mathbb{Z})$) and f is a modular form on Γ, then for every $z \in \mathcal{H}$ we define [BBP87]

$$(2.2) \qquad \mathrm{Ord}_\Gamma(f(z), z) := \frac{\mathrm{ord}(f, z)}{a} \geq 0.$$

A meromorphic modular form on $\Gamma_0(N)$ is called *weakly holomorphic* if its poles, if there are any, are supported at the cusps of $\Gamma_0(N)$. A complete set of representatives for the cusps of $\Gamma_0(N)$ is given by (see [Mar96])

$$(2.3) \qquad \left\{ \frac{a_c}{c} : c | N, 1 \leq a_c \leq N, \gcd(a_c, N) = 1, a_c \text{ distinct } \pmod{\gcd(c, N/c)} \right\}.$$

We now state the modularity of the functions $u_{r,s}$. For this we define the multiplier $\chi_{r,s}$ on matrices in $\Gamma_0([r, s])$:

$$(2.4) \qquad \chi_{r,s}\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right) := \begin{cases} \exp\left(\frac{7\pi i(r+s)}{6} (\beta\delta + (\gamma/rs)(\alpha + \delta - \beta\gamma\delta - 3)) \right) & \text{if } \gamma \text{ odd,} \\ \exp\left(\frac{7\pi i(r+s)}{6} (\beta\delta + (\gamma/rs)(\alpha - \delta - \alpha\delta^2)) \right) & \text{if } \delta \text{ odd.} \end{cases}$$

The following two results are due to Biagioli (Theorems 2.4 and 3.3 resp. of [Bia89]).

Lemma 2.1. *If r, s are positive integers, and $[r, s]$ denotes the least common multiple of r and s, then the functions $u_{r,s}(z)$ are weakly holomorphic modular functions on $\Gamma_0([r, s])$ with multiplier system $\chi_{r,s}$.*

Theorem 2.2. *If $f(z)$ is one of either $g(z)$ or $h(z)$ defined in (1.4) and (1.5), and a, c are integers with $(a, c) = 1$, then*

$$(2.5) \qquad \mathrm{ord}\left(f(z), \frac{a}{c} \right) \in \left\{ \frac{11}{16}, -\frac{1}{60} \right\}.$$

For our purposes we don't need the exact criteria for when each case occurs. Combining this theorem with the fact that for all modular forms $f(z)$ of weight 0, and all $m \in \mathbb{N}$ we have

$$\mathrm{ord}\left(f(mz), \frac{a}{c} \right) = \frac{\gcd(m, c)^2}{m} \cdot \mathrm{ord}\left(f(z), \frac{ma}{c} \right),$$

gives the following bounds for the invariant order of vanishing of the $u_{r,s}(z)$:

$$\mathrm{ord}\left(u_{r,s}(z), \frac{a}{c} \right) \geq -\frac{1}{60} \left(\frac{\gcd(r, c)^2}{r} + \frac{\gcd(s, c)^2}{s} \right),$$

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$$\operatorname{ord}\left(u_{r,s}(z)^{-1}, \frac{a}{c}\right) \geq -\frac{11}{16}\left(\frac{\gcd(r,c)^2}{r} + \frac{\gcd(s,c)^2}{s}\right).$$

For the group $\Gamma_0(N)$ we have the formula [Mar96] for the width of $\Gamma_0(N)$ at the cusp $\frac{a}{c}$,

$$(2.6) \qquad\qquad h_{\frac{a}{c}} := h\left(\Gamma_0(N), \frac{a}{c}\right) = \frac{N}{\gcd(c^2, N)}.$$

The largest possible width at $\frac{a}{c}$ is thus $\frac{N}{c}$, and from this we easily obtain the following proposition.

Proposition 2.3. *Let $N = [r, s]$ and let $\frac{a}{c}$ be a cusp of $\Gamma_0(N)$. Then*

$$\operatorname{Ord}_{\Gamma_0(N)}\left(u_{r,s}(z)^n, \frac{a}{c}\right) \geq \begin{cases} -\frac{nN}{30} & \text{if } n \geq 1, \\ \frac{nN \cdot 11}{8} & \text{if } n \leq -1. \end{cases}$$

In addition, much is known about the modularity of eta-quotients (see [Mar96], [Ono04]). The following theorem (see e.g. [Ono04] page 18) describes when an eta-quotient is modular.

Theorem 2.4. *Assume that $f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta}$ ($N \in \mathbb{N}$, $r_\delta \in \mathbb{Z}$) is an eta-quotient such that $k := \frac{1}{2}\sum_{\delta|N} r_\delta \in \mathbb{Z}$,*

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24},$$

and

$$\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}.$$

Then $f(z)$ is a meromorphic modular form of weight k for the group $\Gamma_0(N)$ with Nebentypus character χ given by

$$\chi(d) := \left(\frac{(-1)^k s}{d}\right)$$

for $s := \prod_{\delta|N} \delta^{r_\delta}$.

We can also calculate the orders of vanishing of eta-quotients at cusps of $\Gamma_0(N)$ by the following theorem.

Theorem 2.5. *If $f(z)$ is an eta-quotient that satisfies the conditions of Theorem 2.4, then*

$$\operatorname{Ord}_{\Gamma_0(N)}\left(f, \frac{a}{c}\right) = \frac{N}{24 \gcd(c^2, N)} \sum_{\delta|N} \frac{\gcd(c, \delta)^2 \cdot r_\delta}{\delta}.$$

3. Proof of Theorem 1.1

3.1. **Overview of method.** To prove identities of the type

$$\prod u_{r,s} = T_{N+S}$$

for a given N, we consider $g_N(z) = \prod u_{r,s} - T_{N+S}$ and prove that $g_N(z) = 0$. To do this, we realize $g_N(z)$ as a weight 0 modular form on $\Gamma_0(N)$, and then contradict the valence formula for $g_N(z)$ (see [Bia89] and equation (3.1)).

The Thompson series T_{N+S} can be viewed as a weakly holomorphic modular function with respect to the group $N + S$ (see [Kim04]). Since this group contains $\Gamma_0(N)$ we may view T_{N+S} as a meromorphic modular function on $\Gamma_0(N)$.

Similarly, Lemma 2.1 implies that each quotient of $u_{r,s}(z)$ functions in Koike's identities is a meromorphic modular function on $\Gamma_0(N)$ for some N with multiplier χ, where χ is the corresponding quotient of multipliers from equation (2.4). A direct calculation in each case shows that in fact

the multiplier system is trivial for each quotient. Thus for each N if $g_N(z)$ is defined to be the difference of the left-hand side and the right-hand side of the corresponding identity from Appendix A, then $g_N(z)$ is a meromorphic modular function on $\Gamma_0(N)$. Thus the valence formula gives that if $g_N(z)$ is not zero, then

$$(3.1) \qquad \sum_{\tau \in \mathcal{H}} \operatorname{Ord}_{\Gamma_0(N)}(g_N(z), \tau) + \sum_{\frac{a}{c} \not\sim \infty} \operatorname{Ord}_{\Gamma_0(N)} \left(g_N(z), \frac{a}{c} \right) + \operatorname{Ord}_{\Gamma_0(N)} (g_N(z), \infty) = 0,$$

where the second sum runs over a set of inequivalent cusps $\frac{a}{c}$ of $\Gamma_0(N)$ that are not equivalent to ∞.

Since $g_N(z)$ is holomorphic on $\mathcal{H}$, the first summand is clearly positive. Our goal is to obtain lower bounds for $\operatorname{Ord}_{\Gamma_0(N)} \left(g_N(z), \frac{a}{c} \right)$ for cups $\frac{a}{c} \not\sim \infty$. Then we can contradict (3.1) by showing that $\operatorname{Ord}_{\Gamma_0(N)} (g_N(z), \infty)$ is "big enough" by computing coefficients of $g_N(z)$ at infinity. In the following sections we will be more precise about the number of coefficients required in each case.

To get a lower bound for $\operatorname{Ord}_{\Gamma_0(N)} \left(g_N(z), \frac{a}{c} \right)$, we simply take the minimum of the lower bounds for the quotient of $u_{r,s}$ functions and the Thompson series on either side of the corresponding identity.

By Proposition 2.3 we easily obtain lower bounds for the relative orders of vanishing of the $u_{r,s}$ quotients that appear in the left-hand sides of the identities. For the Thompson series, we use one of the following methods.

Remark. *We first note that from the Ramanujan identities one can prove the identities for $N = 4, 6, 14, 39, 44, 46, 66$. For example it is not hard to see that (1.3) is equivalent to the identity for $N = 4$.*

3.2. **Group N+ with N square-free.** Here we consider the identities for square-free N where the Thompson series T_{N+} is associated to the group $N+$ with all Atkin-Lehner involutions adjoined. These are the identities for

$$N \in \{19, 21, 26, 29, 31, 34, 41, 46, 51, 59, 69, 71, 119\} .$$

When N is square-free, (2.3) gives that the set of cusps for $\Gamma_0(N)$ is given by

$$\left\{ \frac{1}{m} : m | N \right\} .$$

Proposition 3.1. *For $N \in \{19, 21, 26, 29, 31, 34, 41, 46, 51, 59, 69, 71, 119\}$ we have*

$$(3.2) \qquad \operatorname{Ord}_{\Gamma_0(N)} \left(T_{N+}(z), \frac{1}{m} \right) = -1.$$

Proof. First we recall that if $M := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \operatorname{SL}_2(\mathbb{R})$ and $z \in \mathcal{H}$, then

$$M z := \frac{az + b}{cz + d}.$$

If we let

$$A_m := \left(\begin{array}{cc} 1 & 0 \\ m & 1 \end{array} \right) \in \operatorname{SL}_2(\mathbb{Z}),$$

then the Fourier expansion of $g_N(A_m z)$ at ∞ reflects the behavior of $g_N(z)$ at the cusp $1/m$. Since N is square-free, we can find for $e := N/m$ integers x, y such that $xe - ym = 1$. Thus the matrix

$$W_e := \left(\begin{array}{cc} xe & -y \\ -N & e \end{array} \right)$$

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is an Atkin-Lehner involution with determinant e, and so T_{N+} is invariant under $W_e \in N+$. Since $W_e A_m = \left(\begin{smallmatrix} 1 & y \\ 0 & e \end{smallmatrix} \right)$, we have

$$T_{N+}(A_m z) = T_{N+}\left(\left(\begin{array}{cc} 1 & y \\ 0 & e \end{array} \right) z \right).$$

Recalling that ord $(T_{N+}, \infty) = -1$, this yields that ord $\left(T_{N+}, \frac{1}{m}\right) = -1/e$. Multiplying by the width e from equation (2.6) gives the desired result. $\qquad\square$

3.3. Remaining cases.
The 5 cases not yet covered are $N = 9, 24, 36, 54, 56$. For these we use the fact that we can write the Thompson series as a linear combination of eta-quotients (see [CN79]). Calculating via Theorem 2.5, we obtain the orders of vanishing we need. The case where $N = 54$ must be treated more delicately, and is considered in Section 3.5.

Let $N \neq 4, 6, 14, 39, 44, 46, 66$ (the cases already proved with the Ramanujan identities). After computing the necessary bounds for the orders of vanishing for each Thompson series (or associated eta-quotient) and comparing them with the bounds from Proposition 2.3 for the $u_{r,s}(z)$ quotients, we obtain the following table giving the number of coefficients of the Fourier expansion of $g_N(z)$ at ∞ that we need to compute in order to prove that $g_N(z) = 0$. The resulting computations are done easily using Maple.

N	9	19	21	24	26	29	31	34	36	41	51	54	56	59	69	71	119
coefficients	6	2	13	17	11	2	4	14	558	3	11	8	8	2	14	3	24

Remark. *The reason that we compute so many coefficients for $N = 36$ is due to a combination of the fact that 36 has many divisors, and because the identity in this case has a $u_{r,s}(z)$ function in the denominator. Proposition 2.3 gives poor bounds for $u_{r,s}(z)$ functions in the denominator. We could get better bounds, but the resulting notation and complication is unnecessary for the proof.*

3.4. Example.
Here we demonstrate the argument for the case $N = 9$. Using Lemma 2.1 we see that $(u_{9,1}(z))^6$ is a modular function on $\Gamma_0(9)$ with trivial multiplier. Also, from Theorem 2.4 we see that the functions $\frac{\eta(z)^3}{\eta(9z)^3}$ and $\frac{\eta(9z)^3}{\eta(z)^3}$ are modular functions on $\Gamma_0(9)$ with trivial multiplier, yielding that

$$f_9(z) := \frac{\eta(z)^3}{\eta(9z)^3} + 27\frac{\eta(9z)^3}{\eta(z)^3} + 9$$

is also. Thus we have that $g_9(z) := (u_{9,1}(z))^6 - f_9(z)$ is a modular function on $\Gamma_0(9)$.

The valence formula (3.1) gives that if $g_9(z) \neq 0$, then

$$\sum_{\tau \in \mathcal{H}} \mathrm{Ord}_{\Gamma_0(9)}(g_9(z), \tau) + \sum_{\frac{a}{c} \nsim \infty} \mathrm{Ord}_{\Gamma_0(9)}\left(g_9(z), \frac{a}{c}\right) + \mathrm{Ord}_{\Gamma_0(9)}(g_9(z), \infty) = 0.$$

The first summand is clearly non-negative since $g_9(z)$ is holomorphic on $\mathcal{H}$. We now estimate the second summand. By (2.3), we see that a set of representatives of the cusps of $\Gamma_0(9)$ is

$$\left\{ 1, \frac{1}{3}, \frac{2}{3}, \frac{1}{9} \right\},$$

where the cusp $1/9$ is equivalent to ∞. Using Proposition 2.3 we see that for any cusp a/c of $\Gamma_0(9)$,

$$\mathrm{Ord}_{\Gamma_0(9)}\left((u_{9,1}(z))^6, \frac{a}{c} \right) \geq -\frac{9}{5}.$$

Then we observe from a repeated application of Theorem 2.5 that $\mathrm{Ord}_{\Gamma_0(9)}\left(f_9(z), \frac{a}{c}\right) \geq -1$, and so we have the bound

$$\mathrm{Ord}_{\Gamma_0(9)}\left(g_9(z), \frac{a}{c} \right) \geq -\frac{9}{5}.$$

Since there are 3 cusps of $\Gamma_0(9)$ which are not equivalent to ∞, the second term is larger then -6. One can easily compute Fourier coefficients of g_9 using maple, to show that $\mathrm{Ord}_{\Gamma_0(9)}\left(g_9(z),\infty\right) > 6$, and thus the valence formula (3.1) is contradicted. This gives that $g_9(z) = 0$, which proves (A.3).

3.5. **The case $N = 54$.** In the case $N = 54$, we argue a little differently. Using (2.3), we easily see that a complete set of represenatives of cusps of $\Gamma_0(54)$ is given by

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{5}{3}, \frac{1}{6}, \frac{5}{6}, \frac{1}{9}, \frac{5}{9}, \frac{1}{18}, \frac{5}{18}, \frac{1}{27}, \frac{1}{54}\right\}.$$

As in Section 3.4, we see that $u_{54,1}(z)u_{27,2}(z)$ is a modular function on $(\Gamma_0(54))$ whose order of vanishing is $-\frac{7}{5}$ at the cusps

$$1, \frac{1}{2}, \frac{1}{27}, \frac{1}{54},$$

and is $-\frac{1}{5}$ at the cusps

$$\frac{1}{3}, \frac{5}{3}, \frac{1}{6}, \frac{5}{6}, \frac{1}{9}, \frac{5}{9}, \frac{1}{18}, \frac{5}{18}.$$

Thus we are left to compute the order of vanishing of

$$f_{54}(z) := \frac{\eta(z + 1/3) \cdot \eta(2z + 1/3) \cdot \eta(3z + 2/3) \cdot \eta(6z + 2/3)}{\eta(3z) \cdot \eta(6z) \cdot \eta(9z) \cdot \eta(18z)}$$

at all cusps of $\Gamma_0(54)$. At the cusps $1, \frac{1}{2}, \frac{1}{27}, \frac{1}{54}$, we argue as in Section 3.2. Since the other cusps are all treated in the same manner we will only give the proof for the cusp $\frac{1}{3}$. Clearly the matrix $M := \left(\begin{smallmatrix} 1 & 0 \\ 3 & 1 \end{smallmatrix}\right)$ maps ∞ to $\frac{1}{3}$. We now consider

$$f_{54}(z)^{24} = \frac{\Delta(z + 1/3) \cdot \Delta(2z + 1/3) \cdot \Delta(3z + 2/3) \cdot \Delta(6z + 2/3)}{\Delta(3z) \cdot \Delta(6z) \cdot \Delta(9z) \cdot \Delta(18z)},$$

where $\Delta(z) := \eta^{24}(z)$ is the usual Δ-function. It is easy to see that

$$f_{54}(Mz)^{24} = \frac{\Delta|_{12}\left(\begin{smallmatrix} 3 & 1 \\ 0 & 3 \end{smallmatrix}\right)Mz \cdot \Delta|_{12}\left(\begin{smallmatrix} 6 & 1 \\ 0 & 3 \end{smallmatrix}\right)Mz \cdot \Delta|_{12}\left(\begin{smallmatrix} 9 & 2 \\ 0 & 3 \end{smallmatrix}\right)Mz \cdot \Delta|_{12}\left(\begin{smallmatrix} 18 & 2 \\ 0 & 3 \end{smallmatrix}\right)Mz}{3^{24}\Delta|_{12}\left(\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix}\right)Mz \cdot \Delta|_{12}\left(\begin{smallmatrix} 6 & 0 \\ 0 & 1 \end{smallmatrix}\right)Mz \cdot \Delta|_{12}\left(\begin{smallmatrix} 9 & 0 \\ 0 & 1 \end{smallmatrix}\right)Mz \cdot \Delta|_{12}\left(\begin{smallmatrix} 18 & 0 \\ 0 & 1 \end{smallmatrix}\right)Mz},$$

where for a function $f : \mathcal{H} \to \mathbb{C}$, and a matrix $A := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ we define the weight 12 slash operator $|_{12}$ by

$$f|_{12}Az := (ad - bc)^6 \cdot (cz + d)^{-12} \cdot f(Az).$$

Now we can compute the order of vanishing of $f_{54}(Mz)^{24}$ at ∞ by using the transformation law for the Δ-function. For example we have

$$\Delta|_{12}\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}Mz = \Delta|_{12}\begin{pmatrix} 6 & 1 \\ 9 & 3 \end{pmatrix}z = \Delta|_{12}\begin{pmatrix} -1 & 1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 6 & 1 \\ 9 & 3 \end{pmatrix}z = \Delta|_{12}\begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}z = \zeta_1 \cdot q + \dots,$$

where ζ_1 is some root of unity. In the same way we treat the other Δ-factors. From this we can easily conclude that

$$f_{54}(Mz) = \zeta_2 + O(q),$$

where ζ_2 is some root of unity. Thus we directly obtain that the order of vanishing of f_{54} at $\frac{1}{3}$ is 0. Treating the other cusps in the same way and comparing with our bounds for $u_{54,1}(z)u_{27,2}(z)$ gives us the bound -8 for $g_{54}(z)$ as in the table.

Appendix A

$$(\text{A.1}) \qquad (u_{4,1}(z))^{12} \qquad = T_{4+} + 24 = \frac{\eta(2z)^{48}}{\eta(z)^{24}\eta(4z)^{24}} \qquad N = 4$$

$$(\text{A.2}) \qquad (u_{3,2}(z))^{12} \qquad = T_{6+6} + 12 = \frac{\eta(2z)^{12}\eta(3z)^{12}}{\eta(z)^{12}\eta(6z)^{12}} \qquad N = 6$$

$$(\text{A.3}) \qquad (u_{9,1}(z))^{6} \qquad = T_{9+} + 6 = \frac{\eta(z)^{3}}{\eta(9z)^{3}} + 27\frac{\eta(9z)^{3}}{\eta(z)^{3}} + 9 \qquad N = 9$$

$$(\text{A.4}) \qquad (u_{14,1}(z))^{4} \qquad = T_{14+14} + 4 = \frac{\eta(2z)^{4}\eta(7z)^{4}}{\eta(z)^{4}\eta(14z)^{4}} \qquad N = 14$$

$$(\text{A.5}) \qquad (u_{19,1}(z))^{3} \qquad = T_{19+} + 3 \qquad N = 19$$

$$(\text{A.6}) \qquad (u_{7,3}(z))^{6} \qquad = T_{21+} = \frac{\eta(3z)^{2}\eta(7z)^{2}}{\eta(z)^{2}\eta(21z)^{2}} + \frac{\eta(z)^{2}\eta(21z)^{2}}{\eta(3z)^{2}\eta(7z)^{2}} - 2 \qquad N = 21$$

$$(\text{A.7}) \qquad (u_{24,1}(z))^{2} \cdot u_{6,4}(z) \qquad = T_{24+24} + 2 = \frac{\eta(2z)\eta(3z)^{2}\eta(8z)^{2}\eta(12z)}{\eta(z)^{2}\eta(4z)\eta(6z)\eta(24z)^{2}} \qquad N = 24$$

$$(\text{A.8}) \qquad (u_{13,2}(z))^{4} \qquad = T_{26+} = \frac{\eta(2z)^{2}\eta(13z)^{2}}{\eta(z)^{2}\eta(26z)^{2}} + \frac{\eta(z)^{2}\eta(26z)^{2}}{\eta(2z)^{2}\eta(13z)^{2}} - 2 \qquad N = 26$$

$$(\text{A.9}) \qquad (u_{29,1}(z))^{2} \qquad = T_{29+} + 2 \qquad N = 29$$

$$(\text{A.10}) \qquad (u_{31,1}(z))^{3} \qquad = T_{31+} \qquad N = 31$$

$$(\text{A.11}) \qquad (u_{34,1}(z)u_{17,2}(z))^{2} \qquad = T_{34+} + 2 \qquad N = 34$$

$$(\text{A.12}) \qquad \frac{u_{36,1}(z)}{u_{9,4}(z)} \qquad = T_{36+36} + 1 = \frac{\eta(4z)\eta(9z)}{\eta(z)\eta(36z)} \qquad N = 36$$

$$(\text{A.13}) \qquad \frac{u_{39,1}(z)}{u_{13,3}(z)} \qquad = T_{39+39} + 1 = \frac{\eta(3z)\eta(13z)}{\eta(z)\eta(39z)} \qquad N = 39$$

$$(\text{A.14}) \qquad (u_{41,1}(z))^{2} \qquad = T_{41+} \qquad N = 41$$

$$(\text{A.15}) \qquad u_{44,1}(z)u_{11,4}(z) \qquad = T_{44+} + 1 = \frac{\eta(2z)^{4}\eta(22z)^{4}}{\eta(z)^{2}\eta(4z)^{2}\eta(11z)^{2}\eta(44z)^{2}} - 1 \qquad N = 44$$

$$(\text{A.16}) \qquad u_{46,1}(z)u_{23,2}(z) \qquad = T_{46+} = \frac{\eta(z)\eta(23z)}{\eta(2z)\eta(46z)} + 2\frac{\eta(2z)\eta(46z)}{\eta(z)\eta(23z)} + 1 \qquad N = 46$$

$$(\text{A.17}) \qquad u_{51,1}(z)u_{17,3}(z) \qquad = T_{51+} \qquad N = 51$$

$$(\text{A.18}) \qquad u_{54,1}(z)u_{27,2}(z) \qquad = T_{54+} + 1 = \frac{\eta(z+1/3)\eta(2z+1/3)\eta(3z+2/3)\eta(6z+2/3)}{\eta(3z)\eta(6z)\eta(9z)\eta(18z)} \qquad N = 54$$

$$(\text{A.19}) \qquad u_{56,1}(z)u_{8,7}(z) \qquad = T_{56+} = \frac{\eta(2z)\eta(4z)\eta(14z)\eta(28z)}{\eta(z)\eta(7z)\eta(8z)\eta(56z)} - 1 \qquad N = 56$$

$$(\text{A.20}) \qquad u_{59,1}(z) \qquad = T_{59+} + 1 \qquad N = 59$$

$$(\text{A.21}) \qquad \frac{u_{22,3}(z)u_{33,2}(z)}{u_{11,6}(z)u_{66,1}(z)} \qquad = T_{66+6,11,66} + 1 = \frac{\eta(2z)\eta(3z)\eta(22z)\eta(33z)}{\eta(z)\eta(6z)\eta(11z)\eta(66z)} \qquad N = 66$$

$$(\text{A.22}) \qquad u_{69,1}(z)u_{23,3}(z) \qquad = T_{69+} + 1 \qquad N = 69$$

$$(\text{A.23}) \qquad u_{71,1}(z) \qquad = T_{71+} \qquad N = 71$$

$$(\text{A.24}) \qquad u_{119,1}(z)u_{17,7}(z) \qquad = T_{119+} + 1 \qquad N = 119$$

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