ARITHMETIC PROPERTIES OF NON-HARMONIC WEAK MAASS FORMS

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1. Introduction and Statement of Results

A *partition* of a positive integer n is a non-increasing sequence of positive integers whose sum is n. Let $p(n)$ denote the partition function, i.e., the number of partitions of n, and set $p(0) := 1$. The generating function for $p(n)$ is given by

$$P(q) := 1 + \sum_{n=1}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)^2(1-q^2)^2 \cdots (1-q^n)^2}. \tag{1.1}$$

The arithmetic behavior of the partition function has been of great interest. For example, we have the famous Ramanujan congruences

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11} \end{aligned}$$

for every $n \geq 0$. In a celebrated paper Ono [13] treated this type of congruence systematically. Combining Shimura's theory of modular forms of half-integral weight with results of Serre on modular forms modulo ℓ, he showed that for any prime $\ell \geq 5$ there exist infinitely many non-nested arithmetic progressions $An + B$ such that for every $n \geq 0$,

$$p(An+B) \equiv 0 \pmod{\ell}.$$

Now consider the function

$$f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2},$$

which is one of the mock theta functions Ramanujan [15] defined in his last letter to Hardy in 1920. While $f(q)$ and $P(q)$ have similar shapes, $P(q)$ is (up to a power of q) a modular form and $f(q)$ is not. However, $f(q)$ does constitute the "holomorphic part" of a weak Maass form (see Section 2 for the definition of weak Maass form). The behavior of many arithmetic functions (Hurwitz class numbers, Dyson's ranks, and coefficients of other mock theta functions, to name a few) is governed by the coefficients of weak Maass forms (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12]). For example, Ono and the first author used weak Maass forms [4] to show that Dyson's rank generating functions satisfy congruences of Ramanujan type. Weak Maass forms have expansions involving Whittaker functions, and as in the theory of modular forms one is interested in forms whose expansions have algebraic coefficients. The known examples of such forms are harmonic, i.e., have Laplace eigenvalue zero. Here we show the existence of an infinite family of non-harmonic weak Maass forms of varying weights and

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Laplace eigenvalues possessing algebraic coefficients. In particular, we prove the following result.

Theorem 1.1. *Let w be a non-negative integer and χ a Dirichlet character of conductor b which is even (resp. odd) if w is odd (resp. even). Then there exists a weak Maass form of weight $w+\frac{1}{2}$ on $\Gamma_0(4b^2)$ with Nebentypus χ and Laplace eigenvalue $\frac{1}{4}w(w-1)$ whose coefficients lie in the ring of integers $\mathcal{O}_K$ of some number field K.*

Remark. In Section 3 we explicitly construct forms $f_{w+\frac{1}{2},\chi}(z)$ satisfying the conditions of Theorem 1.1. Writing $z = x+iy$ $(x, y \in \mathbb{R})$, these forms have the shape

$$f_{w+\frac{1}{2},\chi}(z) = f^h_{w+\frac{1}{2},\chi}(z) + f^{nh}_{w+\frac{1}{2},\chi}(z) \tag{1.2}$$

with

$$\begin{aligned} f^h_{w+\frac{1}{2},\chi}(z) &:= \widetilde{a}_{w+\frac{1}{2}}(n_0)\, \widetilde{\mathcal{W}}_{\frac{3}{4}}^{w+\frac{1}{2}}(4\pi n_0 y)\, e^{-2\pi i n_0 x} + \sum_{n>0} a^h_{w+\frac{1}{2}}(n)\, \mathcal{W}_{\frac{3}{4}}^{w+\frac{1}{2}}(4\pi n y)\, e^{2\pi i n x}, \\ f^{nh}_{w+\frac{1}{2},\chi}(z) &:= \sum_{n>0} a^{nh}_{w+\frac{1}{2}}(n)\, \mathcal{W}_{\frac{3}{4}}^{w+\frac{1}{2}}\left(-4\pi d n^2 y\right)\, e^{-2\pi i d n^2 x} \end{aligned}$$

for some positive integers d and n_0 and the functions $\mathcal{W}_{\frac{3}{4}}^{w+\frac{1}{2}}(y)$ and $\widetilde{\mathcal{W}}_{\frac{3}{4}}^{w+\frac{1}{2}}(y)$ are certain modified Whittaker functions defined in Section 2. The forms $f_{w+\frac{1}{2},\chi}(z)$ are examples of weak Maass forms which we call *good* (see Section 2 for the definition). In particular, if we denote the incomplete gamma function by

$$\Gamma(a, y) := \int_y^\infty e^{-t} t^{a-1} dt,$$

, and let $u = w+\frac{1}{2} \in \{\frac{1}{2}, \frac{3}{2}\}$, then one can show that

$$\begin{aligned} f^h_{u,\chi}(z) &= \widetilde{a}_u(n_0)\, q^{-n_0} + \sum_{n>0} a^h_u(n)\, q^n, \\ f^{nh}_u(z) &= \sum_{n>0} a^{nh}_u(n)\, \Gamma\left(\frac{1}{2}, 4\pi d n^2 y\right)\, q^{-dn^2}, \end{aligned}$$

where $q := e^{2\pi i z}$. Note that

$$\frac{\partial}{\partial \bar{z}}\left(f^h_{u,\chi}(z)\right) = 0,$$

while one can show that

$$\frac{\partial}{\partial \bar{z}}\left(f^{nh}_{u,\chi}(z)\right) \neq 0.$$

For this reason we call $f^h_{w+\frac{1}{2},\chi}(z)$ the "holomorphic part" of $f_{w+\frac{1}{2},\chi}(z)$.

We also show that the coefficients of our non-harmonic weak Maass forms satisfy congruences of Ramanujan type.

Theorem 1.2. *Let $f_{w+\frac{1}{2},\chi}(z)$ be the weak Maass form of weight $w+\frac{1}{2}$ and Nebentypus χ constructed in the proof of Theorem 1.1, and decompose $f_{w+\frac{1}{2},\chi}(z)$ as in (1.2). Then for* *any odd prime $p \nmid 3Nd$ and any $j \geq 1$, there exist infinitely many nonnested arithmetic progressions $An + B$ such that for every $n \geq 0$,*

$$a^h_{w+\frac{1}{2}}(An+B) \equiv 0 \pmod{p^j}.$$

Two remarks.
1) From our construction of $f_{w+\frac{1}{2},\chi}(z)$ it follows trivially that

$$a^h_{w+\frac{1}{2}}(pn) \equiv 0 \pmod{p^r},$$

where $r := \left[\frac{w}{2}\right]$.
2) It seems likely that one can give distribution results on the coefficients of the forms $f_{w+\frac{1}{2},\chi}(z)$ modulo p^j as in [11], or asymptotics as in [1].

2. General facts on weak Maass forms

In this section we recall basic facts on weak Maass forms, which were first studied in [8]. For v odd, define ϵ_v by

$$\epsilon_v := \begin{cases} 1 & \text{if } v \equiv 1 \pmod 4, \\ i & \text{if } v \equiv 3 \pmod 4. \end{cases}$$

Suppose that $k \in \frac{1}{2} + \mathbb{Z}$, and let

$$\Delta_k := -4y^2 \frac{\partial^2}{\partial z \partial \bar{z}} + 2iky \frac{\partial}{\partial \bar{z}}$$

be the weight k hyperbolic Laplacian, where $\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ and $\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$.

Definition. Let N be a positive integer, ψ a Dirichlet character modulo $4N$ and $g : \mathbb{H} \to \mathbb{C}$ a smooth function. We call g a *weak Maass form* of weight k and Laplace eigenvalue λ on $\Gamma_0(4N)$ with Nebentypus ψ if it satisfies the following three conditions:

(1) For all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4N)$ and all $z \in \mathbb{H}$, we have

$$g(Az) = \psi(d) \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} \, (cz+d)^k \, g(z).$$

(2) We have $\Delta_k g = \lambda g$.
(3) The function $g(z)$ has at most linear exponential growth at all cusps.

If $\lambda = 0$, we say that $g(z)$ is *harmonic*.

Let us next recall the shape of the Fourier expansion of a weak Maass form. For this let $W_{\nu,\mu}(y)$ and $W_{-\nu,\mu}(-y)$ be the standard W-Whittaker functions. These functions are linearly independent solutions of the differential equation

$$\frac{\partial^2 u}{\partial y^2} + \left(-\frac{1}{4} + \frac{\nu}{y} + \frac{\frac{1}{4} - \mu^2}{y^2}\right) u = 0 \tag{2.1}$$

and can be distinguished by their asymptotic behavior, namely

$$|W_{\pm\nu,\mu}(\pm y)| \sim e^{\mp\frac{y}{2}} |y|^{\mu+\frac{1}{2}} \tag{2.2}$$

as $|y| \to \infty$. For $y \in \mathbb{R} \setminus \{0\}$, define the functions

$$\begin{aligned}\mathcal{W}_s^k(y) &:= |y|^{-\frac{k}{2}} W_{\operatorname{sgn}(y)\frac{k}{2}, s-\frac{1}{2}}(|y|),\\ \widetilde{\mathcal{W}}_s^k(y) &:= |y|^{-\frac{k}{2}} W_{-\operatorname{sgn}(y)\frac{k}{2}, s-\frac{1}{2}}(-|y|).\end{aligned}$$

A computation shows that $\mathcal{W}_s^k(y)\, e^{\frac{ix}{2}}$ and $\widetilde{\mathcal{W}}_s^k(y)\, e^{\frac{ix}{2}}$ are eigenfunctions of Δ_k with eigenvalue $s(1-s) + \frac{k^2-2k}{4}$. Moreover, if $s \neq \frac{1}{2}$, two linearly independent solutions of

$$-y^2 \frac{\partial^2 u}{\partial y^2} - ky\frac{\partial u}{\partial y} = \left(s(1-s) + \frac{k^2-2k}{4}\right) u \tag{2.3}$$

are given by $y^{s-\frac{k}{2}}$ and $y^{1-s-\frac{k}{2}}$. Then by the translation invariance and properties (2) and (3) of a weak Maass form, one can show that each weak Maass form $g(z)$ of weight k and Laplace eigenvalue λ has an expansion of the form

$$g(z) = \sum_{\substack{n=-n_0\\ n\neq 0}}^{m_0} \widetilde{a}(n)\widetilde{\mathcal{W}}_s^k(4\pi ny)e^{2\pi inx} + \sum_{n\in\mathbb{Z}\setminus\{0\}} a(n)\mathcal{W}_s^k(4\pi ny)e^{2\pi inx} + a(0)\, y^{s-\frac{k}{2}} + \widetilde{a}(0)\, y^{1-s-\frac{k}{2}}, \tag{2.4}$$

where s is a solution of

$$s(1-s) + \frac{k^2-2k}{4} = \lambda. \tag{2.5}$$

Note that by (2.2), the first sum in (2.4) is responsible for the possible exponential growth of $g(z)$. We call a weak Maass form $g(z)$ *good* if (1) $\widetilde{a}(n) = 0$ for $n \geq 0$, and (2) there exists a positive integer d such that $a(n) \neq 0$ implies that $n > 0$ or $n = -dm^2$ for some $m > 0$.

For functions $g : \mathbb{H} \to \mathbb{C}$ define the operator R_k by

$$R_k(g) := \frac{1}{2\pi i}\frac{\partial g}{\partial z} - \frac{k}{4\pi y} g. \tag{2.6}$$

One can check (see [8] for the case of harmonic weak Maass forms) that this operator maps a weak Maass form of weight k and Laplace eigenvalue λ to a weak Maass form of weight $k+2$ and Laplace eigenvalue $\lambda + k$ with the same level and Nebentypus. We will refer to R_k as the *raising operator*.

3. Proof of Theorems 1.1 and 1.2

We recall the following relations for Whittaker functions, which we will make use of in the proof of Theorem 1.1.

$$W_{k,m}(y) = y^{\frac{1}{2}}\, W_{k-\frac{1}{2}, m-\frac{1}{2}}(y) + \left(\frac{1}{2} - k + m\right) W_{k-1,m}(y), \tag{3.1}$$

$$W_{k,m}(y) = y^{\frac{1}{2}}\, W_{k-\frac{1}{2}, m+\frac{1}{2}}(y) + \left(\frac{1}{2} - k - m\right) W_{k-1,m}(y), \tag{3.2}$$

$$y\frac{\partial}{\partial y}\left(W_{k,m}(y)\right) = \left(k - \frac{y}{2}\right) W_{k,m}(y) - \left[m^2 - \left(k - \frac{1}{2}\right)^2\right] W_{k-1,m}(y). \tag{3.3}$$

Here and in the following, for all occurring square roots we take a branch of the logarithm with a cut which does not intersect the real axis.

Proof of Theorem 1.1. We first suppose that w is even, and write $w = 2r$. Assume that we have a weight $\frac{1}{2}$ harmonic weak Maass form $f_{\frac{1}{2},\chi}(z)$ on $\Gamma_0(4b^2)$ with Nebentypus χ and coefficients in $\mathcal{O}_K$ (we delay the construction of $f_{\frac{1}{2},\chi}(z)$ to the end). Our proof proceeds by successively applying raising operators (2.6), yielding after ℓ steps ($0 < \ell \le r$) a weight $2\ell + \frac{1}{2}$ weak Maass form $f_{2\ell+\frac{1}{2},\chi}(z)$ on $\Gamma_0(4b^2)$ with Nebentypus χ and Laplace eigenvalue $\frac{\ell(2\ell-1)}{2}$. Computing the effect of the raising operators on Fourier expansions, we find that $f_{2\ell+\frac{1}{2},\chi}(z)$ has coefficients in $\mathcal{O}_K$ as well.

Define the functions $f_{2\ell+\frac{1}{2},\chi}(z)$ inductively by

$$f_{2(\ell+1)+\frac{1}{2},\chi}(z) := R_{2\ell+\frac{1}{2}}\left(f_{2\ell+\frac{1}{2},\chi}(z)\right)$$

for $0 \le \ell < r$, and define the coefficients $a_{2\ell+\frac{1}{2}}(n)$ and $\widetilde{a}_{2\ell+\frac{1}{2}}(n)$ (see (**??**)) for $0 \le \ell \le r$ by

$$f_{2\ell+\frac{1}{2},\chi}(z) =: \sum_{n\in\mathbb{Z}\setminus\{0\}} a_{2\ell+\frac{1}{2}}(n)\, \mathcal{W}_{\frac{3}{4}}^{2\ell+\frac{1}{2}}(4\pi ny)\, e^{2\pi inx} + \sum_{\substack{n=-n_0\\ n\neq 0}}^{m_0} \widetilde{a}_{2\ell+\frac{1}{2}}(n)\, \widetilde{\mathcal{W}}_{\frac{3}{4}}^{2\ell+\frac{1}{2}}(4\pi ny)\, e^{2\pi inx}$$
$$+ a_{2\ell+\frac{1}{2}}(0)(4\pi y)^{\frac{1}{2}-\ell} + \widetilde{a}_{2\ell+\frac{1}{2}}(0)(4\pi y)^{-\ell}$$

(note that we have chosen the $s = \frac{3}{4}$ solution of (2.5)). We now compute the action of $R_{2\ell+\frac{1}{2}}$ on a generic term of this expansion.

For the $n = 0$ case we simply observe that

$$\begin{aligned} R_{2\ell+\frac{1}{2}}\left((4\pi y)^{\frac{1}{2}-\ell}\right) &= -(1+\ell)(4\pi y)^{-\frac{1}{2}-\ell}, \\ R_{2\ell+\frac{1}{2}}\left((4\pi y)^{-\ell}\right) &= -\left(\frac{1}{2}+\ell\right)(4\pi y)^{-\ell-1}. \end{aligned}$$

Next suppose $n < 0$. We first claim that

$$R_{2\ell+\frac{1}{2}}\left(\mathcal{W}_{\frac{3}{4}}^{2\ell+\frac{1}{2}}(4\pi ny)\, e^{2\pi inx}\right) = n\left(\ell^2 + \frac{3\ell}{2} + \frac{1}{2}\right)\mathcal{W}_{\frac{3}{4}}^{2\ell+\frac{5}{2}}(4\pi ny)\, e^{2\pi inx}. \tag{3.4}$$

To see this, recall that for $y > 0$,

$$\mathcal{W}_{\frac{3}{4}}^{2\ell+\frac{1}{2}}(-y) = y^{-\ell-\frac{1}{4}}\, W_{-\ell-\frac{1}{4},\frac{1}{4}}(y).$$

Using (3.3) we find that

$$\frac{\partial}{\partial y}\left(W_{-\ell-\frac{1}{4},\frac{1}{4}}(y)\right) = \frac{1}{y}\left(-\ell - \frac{1}{4} - \frac{y}{2}\right)W_{-\ell-\frac{1}{4},\frac{1}{4}}(y) + \frac{1}{y}\left(\ell^2 + \frac{3\ell}{2} + \frac{1}{2}\right)W_{-\ell-\frac{5}{4},\frac{1}{4}}(y),$$

and therefore

$$\frac{\partial}{\partial y}\left(\mathcal{W}_{\frac{3}{4}}^{2\ell+\frac{1}{2}}(-y)\right) = \left(-2\ell - \frac{1}{2}\right)y^{-\ell-\frac{5}{4}}W_{-\ell-\frac{1}{4},\frac{1}{4}}(y)$$
$$- \frac{1}{2}y^{-\ell-\frac{1}{4}}W_{-\ell-\frac{1}{4},\frac{1}{4}}(y) + \left(\ell^2 + \frac{3}{2} + \frac{\ell}{2}\right)y^{-\ell-\frac{5}{4}}W_{-\ell-\frac{5}{4},\frac{1}{4}}(y).$$

It follows that

$$\frac{\partial}{\partial z}\left(\mathcal{W}_{\frac{3}{4}}^{2\ell+\frac{1}{2}}(-y)\, e^{-\frac{ix}{2}}\right) = i\left(\ell+\frac{1}{4}\right) y^{-\ell-\frac{5}{4}} W_{-\ell-\frac{1}{4},\frac{1}{4}}(y)\, e^{-\frac{ix}{2}} - \frac{i}{2}\left(\ell^2+\frac{3\ell}{2}+\frac{1}{2}\right) y^{-\ell-\frac{5}{4}} W_{-\ell-\frac{5}{4},\frac{1}{4}}(y)\, e^{-\frac{ix}{2}},$$

and hence

$$R_{2\ell+\frac{1}{2}}\left(\mathcal{W}_{\frac{3}{4}}^{2\ell+\frac{1}{2}}(4\pi ny)\, e^{2\pi inx}\right) = n\left(\ell^2+\frac{3\ell}{2}+\frac{1}{2}\right)(4\pi|n|y)^{-\ell-\frac{5}{4}} W_{-\ell-\frac{5}{4},\frac{1}{4}}(4\pi|n|y)\, e^{2\pi inx},$$

which is (3.4). In the same way we find that for $n>0$,

$$R_{2\ell+\frac{1}{2}}\left(\widetilde{\mathcal{W}}_{\frac{3}{4}}^{2\ell+\frac{1}{2}}(4\pi ny)\, e^{2\pi inx}\right) = -n\left(\ell^2+\frac{3\ell}{2}+\frac{1}{2}\right)\widetilde{\mathcal{W}}_{\frac{3}{4}}^{2\ell+\frac{5}{2}}(4\pi ny)\, e^{2\pi inx}.$$

Now suppose $n>0$. We claim that

$$R_{2\ell+\frac{1}{2}}\left(\mathcal{W}_{\frac{3}{4}}^{2\ell+\frac{1}{2}}(4\pi ny)\, e^{2\pi inx}\right) = n\,\mathcal{W}_{\frac{3}{4}}^{2\ell+\frac{5}{2}}(4\pi ny)\, e^{2\pi inx}. \tag{3.5}$$

To see this, begin by noting that for $y>0$,

$$\mathcal{W}_{\frac{3}{4}}^{2\ell+\frac{1}{2}}(y) = y^{-\ell-\frac{1}{4}}\, W_{\ell+\frac{1}{4},\frac{1}{4}}(y).$$

Using (3.3) we obtain

$$\frac{\partial}{\partial y}\left(W_{\ell+\frac{1}{4},\frac{1}{4}}(y)\right) = \frac{1}{y}\left(\ell+\frac{1}{4}-\frac{y}{2}\right) W_{\ell+\frac{1}{4},\frac{1}{4}}(y) + \frac{1}{y}\left(\ell^2-\frac{\ell}{2}\right) W_{\ell-\frac{3}{4},\frac{1}{4}}(y),$$

and thus

$$\frac{\partial}{\partial y}\left(\mathcal{W}_{\frac{3}{4}}^{2\ell+\frac{1}{2}}(y)\right) = -\frac{1}{2}y^{-\ell-\frac{1}{4}} W_{\ell+\frac{1}{4},\frac{1}{4}}(y) + \left(\ell^2-\frac{\ell}{2}\right) y^{-\ell-\frac{5}{4}} W_{\ell-\frac{3}{4},\frac{1}{4}}(y).$$

Hence we find that

$$\frac{\partial}{\partial z}\left(\mathcal{W}_{\frac{3}{4}}^{2\ell+\frac{1}{2}}(y)\, e^{\frac{ix}{2}}\right) = \frac{i}{2}y^{-\ell-\frac{1}{4}} W_{\ell+\frac{1}{4},\frac{1}{4}}(y)\, e^{\frac{ix}{2}} - \frac{i}{2}\left(\ell^2-\frac{\ell}{2}\right) y^{-\ell-\frac{5}{4}} W_{\ell-\frac{3}{4},\frac{1}{4}}(y)\, e^{\frac{ix}{2}},$$

which gives

$$R_{2\ell+\frac{1}{2}}\left(\mathcal{W}_{\frac{3}{4}}^{2\ell+\frac{1}{2}}(4\pi ny)\, e^{2\pi inx}\right) = n(4\pi ny)^{-\ell-\frac{1}{4}} W_{\ell+\frac{1}{4},\frac{1}{4}}(4\pi ny)\, e^{2\pi inx} \tag{3.6}$$
$$-n\left(\ell^2-\frac{\ell}{2}\right)(4\pi ny)^{-\ell-\frac{5}{4}} W_{\ell-\frac{3}{4},\frac{1}{4}}(4\pi ny)\, e^{2\pi inx} - n\left(2\ell+\frac{1}{2}\right)(4\pi ny)^{-\ell-\frac{5}{4}} W_{\ell+\frac{1}{4},\frac{1}{4}}(4\pi ny)\, e^{2\pi inx}.$$

Now, equations (3.1) and (3.2) yield

$$\begin{aligned} W_{\ell+\frac{5}{4},\frac{1}{4}}(y) &= y^{\frac{1}{2}}\, W_{\ell+\frac{3}{4},\frac{3}{4}}(y) + (-\ell-1)\, W_{\ell+\frac{1}{4},\frac{1}{4}}(y),\\ W_{\ell+\frac{3}{4},\frac{3}{4}}(y) &= y^{\frac{1}{2}}\, W_{\ell+\frac{1}{4},\frac{1}{4}}(y) + \left(\frac{1}{2}-\ell\right) W_{\ell-\frac{1}{4},\frac{3}{4}}(y),\\ y^{\frac{1}{2}}\, W_{\ell-\frac{1}{4},\frac{3}{4}}(y) &= W_{\ell+\frac{1}{4},\frac{1}{4}}(y) + \ell\, W_{\ell-\frac{3}{4},\frac{1}{4}}(y). \end{aligned}$$

Using these relations, we find that

$$W_{\ell+\frac{5}{4},\frac{1}{4}}(y) = -\left(2\ell+\frac{1}{2}\right) W_{\ell+\frac{1}{4},\frac{1}{4}}(y) + yW_{\ell+\frac{1}{4},\frac{1}{4}}(y) - \left(\ell^2-\frac{\ell}{2}\right) W_{\ell-\frac{3}{4},\frac{1}{4}}(y),$$

and combining this with (3.6) gives (3.5). In the same way we find that for $n<0$,

$$R_{2\ell+\frac{1}{2}}\left(\widetilde{\mathcal{W}}_{\frac{3}{4}}^{2\ell+\frac{1}{2}}(4\pi ny)\, e^{2\pi inx}\right) = -n\widetilde{\mathcal{W}}_{\frac{3}{4}}^{2\ell+\frac{5}{2}}(4\pi ny)\, e^{2\pi inx}.$$

Putting these results together gives, by linearity,

$$R_{2\ell+\frac{1}{2}}\left(f_{2\ell+\frac{1}{2},\chi}(z)\right) = \sum_{n=-\infty}^{-1} n\left(\ell^2+\frac{3\ell}{2}+\frac{1}{2}\right) a_{2\ell+\frac{1}{2}}(n)\, \mathcal{W}_{\frac{3}{4}}^{2\ell+\frac{5}{2}}(4\pi ny)\, e^{2\pi inx}$$
$$+\sum_{n=1}^{\infty} n\, a_{2\ell+\frac{1}{2}}(n)\, \mathcal{W}_{\frac{3}{4}}^{2\ell+\frac{5}{2}}(4\pi ny)\, e^{2\pi inx} - a_{2\ell+\frac{1}{2}}(0)(\ell+1)(4\pi y)^{-\frac{1}{2}-\ell} - \widetilde{a}_{2\ell+\frac{1}{2}}(0)\left(\frac{1}{2}+\ell\right)(4\pi y)^{-\ell-1}$$
$$-\sum_{n=-n_0}^{-1} n\, a_{2\ell+\frac{1}{2}}(n)\, \widetilde{\mathcal{W}}_{\frac{3}{4}}^{2\ell+\frac{5}{2}}(4\pi ny)\, e^{2\pi inx} - \sum_{n=1}^{m_0} n\left(\ell^2+\frac{3\ell}{2}+\frac{1}{2}\right) a_{2\ell+\frac{1}{2}}(n)\, \widetilde{\mathcal{W}}_{\frac{3}{4}}^{2\ell+\frac{5}{2}}(4\pi ny)\, e^{2\pi inx}.$$

From this we conclude that

$$\begin{aligned}(3.7)\quad f_{w+\frac{1}{2},\chi}(z) &= \mu_r\, a_{\frac{1}{2}}(0)(4\pi y)^{-\frac{1}{2}-r} + \tau_r\, \widetilde{a}_{\frac{1}{2}}(0) \\ &+ \sum_{n=-\infty}^{-1} n^r\, \rho_r\, a_{\frac{1}{2}}(n)\, \mathcal{W}_{\frac{3}{4}}^{2r+\frac{1}{2}}(4\pi ny)\, e^{2\pi inx} + \sum_{n=1}^{\infty} n^r\, a_{\frac{1}{2}}(n)\, \mathcal{W}_{\frac{3}{4}}^{2r+\frac{1}{2}}(4\pi ny)\, e^{2\pi inx} \\ &+ \sum_{n=-n_0}^{-1} n^r\, a_{\frac{1}{2}}(n)\, \widetilde{\mathcal{W}}_{\frac{3}{4}}^{2r+\frac{1}{2}}(4\pi ny) + \sum_{n=1}^{m_0} n^r\, \rho_r\, a_{\frac{1}{2}}(n)\, \widetilde{\mathcal{W}}_{\frac{3}{4}}^{2r+\frac{1}{2}}(4\pi ny)\, e^{2\pi inx}\ ,\end{aligned}$$

where

$$\begin{aligned}\rho_r &:= \prod_{j=0}^{r-1}\left(j^2+\frac{3}{2}j+\frac{1}{2}\right), \\ \mu_r &:= (-1)^r\prod_{j=0}^{r-1}(1+j), \\ \tau_r &:= (-1)^r\prod_{j=0}^{r-1}\left(\frac{1}{2}+j\right).\end{aligned}$$

We finish by showing how to construct $f_{\frac{1}{2},\chi}(z)$. By work of Shimura [16] the function

$$\Theta_\chi(z) := \sum_{n\in\mathbb{Z}} \chi(n)\, n\, q^{n^2}$$

is a weight $\frac{3}{2}$ cusp form on $\Gamma_0(4b^2)$ with Nebentypus χ. Using ideas from [4] and [5], one can construct a weak Maass form $f_{\frac{1}{2},\chi}(z)$ of weight $\frac{1}{2}$ and Nebentypus χ on $\Gamma_0(4b^2)$ such that

$f^{nh}_{\frac{1}{2},\chi}(z)$ is, up to change of variables, a constant multiple of

$$\int_{-\bar{z}}^{i\infty} \frac{\Theta_\chi(\tau)}{\sqrt{-i(\tau+z)}}\, d\tau. \tag{3.8}$$

To do this, one can go "backwards" in the proof of Theorem 1.1 of [4], where Ono and the first author constructed a harmonic weak Maass form starting with a holomorphic q-series. As in [5] one can instead start with the non-holomorphic function (3.8). Then, using the theory of Mittag-Leffler partial fraction decomposition one can rewrite the error integral produced in (3.8) by the transformation $z \mapsto -\frac{1}{z}$ as an integral of hyperbolic functions. One can then shift the path of integration, producing two q-series, one being the shifted integral (a series in $\widetilde{q} := e^{-2\pi i/z}$) and the other one coming from the residues (a series in $q := e^{2\pi i z}$). It is not hard to see that the expansion of (3.8) has coefficients in $\mathcal{O}_K$ for some number field K (see Proposition 4.1 of [4] for the computation of such an expansion). Moreover, the form $f^{h}_{\frac{1}{2},\chi}(z)$ constructed in this way has coefficients in $\mathcal{O}_K$. We leave the details to the reader.

We deal with the case that w is odd similarly, starting with a weight $\frac{3}{2}$ weak Maass form $f_{\frac{3}{2},\chi}(z)$. The corresponding theta series in this case is the weight $\frac{1}{2}$ form

$$\Theta_\chi(z) := \sum_{n\in\mathbb{Z}} \chi(n)\, q^{n^2}.$$

It turns out that the function $f^{h}_{\frac{3}{2},\chi}(z)$ corresponding to $f^{nh}_{\frac{3}{2},\chi}(z)$, which is essentially (3.8), is given as a linear combination of basic hypergeometric functions. For details we refer the reader to [4]. □

Proof of Theorem 1.2. Assume that $w = 2r$ is even (the odd case can be dealt with in an analogous way). As we have seen above, the function $f^{h}_{\frac{1}{2},\chi}(z)$ has coefficients in $\mathcal{O}_K$ for some number field K. Moreover, as in the proof of Proposition 4.1 in [4], we have that $f^{nh}_{\frac{1}{2},\chi}(z)$ has the form

$$f^{nh}_{\frac{1}{2},\chi}(z) = \sum_{n=1}^{\infty} \beta_n(y) q^{-dn^2}$$

for functions $\beta_n : \mathbb{R}^+ \to \mathbb{C}$ and some positive integer d. Then by Theorem 1.1 of [11], there are infinitely many nonnnested arithmetic progressions $An + B$ such that the coefficients $a^h_{\frac{1}{2}}(n)$ of $f^h_0(z)$ satisfy

$$a^h_{\frac{1}{2}}(An + B) \equiv 0 \pmod{p^j}$$

for all $n \geq 0$. By (3.7), we have that $a^h_{2r+\frac{1}{2}}(n) = n^r a^h_{\frac{1}{2}}(n)$, and therefore

$$a^h_{2r+\frac{1}{2}}(An + B) \equiv 0 \pmod{p^j}$$

for all $n \geq 0$. □

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