DYSON'S RANK, OVERPARTITIONS, AND WEAK MAASS FORMS

KATHRIN BRINGMANN AND JEREMY LOVEJOY

Abstract. In a series of papers the first author and Ono connected the rank, a partition statistic introduced by Dyson, to weak Maass forms, a new class of functions which are related to modular forms. Naturally it is of wide interest to find other explicit examples of Maass forms. Here we construct a new infinite family of such forms, arising from overpartitions. As applications we obtain combinatorial decompositions of Ramanujan-type congruences for overpartitions as well as the modularity of rank differences in certain arithmetic progressions.

1. Introduction and Statement of Results

A *partition* of a positive integer n is any non-increasing sequence of positive integers whose sum is n. Let $p(n)$ denote the number of partitions of n (with the usual convention that $p(0) := 1$, and $p(n) := 0$ for $n \notin \mathbb{N}_0$).

Ramanujan proved that for every positive integer n, we have:

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5},\\ p(7n+5) &\equiv 0 \pmod{7},\\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned} \tag{1.1}$$

In a celebrated paper Ono [27] treated these kinds of congruences systematically (also see [29]). Combining Shimura's theory of modular forms of half-integral weight with results of Serre on modular forms modulo ℓ he showed that for any prime $\ell \geq 5$ there exist infinitely many non-nested arithmetic progressions of the form $An + B$ such that

$$p(An+B) \equiv 0 \pmod{\ell}.$$

In order to explain the congruences in (1.1) with moduli 5 and 7 combinatorially, Dyson [16] introduced the rank of a partition. The *rank* of a partition is defined to be its largest part minus the number of its parts. Dyson conjectured that the partitions of $5n+4$ (resp. $7n+5$) form 5 (resp. 7) groups of equal size when sorted by their ranks modulo 5 (resp. 7). This conjecture was proven in 1954 by Atkin and Swinnerton-Dyer [2]. In [9] and [6], Ono and the first author showed that Dyson's rank partition function also satisfies congruences of Ramanujan type. One of the main steps in their proof is to show that generating functions related to the rank are the "holomorphic parts" of "weak Maass forms", a notion we will explain later. This new theory has many applications, such as congruences [9, 6] and asymptotics [5] for ranks as well as modularity for rank differences [10].

Naturally it is of wide interest to find other explicit examples of weak Maass forms. After partitions, the next place to look is overpartitions. Recall that an *overpartition* is a partition where the first occurrence of a summand may be overlined (see [13]). For example, there are 14 overpartitions of 4:

$$4, \overline{4}, 3+1, \overline{3}+1, 3+\overline{1}, \overline{3}+\overline{1}, 2+2, \overline{2}+2,$$

$$2+1+1, \overline{2}+1+1, 2+\overline{1}+1, \overline{2}+\overline{1}+1, 1+1+1+1, \overline{1}+1+1+1.$$

Overpartitions have arisen in many areas where ordinary partitions play an important role, most notably in q-series and combinatorics (e.g. [3, 11, 12, 13, 21, 30, 34]), but also in mathematical physics (e.g.

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[17, 18]), symmetric functions (e.g. [4, 15], representation theory (e.g. [19]) and algebraic number theory (e.g. [20, 24]). To give a few specific examples, the combinatorial theory of overpartitions leads to natural and straightforward bijective proofs of q-series identities like Ramanujan's ${}_1\psi_1$ summation [12, 34]; in the theory of symmetric functions in superspace, overpartitions play the role that partitions play in the classical theory of symmetric functions [17, 18]; and certain Dedekind zeta functions associated to rings of integers of real quadratic fields can be regarded as generating functions for weighted counts of overpartitions [20, 24].

Returning to Dyson's rank, this statistic applies just as well to overpartitions. Indeed, this rank and its generalizations have already proven fundamental in the combinatorial theory of overpartitions [14, 22, 23]. The main result of the present paper will be the construction of an infinite family of weak Maass forms whose holomorphic parts are related to the generating function for Dyson's rank of an overpartition. As applications, we discuss congruence properties of overpartitions and the modularity of rank differences in arithmetic progressions.

For a positive integer n we denote by $\overline{p}(n)$ the number of overpartions of n. We have the generating function [13]

$$\overline{P}(q) := \sum_{n\geq 0} \overline{p}(n)\, q^n = \frac{\eta(2z)}{\eta^2(z)} = 1 + 2q + 4q^2 + 8q^3 + 14q^4 + \cdots . \tag{1.2}$$

Here $\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n)$ is Dedekind's eta function and we write $q := e^{2\pi i z}$. Moreover we denote by $\overline{N}(m,n)$ the number of overpartitions of n with rank m. It is shown in [22] that

$$\begin{aligned} \mathcal{O}(u;q) &:= 1 + \sum_{n=1}^{\infty} \overline{N}(m,n) u^m q^n = \sum_{n=0}^{\infty} \frac{(-1)_n q^{\frac{1}{2}n(n+1)}}{(uq, q/u)_n} \\ &= \frac{(-q)_\infty}{(q)_\infty} \left(1 + 2 \sum_{n\geq 1} \frac{(1-u)\left(1-u^{-1}\right)(-1)^n q^{n^2+n}}{(1-uq^n)\left(1-u^{-1}q^n\right)} \right). \end{aligned} \tag{1.3}$$

Here for $a, b \in \mathbb{C}$, $n \in \mathbb{N} \cup \{\infty\}$, we employ the standard q-series notation:

$$\begin{aligned} (a)_n : &= \prod_{r=0}^{n-1} (1-aq^r), \\ (a,b)_n : &= \prod_{r=0}^{n-1} (1-aq^r)(1-bq^r), \\ (a)_\infty : &= \lim_{n\to\infty} (a)_n. \end{aligned}$$

It turns out that the function $\mathcal{O}(u;q)$ for u a root of unity $\neq 1$ is the holomorphic part of a weak Maass form.

To make this precise, we recall the notion of a weak Maass form of half-integral weight $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$. If $z = x + iy$ with $x, y \in \mathbb{R}$, then the weight k hyperbolic Laplacian is given by

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right). \tag{1.4}$$

If v is odd, then define ϵ_v by

$$\epsilon_v := \begin{cases} 1 & \text{if } v \equiv 1 \pmod 4, \\ i & \text{if } v \equiv 3 \pmod 4. \end{cases} \tag{1.5}$$

A *(harmonic) weak Maass form* of weight k and Nebentypus χ on a subgroup $\Gamma \subset \Gamma_0(4)$ is any smooth function $f : \mathbb{H} \to \mathbb{C}$ satisfying the following:

(1) For all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and all $z \in \mathbb{H}$, we have

$$f(Az) = \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} \chi(d)(cz+d)^k \ f(z).$$

(2) We have that $\Delta_k f = 0$.

(3) The function $f(z)$ has at most linear exponential growth at all the cusps of Γ.

Suppose that $0 < a < c$ are integers, and let $\zeta_c := e^{\frac{2\pi i}{c}}$. Define the theta function of weight $\frac{3}{2}$

$$\theta(\alpha, \beta; \tau) := \sum_{n \equiv \alpha \pmod{\beta}} n e^{\frac{2\pi i \tau n^2}{2\beta}}, \tag{1.6}$$

and let

$$\Theta_{a,c}(\tau) := \begin{cases} \theta\left(4a+c, 2c; \frac{\tau}{4c}\right) & \text{if } c \text{ is odd}, \\ 2\theta\left(2a+\frac{c}{2}, c; \frac{\tau}{2c}\right) & \text{if } 2 \parallel c, \\ 4\theta\left(a+\frac{c}{4}, \frac{c}{2}; \frac{\tau}{c}\right) & \text{if } 4|c. \end{cases}$$

Using these cuspidal theta functions, we define for $c \neq 2$, the non-holomorphic integral

$$J\left(\frac{a}{c}; z\right) := \frac{\pi i \cdot \tan\left(\frac{\pi a}{c}\right)}{4c} \int_{-\bar{z}}^{i\infty} \frac{(-i\tau)^{-\frac{3}{2}} \cdot \Theta_{a,b}\left(-\frac{1}{\tau}\right)}{\sqrt{-i(\tau+z)}} \, d\tau. \tag{1.7}$$

Moreover define $\mathcal{M}\left(\frac{a}{c}; z\right)$ by

$$\mathcal{M}\left(\frac{a}{c}; z\right) := \mathcal{O}\left(\frac{a}{c}; q\right) - J\left(\frac{a}{c}; z\right), \tag{1.8}$$

where $\mathcal{O}\left(\frac{a}{c}; q\right) := \mathcal{O}\left(\zeta_c^a; q\right)$. If $u = -1$, we define

$$\mathcal{M}(-1; z) := \mathcal{O}(-1; z) - I(-1; z).$$

with

$$I(-1; z) := \frac{\sqrt{2}}{\pi i} \int_{-\bar{z}}^{i\infty} \frac{\eta^2(\tau)}{\eta(2\tau) \cdot (-i(\tau+z))^{\frac{3}{2}}} \, d\tau.$$

The main result of this paper is the following theorem which establishes that those real analytic functions are Maass forms.

Theorem 1.1. *The following statements are true:*

(1) *If $0 < a < c$ with $(a, c) = 1$ and $c \neq 2$, then $\mathcal{M}\left(\frac{a}{c}; z\right)$ is a weak Maass form of weight $\frac{1}{2}$ on $\Gamma_1(16c^2)$. If $2|c$ and $4|c$, then it is a weak Maass form on $\Gamma_1(4c^2)$ and $\Gamma_1(c^2)$, respectively.*

(2) *The function $\mathcal{M}(-1; z)$ is a weak Maass form of weight $\frac{3}{2}$ on $\Gamma_0(16)$.*

Five remarks.

1) If c is odd, we actually obtain Maass forms for the larger group

$$\left\{ \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \middle| \alpha \equiv \delta \equiv 1 \pmod{4c}, \gamma \equiv 0 \pmod{16c^2} \right\}.$$

2) The proof of the second part of Theorem 1.1 is harder than the first since the generating function has double poles. To overcome this problem, we introduce new functions $\mathcal{O}_r(q)$ having an additional parameter r but only simple poles such that one can obtain $\mathcal{O}(-1; z)$ by a process of differentiation. This differentiation accounts for the augmentation of the weight by 1 in this case. It is worth mentioning that for the case of the classical Dyson's rank generating functions in [9], the weak Maass forms have weight 1/2 for every root of unity $\neq 1$.

3) The authors [7] show that in the context of overpartition pairs, the analogous generating functions associated to the appropriate generalization of Dyson's rank are not weak Maass forms, but classical modular forms.

4) We should stress that the analysis of the transformation behavior of $\mathcal{O}(u;q)$ is much more involved than in the case of the Dyson's rank generating functions in [9]. One of the reasons is that the half-integer weight modular form $\frac{(-q)_\infty}{(q)_\infty}$ that shows up in (1.3) is not mapped to itself as in the case of the usual ranks. This prohibits "guessing" images under Möbius transformations as in [9]. There the first author and Ono started with part of images of the generating function that they were able to guess. Thus the idea of proof in [9] which builds on old results of Watson, cannot be employed here. Instead we have to determine explicitly the images under all Möbius transformations with different techniques.

In view of Theorem 1.1 one can obtain results on overpartitions by arguing as in work of Ono and the first author [5, 6, 8, 9, 10]. In this direction we exhibit congruences for $\overline{N}(r,t;n)$, the number of overpartitions of n whose rank is congruent to $r \pmod{t}$, and provide a theoretical framework for proving identities for rank differences in arithmetic progressions. Other possible applications, which we do not address here, would be to asymptotics or inequalities for ranks, exact formulas or distribution questions. We first consider congruences satisfied by $\overline{N}(r,t;n)$. For ease of notation we restrict to the case that t is odd, the case t even can be considered similarly.

Theorem 1.2. *Let t be a positive odd integer, and let $\ell \nmid 6t$ be a prime. If j is a positive integer, then there are infinitely many non-nested arithmetic progressions $An+B$ such that for every $0 \leq r < t$ we have*

$$\overline{N}(r,t;An+B) \equiv 0 \pmod{\ell^j}.$$

Theorem 1.3. *Suppose that $\ell \geq 5$ is a prime, $m,u,\beta \in \mathbb{N}$ with $\left(\frac{-\beta}{\ell}\right) = -1$. Then a positive proportion of primes $p \equiv -1 \pmod{\ell}$ have the property that for every $0 \leq r \leq \ell^m - 1$*

$$\overline{N}\left(r,\ell^m;p^3 n\right) \equiv 0 \pmod{\ell^u}$$

for all $n \equiv \beta \pmod{\ell}$ that are not divisible by p.

This directly implies.

Corollary 1.4. *If $\ell \geq 5$ is a prime, $m,u \in \mathbb{N}$, then there are infinitely many non-nested arithmetic progressions $An+B$ such that*

$$\overline{N}\left(r,\ell^m;An+B\right) \equiv 0 \pmod{\ell^u}$$

for all $0 \leq r \leq \ell^m - 1$.

Remark.
The congruences in Theorems 1.2 and 1.3 may be viewed as a combinatorial decomposition of the overpartition function congruence

$$\overline{p}(An+B) \equiv 0 \pmod{\ell^u}. \tag{1.9}$$

That (1.9) holds for infinitely many non-nested arithmetic progressions $An+B$ was first observed by Treneer [32].

We next put identities involving rank differences for overpartitions in the framework of weak Maass forms (see also [10]). For this define for a prime ℓ and integers s_1 and s_2 the function

$$R_{s_1,s_2}(d) := \sum_{n=0}^{\infty} \left(\overline{N}(s_1,\ell,\ell n+d) - \overline{N}(s_2,\ell,\ell n+d)\right) q^{\ell n+d}.$$

We provide a framework that could be used to show an infinite family of identities (see also [10] for related results for usual ranks).

Theorem 1.5. *If $\left(\frac{d}{\ell}\right) = -\left(\frac{-1}{\ell}\right)$, then the function $R_{s_1,s_2}(d)$ is a weakly holomorphic modular form on $\Gamma_1\left(16\ell^4\right)$.*

Using Theorem 1.5, we could prove concrete identities using the valence formula. Since the computations are straightforward but lengthy (coming from the fact that $\Gamma_1\left(16\ell^4\right)$ has a lot of cusps), we chose not to prove individual identities. Instead we just list some identities, and their truth follows from work of the second author and Osburn [25].

The paper is organized as follows. In Section 2, we prove a transformation law for the rank generating functions in the case $c \neq 2$. In Section 3, we show the first part of Theorem 1.1. The main step is to recognize the Mordell type integrals occurring the transformation law of the rank generating functions as integrals of theta functions. In Section 4 we treat the case $c = 2$ which is more complicated due to double poles of the generating function. In Sections 5 and 6 we show congruences for $\overline{N}(r,t,n)$. Section 7 is dedicated the proof of Theorem 1.5.

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2. A transformation law

Here we consider modularity properties for $\mathcal{O}\left(\frac{a}{c};q\right)$. For this we need some notation. Let $c > 2$ and k be positive integers. Let $\widetilde{k}$ be either 0 or 1 depending on whether k is even or odd. Moreover let $k_1 := \frac{k}{(k,c)}$, $c_1 = \frac{c}{(c,k)}$, and define the integer $0 \leq l < c_1$ by the congruence $l \equiv ak_1 \pmod{c_1}$. If $\frac{b}{c} \in (0,1)$, then define the integers $s(b,c)$ and $t(b,c)$ (for $\frac{b}{c} \neq \frac{1}{2}$) by

$$s(b,c) := \begin{cases} 0 & \text{if } 0 < \frac{b}{c} \leq \frac{1}{4}, \\ 1 & \text{if } \frac{1}{4} < \frac{b}{c} \leq \frac{3}{4}, \\ 2 & \text{if } \frac{3}{4} < \frac{b}{c} < 1, \end{cases} \qquad t(b,c) := \begin{cases} 1 & \text{if } 0 < \frac{b}{c} < \frac{1}{2}, \\ 3 & \text{if } \frac{1}{2} < \frac{b}{c} < 1. \end{cases}$$

In particular let $s := s(l,c_1)$ and $t := t(l,c_1)$. Let h' be defined by $hh' \equiv -1 \pmod{k}$. Moreover let $\omega_{h,k}$ be given by

$$\omega_{h,k} := \exp\left(\pi i \sum_{\mu \pmod{k}} \left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{h\mu}{k}\right)\right)\right), \tag{2.1}$$

where

$$((x)) := \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

Define for $q = e^{2\pi i z}$ the following functions.

$$\begin{aligned}
\mathcal{U}\left(\frac{a}{c};q\right) &:= \mathcal{U}\left(\frac{a}{c};z\right) := \sin\left(\frac{\pi a}{c}\right)\frac{\eta\left(\frac{z}{2}\right)}{\eta^2(z)}\sum_{n\in\mathbb{Z}}\frac{(1+q^n)\,q^{n^2+\frac{n}{2}}}{1-2q^n\cos\left(\frac{2\pi a}{c}\right)+q^{2n}},\\
\mathcal{U}(a,b,c;q) &:= \mathcal{U}(a,b,c;z) := \frac{\eta\left(\frac{z}{2}\right)}{\eta^2(z)}e^{\frac{\pi i a}{c}\left(\frac{4b}{c}-1-2s(b,c)\right)}q^{\frac{s(b,c)}{c}+\frac{b}{2c}-\frac{b^2}{c^2}}\sum_{m\in\mathbb{Z}}\frac{q^{\frac{m}{2}(2m+1)+ms(b,c)}}{1-e^{-\frac{2\pi i a}{c}}\,q^{m+\frac{b}{c}}},\\
\mathcal{V}(a,b,c;q) &:= \mathcal{V}(a,b,c;z) := \frac{\eta\left(\frac{z}{2}\right)}{\eta^2(z)}e^{\frac{\pi i a}{c}\left(\frac{4b}{c}-1-2s(b,c)\right)}\cdot q^{\frac{s(b,c)b}{c}+\frac{b}{2c}-\frac{b^2}{c^2}}\sum_{m\in\mathbb{Z}}\frac{q^{\frac{m}{2}(2m+1)+ms(b,c)}}{1-e^{-\frac{2\pi i a}{c}}\cdot q^{m+\frac{b}{c}}},\\
\mathcal{O}(a,b,c;q) &:= \mathcal{O}(a,b,c;z) := \frac{\eta(2z)}{\eta^2(z)}e^{\frac{\pi i a}{c}\left(\frac{4b}{c}-1-t(b,c)\right)}\cdot q^{\frac{t(b,c)b}{2c}+\frac{b}{2c}-\frac{b^2}{c^2}}\sum_{m\in\mathbb{Z}}(-1)^m\frac{q^{\frac{m}{2}(2m+1)+\frac{mt(b,c)}{2}}}{1-e^{-\frac{2\pi i a}{c}}\cdot q^{m+\frac{b}{c}}},\\
\mathcal{V}\left(\frac{a}{c};q\right) &:= \mathcal{V}\left(\frac{a}{c};z\right) := \frac{\eta(2z)}{\eta^2(z)}q^{\frac{1}{4}}\sum_{m\in\mathbb{Z}}\frac{q^{m^2+m}\cdot\left(1+e^{-\frac{2\pi i a}{c}}\cdot q^{m+\frac{1}{2}}\right)}{1-e^{-\frac{2\pi i a}{c}}\cdot q^{m+\frac{1}{2}}}.
\end{aligned}$$

Moreover let

$$H_{a,c}(x) := \frac{e^x}{1-2\cos\left(\frac{2\pi a}{c}\right)e^x+e^{2x}}.$$

Then

$$\begin{aligned}
H_{a,c}(-x) &= H_{a,c}(x),\\
H_{a,c}(x+2\pi i) &= H_{a,c}(x).
\end{aligned}$$

Moreover define for an integer ν the Mordell type integral

$$I_{a,c,k,\nu}(w) := \int_{\mathbb{R}} e^{-\frac{2\pi w x^2}{k}}H_{a,c}\left(\frac{2\pi i\nu}{k}-\frac{2\pi w x}{k}-\frac{\widetilde{k}\pi i}{2k}\right)dx.$$

For k even we have to take the principal part of the integral. We are now ready to show the transformation law of $\mathcal{O}\left(\frac{a}{c};q\right)$.

Theorem 2.1. *Assume the notation above. Moreover, let $w\in\mathbb{C}$ with $\mathrm{Re}(w)>0$, $q := e^{\frac{2\pi i}{k}(h+iw)}$, and $q_1 := e^{\frac{2\pi i}{k}\left(h'+\frac{i}{w}\right)}$.*

(1) *If $c|k$ and k is even, then we have*

$$\mathcal{O}\left(\frac{a}{c};q\right) = (-1)^{k_1}i\cdot e^{-\frac{2\pi i a^2h'k_1}{c}}\cdot\tan\left(\frac{\pi a}{c}\right)\cdot\cot\left(\frac{\pi a h'}{c}\right)\frac{\omega_{h,k}^2}{\omega_{h,k/2}}w^{-\frac{1}{2}}\cdot\mathcal{O}\left(\frac{ah'}{c};q_1\right)$$
$$+\frac{4\cdot\sin^2\left(\frac{\pi a}{c}\right)\cdot\omega_{h,k}^2}{\omega_{h,k/2}\cdot k}\cdot w^{\frac{1}{2}}\sum_{\nu\pmod k}(-1)^{\nu}e^{-\frac{2\pi i h'\nu^2}{k}}\cdot I_{a,c,k,\nu}(w).$$

(2) *If $c|k$ and k is odd, then we have*

$$\mathcal{O}\left(\frac{a}{c};q\right) = \sqrt{2}i\cdot e^{\frac{\pi i h'}{8k}-\frac{2\pi i a^2h'k_1}{c}}\tan\left(\frac{\pi a}{c}\right)\frac{\omega_{h,k}^2}{\omega_{2h,k}}\cdot w^{-\frac{1}{2}}\cdot\mathcal{U}\left(\frac{ah'}{c};q_1\right)$$
$$+\frac{4\sqrt{2}\cdot\sin^2\left(\frac{\pi a}{c}\right)\cdot\omega_{h,k}^2}{\omega_{2h,k}\cdot k}\cdot w^{\frac{1}{2}}\sum_{\nu\pmod k}e^{-\frac{\pi i h'}{k}(2\nu^2-\nu)}\cdot I_{a,c,k,\nu}(w).$$

(3) *If $c \nmid k$, $2|k$, and $c_1 \neq 2$, then we have*

$$\mathcal{O}\left(\frac{a}{c};q\right) = -2e^{-\frac{2\pi i a^2 h' k_1}{c_1 c}} \tan\left(\frac{\pi a}{c}\right) \frac{\omega_{h,k}^2}{\omega_{h,k/2}} w^{-\frac{1}{2}} \cdot (-1)^{c_1(l+k_1)} \mathcal{O}\left(ah', \frac{lc}{c_1}, c; q_1\right) + \frac{4\sin^2\left(\frac{\pi a}{c}\right)\cdot \omega_{h,k}^2}{\omega_{h,k/2}\cdot k} \cdot w^{\frac{1}{2}} \sum_{\nu \pmod{k}} (-1)^\nu e^{-\frac{2\pi i h' \nu^2}{k}} \cdot I_{a,c,k,\nu}(w).$$

(4) *If $c \nmid k$, $2|k$, and $c_1 = 2$, then we have*

$$\mathcal{O}\left(\frac{a}{c};q\right) = -e^{-\frac{\pi i a^2 h' k_1}{c}} \cdot \tan\left(\frac{\pi a}{c}\right) \frac{\omega_{h,k}^2}{\omega_{h,k/2}} \cdot w^{-\frac{1}{2}} \cdot \mathcal{V}\left(\frac{ah'}{c}; q_1\right) + \frac{4\sin^2\left(\frac{\pi a}{c}\right)\cdot \omega_{h,k}^2}{\omega_{h,k/2}\cdot k} \cdot w^{\frac{1}{2}} \sum_{\nu \pmod{k}} (-1)^\nu e^{-\frac{2\pi i h' \nu^2}{k}} \cdot I_{a,c,k,\nu}(w).$$

(5) *If $c \nmid k$, $2 \nmid k$, and $c_1 \neq 4$, then we have*

$$\mathcal{O}\left(\frac{a}{c};q\right) = -\sqrt{2}e^{\frac{\pi i h'}{8k} - \frac{2\pi i h' a^2 k_1}{cc_1}} \cdot \tan\left(\frac{\pi a}{c}\right) \frac{\omega_{h,k}^2}{\omega_{2h,k}} w^{-\frac{1}{2}} \cdot \mathcal{U}\left(ah', \frac{lc}{c_1}, c; q_1\right) + \frac{4\sqrt{2}\cdot\sin^2\left(\frac{\pi a}{c}\right)\cdot \omega_{h,k}^2}{\omega_{2h,k}\cdot k} \cdot w^{\frac{1}{2}} \sum_{\nu \pmod{k}} e^{-\frac{\pi i h'}{k}(2\nu^2-\nu)} \cdot I_{a,c,k,\nu}(w).$$

(6) *If $c \nmid k$, $2 \nmid k$, and $c_1 = 4$, then we have*

$$\mathcal{O}\left(\frac{a}{c};q\right) = -e^{\frac{\pi i h'}{8k} - \frac{2\pi i h' a^2 k_1}{cc_1}} \cdot \tan\left(\frac{\pi a}{c}\right) \frac{\omega_{h,k}^2}{\sqrt{2}\cdot\omega_{2h,k}} \cdot w^{-\frac{1}{2}} \cdot \mathcal{V}\left(ah', \frac{lc}{c_1}, c; q_1\right) + \frac{4\sqrt{2}\cdot\sin^2\left(\frac{\pi a}{c}\right)\cdot \omega_{h,k}^2}{\omega_{2h,k}\cdot k} \cdot w^{\frac{1}{2}} \sum_{\nu \pmod{k}} e^{-\frac{\pi i h'}{k}(2\nu^2-\nu)} \cdot I_{a,c,k,\nu}(w).$$

Corollary 2.2. *Assume that $z \in \mathbb{H}$, $0 < a < c$ with $c \neq 2$.*

(1) *If $c \neq 4$, then we have*

$$\mathcal{O}\left(\frac{a}{c}; -\frac{1}{z}\right) = -\sqrt{2}\tan\left(\frac{\pi a}{c}\right)\cdot(-iz)^{\frac{1}{2}}\cdot\mathcal{U}(0,a,c;z) + 4\sqrt{2}\cdot\sin^2\left(\frac{\pi a}{c}\right)\cdot(-iz)^{-\frac{1}{2}}\cdot I_{a,c,1,0}\left(\frac{i}{z}\right).$$

(2) *If $c = 4$, then we have*

$$\mathcal{O}\left(\frac{a}{c}; -\frac{1}{z}\right) = -\frac{\tan\left(\frac{\pi a}{c}\right)}{\sqrt{2}}(-iz)^{\frac{1}{2}}\cdot\mathcal{V}(0,a,c;z) + 4\sqrt{2}\cdot\sin^2\left(\frac{\pi a}{c}\right)\cdot(-iz)^{-\frac{1}{2}}\cdot I_{a,c,1,0}\left(\frac{i}{z}\right).$$

Corollary 2.3. *Assume that $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \Gamma_0(c)$ with c odd, $z \in \mathbb{H}$, and let $\gamma_1 := \frac{\gamma}{(c,\gamma)}$.*

(1) *If $2|\gamma$, then the holomorphic part of $\mathcal{O}\left(\frac{a}{c}; \frac{\alpha z+\beta}{\gamma z+\delta}\right)$ is given by*

$$-i\cdot e^{\frac{2\pi i a^2\delta\gamma_1}{c}}\cdot\tan\left(\frac{\pi a}{c}\right)\cdot\cot\left(\frac{\pi a\delta}{c}\right)\frac{\omega_{\alpha,\gamma}^2}{\omega_{\alpha,\gamma/2}}(-i(\gamma z+\delta))^{\frac{1}{2}}\cdot\mathcal{O}\left(\frac{a\delta}{c};z\right).$$

(2) *If γ is odd, then the holomorphic part of $\mathcal{O}\left(\frac{a}{c}; \frac{\alpha z+\beta}{\gamma z+\delta}\right)$ is given by*

$$\sqrt{2}i\cdot e^{-\frac{\pi i\delta}{8\gamma}+\frac{2\pi i a^2\delta\gamma_1}{c}}\tan\left(\frac{\pi a}{c}\right)\frac{\omega_{\alpha,\gamma}^2}{\omega_{2\alpha,\gamma}}\cdot(-i(\gamma z+\delta))^{\frac{1}{2}}\cdot\mathcal{U}\left(-\frac{a\delta}{c};z\right).$$

Proof of Theorem 2.1. We proceed similarly as in [1, 5]. First we rewrite (1.3) as

$$\mathcal{O}\left(\frac{a}{c};q\right)=4\sin^2\left(\frac{\pi a}{c}\right)\frac{\eta\left(\frac{2}{k}(h+iw)\right)}{\eta^2\left(\frac{1}{k}(h+iw)\right)}\sum_{n\in\mathbb{Z}}(-1)^n\,e^{\frac{2\pi i n^2(h+iw)}{k}}\cdot H_{a,c}\left(\frac{2\pi i n}{k}(h+iw)\right). \tag{2.2}$$

Now let

$$\widetilde{\mathcal{O}}\left(\frac{a}{c};q\right):=\frac{\eta^2\left(\frac{1}{k}(h+iw)\right)}{4\sin^2\left(\frac{\pi a}{c}\right)\cdot\eta\left(\frac{2}{k}(h+iw)\right)}\cdot\mathcal{O}\left(\frac{a}{c};q\right). \tag{2.3}$$

Writing $n=km+\nu$ with $0\leq\nu<k$, $m\in\mathbb{Z}$ gives that $\widetilde{\mathcal{O}}\left(\frac{a}{c};q\right)$ equals

$$\sum_{\nu=0}^{k-1}(-1)^\nu\,e^{\frac{2\pi i h\nu^2}{k}}\sum_{m\in\mathbb{Z}}(-1)^{km}\;H_{a,c}\left(\frac{2\pi i h\nu}{k}-\frac{2\pi w}{k}(km+\nu)\right)\cdot e^{-\frac{2\pi w}{k}(km+\nu)^2}. \tag{2.4}$$

Using Poisson summation and substituting $x\mapsto kx+\nu$ gives that the inner sum equals

$$\frac{1}{k}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}H_{a,c}\left(\frac{2\pi i h\nu}{k}-\frac{2\pi xw}{k}\right)\cdot e^{\frac{\pi i}{k}(2n+\widetilde{k})(x-\nu)-\frac{2\pi w x^2}{k}}dx. \tag{2.5}$$

Strictly speaking for $c|k$ there may lie a pole at $x=0$. In this case we take the principal part of the integral. Inserting (2.5) into (2.4) we see that the summation only depends on $\nu \pmod k$. Moreover, by changing ν into $-\nu$, x into $-x$, and n into $-(n+\widetilde{k})$, we see that the part of the sum over n with $n\leq -1$ equals the part of the sum with $n\geq 0$. Thus (2.4) equals

$$\frac{2}{k}\sum_{\nu \pmod k}(-1)^\nu e^{\frac{2\pi i h\nu^2}{k}}\sum_{n\in\mathbb{N}}\int_{\mathbb{R}}H_{a,c}\left(\frac{2\pi i h\nu}{k}-\frac{2\pi xw}{k}\right)\cdot e^{\frac{\pi i}{k}(2n+\widetilde{k})(x-\nu)-\frac{2\pi w x^2}{k}}dx. \tag{2.6}$$

Next we introduce the function

$$S_{a,c,k}(x):=\frac{\sinh(c_1x)}{\sinh\left(\frac{x}{k}+\frac{\pi i a}{c}\right)\cdot\sinh\left(\frac{x}{k}-\frac{\pi i a}{c}\right)}$$

which is entire as a function of x. Here we need that $c\neq 2$. We rewrite the integrand in (2.6) as

$$\frac{(-1)^{hc_1\nu}e^{\frac{\pi i(2n+\widetilde{k})(x-\nu)}{k}-\frac{2\pi w x^2}{k}}\cdot S_{a,c,k}\left(\pi xw-\pi i h\nu\right)}{4\sinh(\pi c_1 xw)}.$$

From this we see that the only poles can lie in the points

$$x_m:=\frac{im}{c_1w}\qquad(m\in\mathbb{Z}).$$

If $c|k$, then $c_1=1$; thus poles can only lie in points of the form $x_m=\frac{im}{w}$. One can easily compute that each choice $\pm$ leads at most for one $\nu \pmod k$ to a non-zero residue, and that this ν can be chosen as

$$\nu_m^{\pm}:=-h'(m\mp ak_1).$$

If $c\nmid k$, then we can only have a nontrivial residue if $m\equiv\pm ak_1 \pmod{c_1}$. We write $c_1m\pm l$ instead of m with $m\geq\frac{1}{2}(1\mp 1)$. We see that to each choice $+$ or $-$ there corresponds exactly one $\nu \pmod k$ and we can choose ν as

$$\nu_m^{\pm}:=-h'\left(m\pm\frac{1}{c_1}\left(l-ak_1\right)\right).$$

Now shift the path of integration through the points

$$\omega_n:=\frac{\left(2n+\widetilde{k}\right)i}{4w}.$$

Which points x_m $(m \geq 0)$ we have to take into account when we use the Residue Theorem depends on whether $c|k$ or not and on whether k is even or odd. The cases that $c_1 = 2, 4$ require special care.

- If $c|k$ and k is even, then we have to take those x_m into account for which $2m \leq n$. The poles on the path of integration are x_0 and ω_n.
- If $c|k$ and k is odd, then we have to take those x_m into account for which $2m \leq n$. The point x_0 is the only pole on the path of integration .
- If $c \nmid k$, k is even, and $c_1 \neq 2$, then there is no pole on the path of integration. Moreover in this case we have to take those x_m into account for which $n \geq 2m + \frac{1}{2}(1 \pm t)$.
- If $c \nmid k$, k is even, and $c_1 = 2$, then the only pole on the path of integartion lies in ω_n. We have to take those x_m into account for which $n \geq 2m \pm 1$.
- If $c \nmid k$, k is odd, and $c_1 \neq 4$, then there is no pole on the path of integration. Moreover we have to take those x_m into account for which $n \geq 2m \pm s$.
- If $c \nmid k$, k is odd, and $c_1 = 4$, then there lies a pole in ω_n and we have to take those x_m into account for which $n \geq 2m \pm s + \frac{1}{2}(-1 \pm 1)$.

In the following we denote the residues of the integrand by $\lambda^{\pm}_{n,m}$. It is not hard to compute

$$\lambda^{\pm}_{n,m} = \pm \frac{ik}{4\pi w \sin\left(\frac{2\pi a}{c}\right)} \, e^{\frac{\pi i}{k}(2n+\widetilde{k})\left(x_m - \nu^{\pm}_m\right) - \frac{2\pi w x_m^2}{k}}.$$

¿From this one directly sees that

$$\lambda^{\pm}_{n+1,m} = \exp\left(\frac{2\pi i}{k}\left(x_m - \nu^{\pm}_m\right)\right) \cdot \lambda^{\pm}_{n,m}.$$

Shifting the path of integration through ω_n, we obtain by the Residue Theorem

$$\widetilde{O}\left(\frac{a}{c}; q\right) = \sum_1 + \sum_2,$$

where

$$\sum_2 := \frac{2}{k} \sum_{\nu \pmod{k}} (-1)^{\nu} \, e^{\frac{2\pi i h \nu^2}{k}} \sum_{n \in \mathbb{N}} \int_{-\infty+\omega_n}^{\infty+\omega_n} H_{a,c}\left(\frac{2\pi i h \nu}{k} - \frac{2\pi x w}{k}\right) \cdot e^{\frac{\pi i (2n+\widetilde{k})(x-\nu)}{k} - \frac{2\pi x^2 w}{k}} \, dx.$$

For the definition of $\sum_1$ we have to distinguish several cases. We set $r_0 := \frac{1}{2}$ and $r_m := 1$ for $m \in \mathbb{N}$. If $c|k$ and k is even, then

$$\sum_1 := \frac{4\pi i}{k} \sum_{\substack{m \geq 0 \\ \epsilon \in \{\pm\}}} r_m (-1)^{\nu^{\epsilon}_m} \, e^{\frac{2\pi i h (\nu^{\epsilon}_m)^2}{k}} \left(\frac{\lambda^{\epsilon}_{2m+1,m}}{1 - \exp\left(\frac{2\pi i}{k}\left(x_m - \nu^{\epsilon}_m\right)\right)} + \frac{1}{2}\lambda^{\epsilon}_{2m,m}\right).$$

An easy calculation shows that this equals

$$(-1)^{k_1} i e^{-\frac{2\pi i h' a^2 k_1}{c}} \frac{\sin\left(\frac{2\pi h' a}{c}\right)}{w \cdot \sin\left(\frac{2\pi a}{c}\right)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q_1^{n^2+n}}{1 - 2q_1^n \cos\left(\frac{2\pi a h'}{c}\right) + q_1^{2n}}. \tag{2.7}$$

If $c|k$ and $2 \nmid k$, then

$$\sum_1 := \frac{4\pi i}{k} \sum_{\substack{m \geq 0 \\ \epsilon \in \{\pm\}}} r_m (-1)^{\nu^{\epsilon}_m} \, e^{\frac{2\pi i h (\nu^{\epsilon}_m)^2}{k}} \frac{\lambda^{\epsilon}_{2m,m}}{1 - \exp\left(\frac{2\pi i}{k}\left(x_m - \nu^{\epsilon}_m\right)\right)}. \tag{2.8}$$

We assume without loss of generality that h' is even. Then we can show that (2.8) equals

$$\frac{i \sin\left(\frac{\pi h' a}{c}\right)}{w \cdot \sin\left(\frac{2\pi a}{c}\right)} \, e^{-\frac{2\pi i h' a^2 k_1}{c}} \sum_{n \in \mathbb{Z}} \frac{\left(1 + q_1^n\right) q_1^{n^2+\frac{n}{2}}}{1 - 2q_1^n \cos\left(\frac{2\pi a h'}{c}\right) + q_1^{2n}}. \tag{2.9}$$

If $c \nmid k$, $2|k$, and $c_1 \neq 2$, then

$$\sum_1 := \frac{4\pi i}{k} \sum_{\substack{m\geq 0\\ \epsilon\in\{\pm\}}} (-1)^{\nu_m^\epsilon}\ e^{\frac{2\pi i h (\nu_m^\epsilon)^2}{k}} \frac{\lambda^\epsilon_{2m+\frac12(1\pm t),m}}{1-\exp\left(\frac{2\pi i}{k}\left(x_m-\nu_m^\epsilon\right)\right)}.$$

It can be calculated that this equals

$$-\frac{1}{w\sin\left(\frac{2\pi a}{c}\right)} q_1^{\frac{tl}{2c_1}-\frac{l^2}{c_1^2}+\frac{l}{2c_1}} \cdot e^{\frac{\pi i h' a}{c}\left(\frac{4l}{c_1}-\frac{2ak_1}{c_1}-t-1\right)} (-1)^{c_1(l+k_1)} \sum_{m\in\mathbb{Z}} (-1)^m \frac{q_1^{\frac{m}{2}(2m+1)+\frac{mt}{2}}}{1-e^{-\frac{2\pi i a h'}{c}}\cdot q_1^{m+\frac{l}{c_1}}}.$$

If $c \nmid k$, $2|k$, and $c_1 = 2$, then

$$\sum_1 := \frac{4\pi i}{k}\left(\sum_{m\geq 0}(-1)^{\nu_m^+}\ e^{\frac{2\pi i h(\nu_m^+)^2}{k}}\left(\frac{\lambda^+_{2m+2,m}}{1-\exp\left(\frac{2\pi i}{k}\left(x_m-\nu_m^+\right)\right)}+\frac12\lambda^+_{2m+1,m}\right)\right.$$
$$\left.+\sum_{m\geq 1}(-1)^{\nu_m^-}\ e^{\frac{2\pi i h(\nu_m^-)^2}{k}}\left(\frac{\lambda^-_{2m,m}}{1-\exp\left(\frac{2\pi i}{k}\left(x_m-\nu_m^-\right)\right)}+\frac12\lambda^-_{2m-1,m}\right)\right).$$

One can show that this equals.

$$-\frac{q_1^{\frac14}}{2w\sin\left(\frac{2\pi a}{c}\right)}\, e^{-\frac{\pi i a^2 h' k_1}{c}} \sum_{m\in\mathbb{Z}} \frac{q_1^{m^2+m}\left(1+e^{-\frac{2\pi i h' a}{c}} q_1^{m+\frac12}\right)}{1-e^{-\frac{2\pi i a h'}{c}}\cdot q_1^{m+\frac12}}. \tag{2.10}$$

If $c \nmid k$, k is odd, and $c_1 \neq 4$, then

$$\sum_1 := \frac{4\pi i}{k}\left(\sum_{m\geq 0}(-1)^{\nu_m^+}\ e^{\frac{2\pi i h(\nu_m^+)^2}{k}}\frac{\lambda^+_{2m+s,m}}{1-\exp\left(\frac{2\pi i}{k}\left(x_m-\nu_m^+\right)\right)}\right.$$
$$\left.+\sum_{m\geq 1}(-1)^{\nu_m^-}\ e^{\frac{2\pi i h(\nu_m^-)^2}{k}}\frac{\lambda^-_{2m-s,m}}{1-\exp\left(\frac{2\pi i}{k}\left(x_m-\nu_m^-\right)\right)}\right).$$

Without loss of generality we may assume that h' is even. With this assumption we can compute that $\sum_1$ equals

$$-\frac{1}{w\sin\left(\frac{2\pi a}{c}\right)} q_1^{\frac{sl}{c_1}-\frac{l^2}{c_1^2}+\frac{l}{2c_1}} \cdot e^{\frac{\pi i h' a}{c}\left(\frac{4l}{c_1}-\frac{2ak_1}{c_1}-2s-1\right)} \sum_{m\in\mathbb{Z}} \frac{q_1^{\frac{m}{2}(2m+1)+ms}}{1-e^{-\frac{2\pi i a h'}{c}}\cdot q_1^{m+\frac{l}{c_1}}}.$$

If $c \nmid k$, k is odd, and $c_1 = 4$, then

$$\sum_1 := \frac{4\pi i}{k}\left(\sum_{m=0}^{\infty}(-1)^{\nu_m^+}\ e^{\frac{2\pi i h(\nu_m^+)^2}{k}}\left(\frac{\lambda^+_{2m+s+1,m}}{1-\exp\left(\frac{2\pi i}{k}\left(x_m-\nu_m^+\right)\right)}+\frac12\lambda_{2m+s,m}\right)\right.$$
$$\left.+\sum_{m=1}^{\infty}(-1)^{\nu_m^-}\ e^{\frac{2\pi i h(\nu_m^-)^2}{k}}\left(\frac{\lambda^-_{2m-s,m}}{1-\exp\left(\frac{2\pi i}{k}\left(x_m-\nu_m^-\right)\right)}+\frac12\lambda_{2m-s-1,m}\right)\right).$$

One can show that this equals

$$-\frac{1}{2w\sin\left(\frac{2\pi a}{c}\right)} q_1^{\frac{sl}{c_1}-\frac{l^2}{c_1^2}+\frac{l}{2c_1}} e^{\frac{\pi i h' a}{c}\left(\frac{4l}{c_1}-\frac{ak_1}{2c_1}-2s\right)} \sum_{m\in\mathbb{Z}} \frac{q_1^{\frac{m}{2}(2m+1)+ms}\left(1+e^{-\frac{2\pi i a h'}{c}} q_1^{m+\frac{l}{c_1}}\right)}{1-e^{-\frac{2\pi i a h'}{c}}\cdot q_1^{m+\frac{l}{c_1}}}. \tag{2.11}$$

We next turn to the computation of $\sum_2$. If there is a pole in ω_n we take the principal part of the integral. With the same argument as before we can change the sum over $\mathbb{N}$ into a sum over $\mathbb{Z}$. Moreover we make the translation $x \mapsto x + \omega_n$ and write $n = 2p + \delta$ with $p \in \mathbb{Z}$ and $\delta \in \{0, -1\}$. This gives

$$\sum_2 = \frac{1}{k} \sum_{\nu \pmod{k}} (-1)^\nu \, e^{\frac{2\pi i h \nu^2}{k}} \sum_{\substack{p\in\mathbb{Z}\\ \delta\in\{0,-1\}}} e^{-\frac{\pi i}{k}(4p+2\delta+\widetilde{k})\nu - \frac{\pi}{8kw}(4p+2\delta+\widetilde{k})^2}$$
$$\int_{-\infty}^{\infty} H_{a,c}\left(\frac{2\pi i h\nu}{k} - \frac{2\pi w x}{k} - \frac{\pi i}{2k}\left(4p+2\delta+\widetilde{k}\right)\right) \cdot e^{-\frac{2\pi w x^2}{k}}\, dx.$$

Next we change ν into $-h'(\nu+p)$ and distinguish whether k is even or odd.

If k is even, then we have, since h' is odd,

$$\sum_2 = \frac{1}{k} \sum_{\substack{\nu \pmod{k}\\ \delta\in\{0,-1\}}} (-1)^\nu \, e^{\frac{\pi i h'}{2k}\left(-\delta^2+4\nu(\delta-\nu)\right)} \sum_{p\in\mathbb{Z}} (-1)^p \, q_1^{\frac{(4p+2\delta)^2}{16}}$$
$$\int_{-\infty}^{\infty} H_{a,c}\left(\frac{2\pi i\nu}{k} - \frac{2\pi w x}{k} - \frac{\pi i\delta}{k}\right) \cdot e^{-\frac{2\pi w x^2}{k}}\, dx.$$

Now the integral is independent of p. Moreover the sum over p vanishes for $\delta = -1$ since the pth and the $(-p+1)$th term cancel. For $\delta = 0$ the sum over p equals

$$\sum_{p\in\mathbb{Z}} (-1)^p \cdot q_1^{p^2} = \frac{\eta^2\left(\frac{1}{k}\left(h' + \frac{i}{w}\right)\right)}{\eta\left(\frac{2}{k}\left(h' + \frac{i}{w}\right)\right)}. \tag{2.12}$$

Thus

$$\sum_2 = \frac{\eta^2\left(\frac{1}{k}\left(h' + \frac{i}{w}\right)\right)}{k \cdot \eta\left(\frac{2}{k}\left(h' + \frac{i}{w}\right)\right)} \sum_{\nu \pmod{k}} (-1)^\nu \, e^{-\frac{2\pi i h' \nu^2}{k}} \cdot I_{a,c,k,\nu}(w). \tag{2.13}$$

If k is odd, then we may without loss of generality assume that h' is even. In this case we obtain

$$\sum_2 = \frac{1}{k} e^{-\frac{\pi i h'}{8k}} \sum_{\substack{\nu \pmod{k}\\ \delta\in\{0,-1\}}} e^{\frac{\pi i h'}{k}\left(-2\nu^2+\nu(2\delta+1)\right)} \sum_{p\in\mathbb{Z}} q_1^{\frac{(4p+2\delta+1)^2}{16}}$$
$$\int_{-\infty}^{\infty} H_{a,c}\left(\frac{2\pi i\nu}{k} - \frac{2\pi w x}{k} - \frac{\pi i(2\delta+1)}{2k}\right) \cdot e^{-\frac{2\pi w x^2}{k}}\, dx.$$

Making the substitutions $p \mapsto -p$, $x \mapsto -x$, and $\nu \mapsto -\nu$, one can easily see that the contribution for $\delta = 0$ and for $\delta = -1$ coincide. Moreover the sum over p equals

$$\sum_{p\in\mathbb{Z}} q_1^{\frac{(4p+1)^2}{16}} = \frac{\eta^2\left(\frac{1}{k}\left(h' + \frac{i}{w}\right)\right)}{\eta\left(\frac{1}{2k}\left(h' + \frac{i}{w}\right)\right)}. \tag{2.14}$$

Thus

$$\sum_2 = \frac{2}{k} \cdot e^{-\frac{\pi i h'}{8k}} \frac{\eta^2\left(\frac{1}{k}\left(h' + \frac{i}{w}\right)\right)}{\eta\left(\frac{1}{2k}\left(h' + \frac{i}{w}\right)\right)} \sum_{\nu \pmod{k}} e^{\frac{\pi i h'}{k}\left(-2\nu^2+\nu\right)} \cdot I_{a,c,k,\nu}(w). \tag{2.15}$$

To finish the proof of Theorem 2.1, we require the well-known transformation law of Dedekind's η-function.

$$\eta\left(\frac{1}{k}(h+iw)\right) = e^{\frac{\pi i}{12k}(h-h')} \cdot \omega_{h,k}^{-1} \cdot w^{-\frac{1}{2}} \cdot \eta\left(\frac{1}{k}\left(h' + \frac{i}{w}\right)\right). \tag{2.16}$$

This implies that for k even, we have

$$\eta\left(\frac{2}{k}(h+iw)\right)=e^{\frac{\pi i}{6k}(h-\widetilde{h})}\cdot\omega_{h,k/2}^{-1}\cdot w^{-\frac{1}{2}}\cdot\eta\left(\frac{2}{k}\left(\widetilde{h}+\frac{i}{w}\right)\right), \tag{2.17}$$

where $h\widetilde{h}\equiv -1 \pmod{k/2}$. Moreover if k is odd, (2.16) implies that

$$\eta\left(\frac{2}{k}(h+iw)\right)=e^{\frac{\pi i}{12k}(2h-(2h)')}\cdot\omega_{2h,k}^{-1}\cdot(2w)^{-\frac{1}{2}}\cdot\eta\left(\frac{1}{k}\left((2h)'+\frac{i}{2w}\right)\right). \tag{2.18}$$

Combining (2.7), (2.9), (2.10), (2.11), (2.13), (2.15), (2.16), (2.17), and (2.18) gives (after a lengthy but straightforward calculation) the theorem. □

3. Construction of the weak Maass forms

In this section we prove the first part of Theorem 1.1. First we interpret the Mordell type integral occurring in Corollary 2.2 as an integral of theta functions. For this let

$$I_z:=\frac{4\sqrt{2}\sin^2\left(\frac{\pi a}{c}\right)}{(-iz)}\int_{\mathbb{R}}e^{-\frac{2\pi i x^2}{z}}\cdot H_{a,c}\left(\frac{2\pi i x}{z}+\frac{\pi i}{2}\right)dx. \tag{3.1}$$

Lemma 3.1. *We have*

$$I_z=\frac{\pi\tan\left(\frac{\pi a}{c}\right)}{4c}\int_0^\infty\frac{\Theta_{a,c}(iu)}{\sqrt{-i(iu+z)}}\,du.$$

Proof. We modify a proof of [9, 35]. By analytic continuation it is enough to show the claim for $z=it$ with $t>0$. Making the change of variables $x\mapsto\frac{x}{t}$, we find that

$$I_{it}=4\sqrt{2}\sin^2\left(\frac{\pi a}{c}\right)\int_{\mathbb{R}}e^{-2\pi t x^2}\cdot H_{a,c}\left(2\pi x+\frac{\pi i}{2}\right)dx. \tag{3.2}$$

We next rewrite $H_{a,c}\left(2\pi x+\frac{\pi i}{2}\right)$ using the Mittag-Leffler theory of partial fraction decomposition. This easily gives that

$$H_{a,c}\left(2\pi x+\frac{\pi i}{2}\right)=\frac{i}{4\pi\sin\left(\frac{2\pi a}{c}\right)}\sum_{m\in\mathbb{Z}}\left(\frac{1}{x-i\left(m-\frac{a}{c}-\frac{1}{4}\right)}-\frac{1}{x-i\left(m+\frac{a}{c}-\frac{1}{4}\right)}\right). \tag{3.3}$$

We plug (3.3) back into (3.2) and interchange summation and integration. For this we introduce the extra summands $\frac{1}{i\left(m-\frac{a}{c}-\frac{1}{4}\right)}$ and $\frac{1}{i\left(m+\frac{a}{c}-\frac{1}{4}\right)}$ which enforce absolute convergence and cancel when we integrate. This gives

$$I_{it}=\frac{i\tan\left(\frac{\pi a}{c}\right)}{\sqrt{2}}\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}}e^{-2\pi t x^2}\left(\frac{1}{x-i\left(m-\frac{a}{c}-\frac{1}{4}\right)}-\frac{1}{x-i\left(m+\frac{a}{c}-\frac{1}{4}\right)}\right)dx.$$

Next use that for all $s\in\mathbb{R}\setminus\{0\}$, we have the identity

$$\int_{-\infty}^{\infty}\frac{e^{-\pi t x^2}}{x-is}\,dx=\pi i s\int_0^\infty\frac{e^{-\pi u s^2}}{\sqrt{u+t}}\,du$$

(this follows since both sides are solutions of $\left(-\frac{\partial}{\partial t}+\pi s^2\right)f(t)=\frac{\pi i s}{\sqrt{t}}f(t)$ and have the same limit 0 as $t\mapsto\infty$ and hence are equal). Again interchanging summation and integration and making the change of variables $u \mapsto \frac{u}{2}$ gives

$$I_{it} = -\pi \tan\left(\frac{\pi a}{c}\right) \int_0^\infty \frac{1}{\sqrt{u+t}} \sum_{m\in\mathbb{Z}} \left(\left(m - \frac{a}{c} - \frac{1}{4}\right) \cdot e^{-2\pi u \left(m - \frac{a}{c} - \frac{1}{4}\right)^2} - \left(m + \frac{a}{c} - \frac{1}{4}\right) \cdot e^{-2\pi u \left(m + \frac{a}{c} - \frac{1}{4}\right)^2} \right) du.$$

From this the claim can be easily deduced. □

Lemma 3.2. *For $z \in \mathbb{H}$, we have*

$$\begin{aligned} J\left(\frac{a}{c}; z+1\right) &= J\left(\frac{a}{c}; z\right), \\ \frac{1}{\sqrt{-iz}} \cdot J\left(\frac{a}{c}; -\frac{1}{z}\right) &= I_z + \frac{\pi i \tan\left(\frac{\pi a}{c}\right)}{4c} \int_{-\bar{z}}^{i\infty} \frac{\Theta_{a,c}(\tau)}{\sqrt{-i(\tau+z)}}\, d\tau. \end{aligned}$$

Proof. We only show the lemma in the case that c is odd, the case c even is shown similarly. The first claim follows from the fact that $\Theta_{a,c}\left(-\frac{1}{\tau}\right)$ is invariant under $\tau \mapsto \tau + 1$. Indeed Shimura's work [31] implies that

$$(3.4) \qquad (-i4c\tau)^{-\frac{3}{2}} \cdot \Theta_{a,c}\left(-\frac{1}{\tau}\right) = -i(2c)^{-\frac{1}{2}} \sum_{k \pmod{2c}} \exp\left(\frac{2\pi i k(4a+c)}{2c}\right) \cdot \Theta(k, 2c; 4c\tau).$$

To prove the second transformation law we directly compute

$$\frac{1}{\sqrt{-iz}} \cdot J_{a,c}\left(-\frac{1}{z}\right) = -\frac{\pi i \tan\left(\frac{\pi a}{c}\right)}{4c \cdot \sqrt{-iz}} \int_{\frac{1}{\bar{z}}}^{i\infty} \frac{(-i\tau)^{-\frac{3}{2}} \cdot \Theta_{a,c}\left(-\frac{1}{\tau}\right)}{\sqrt{-i\left(\tau - \frac{1}{z}\right)}}\, d\tau.$$

Making the change of variable $\tau \mapsto -\frac{1}{\tau}$ now easily gives the claim. □

Proof of Theorem 1.1 (1). Again we assume that c is odd. By [31] Proposition 2.1, the functions $\Theta(k, 2c; \tau)$ are cusp forms for $\Gamma(4c)$. Thus the functions $\Theta(k, 2c; 4c\tau)$ are cusp forms for $\Gamma_1(16c^2)$. Using Theorem 2.1 one can conclude that also $\mathcal{M}\left(\frac{a}{c}; z\right)$ transforms correctly under $\Gamma_1(16c^2)$. That it is an eigenfunction under $\Delta_{\frac{1}{2}}$ follows as in [9] page 21. □

4. The case $u = -1$

Here we consider the case $u = -1$. We assume the same notation as in Section 2. Equation (1.3) gives

$$\mathcal{O}(-1; q) = 4\frac{(-q)_\infty}{(q)_\infty} \sum_{n\in\mathbb{Z}} (-1)^n \frac{q^{n^2+n}}{(1+q^n)^2}.$$

This function is more complicated than $\mathcal{O}\left(\frac{a}{c}; q\right)$ with $c \neq 2$ since double poles occur. To overcome this problem, we first prove a transformation law for the function

$$\mathcal{O}_r(q) := 4\frac{(-q)_\infty}{(q)_\infty} \sum_{n\in\mathbb{Z}} (-1)^{n+1} \frac{q^{n^2}}{(1+e^{2\pi i r}\, q^n)}.$$

This function is related to $\mathcal{O}(-1; q)$ by

$$\frac{1}{2\pi i}\left[\frac{\partial}{\partial r}\mathcal{O}_r(q)\right]_{r=0} = \mathcal{O}(-1; q).$$

To state the transformation law for $\mathcal{O}_r(q)$ we additionally need the functions

$$\mathcal{U}_r(q) := e^{\pi i r}\frac{\eta(z)}{\eta^2(2z)}\sum_{\substack{m\in\mathbb{Z}\\ m \text{ odd}}}\frac{q^{\frac{1}{2}(m^2+m)}}{1-e^{2\pi i r}q^m},$$

$$I^{\pm}_{k,\nu,r}(w) := \int_{\mathbb{R}}\frac{e^{-\frac{2\pi w x^2}{k}}}{1+e^{\pm\frac{2\pi i\nu}{k}\mp\frac{\pi i\widetilde{k}}{2k}+2\pi i r-\frac{2\pi w x}{k}}}\,dx.$$

Theorem 4.1. *Assume that $r \in \mathbb{R}$ with $|r|$ sufficiently small.*

(1) *If k is odd, then*

$$\mathcal{O}_r(q) = -2\sqrt{2}iw^{-\frac{1}{2}}\frac{\omega^2_{h,k}}{\omega_{2h,k}}\,e^{\frac{\pi i h'}{8k}+\frac{2\pi k r^2}{w}}\,\mathcal{U}_{\frac{ir}{w}}\left(q_1^{\frac{1}{2}}\right) - 2\sqrt{2}w^{\frac{1}{2}}\frac{\omega^2_{h,k}}{\omega_{2h,k}}\,\frac{1}{k}\sum_{\substack{\nu \pmod k\\ \pm}} e^{\frac{2\pi i h'}{k}\left(-\nu^2+\frac{\nu}{2}\right)}I^{\pm}_{k,\nu,r}(w).$$

(2) *If k is even, then*

$$\mathcal{O}_r(q) = -i\frac{\omega^2_{h,k}}{\omega_{h,k/2}}\;w^{-\frac{1}{2}}e^{\frac{2\pi k r^2}{w}}\mathcal{O}_{\frac{r}{iw}}(q_1) - \frac{4\omega^2_{h,k}w^{\frac{1}{2}}}{k\omega_{h,k/2}}\sum_{\nu \pmod k}(-1)^\nu e^{-\frac{2\pi i h'\nu^2}{k}}I^{+}_{k,\nu,r}(w).$$

Proof. We proceed similarly as in the proof of Theorem 2.1 and therefore we skip most of the details. Moreover, we only show the claim for k odd, the case k even is treated similarly. Define

$$\widetilde{\mathcal{O}}_r(q) := \sum_{n\in\mathbb{Z}}(-1)^n\frac{q^{n^2}}{(1+e^{2\pi i r}\,q^n)}.$$

Changing n into $\nu + km$ with ν running modulo k and $m \in \mathbb{Z}$, we obtain, using Poisson summation and making a change of variables,

$$\widetilde{\mathcal{O}}_r(q) = \frac{1}{k}\sum_{\nu \pmod k}(-1)^\nu\,e^{\frac{2\pi i h\nu^2}{k}}\sum_{\substack{n\in\mathbb{N}_0\\ \pm}}\int_{\mathbb{R}}\frac{e^{-\frac{2\pi w x^2}{k}+\frac{\pi i}{k}(2n+1)(x-\nu)}}{1+e^{2\pi i r\pm\frac{2\pi i h\nu}{k}\mp\frac{2\pi w}{k}x}}\,dx.$$

One can show that poles of the integrand only lie in points

$$x_m := \frac{i}{w}\left(m+\frac{1}{2}\pm kr\right)$$

and we have nontrivial residues at most for one $\nu \pmod k$ which can can chose as

$$\nu_m := -\frac{1+k}{2}(2m+1)h'.$$

Using the Residue Theorem, we shift the path of integration through

$$\omega_n := \frac{2n+1}{4w}.$$

There are no poles on the real axis or in ω_n. Moreover we have to take those x_m with $m \geq 0$ into account that satisfy $n \geq 2m+1$. We denote by $\lambda^{\pm}_{n,m}$ the residue of each summand and compute

$$\lambda^{\pm}_{n,m} = \pm\frac{k}{2\pi w}\,e^{-\frac{2\pi w x_m^2}{k}+\frac{\pi i}{k}(2n+1)(x_m-\nu_m)}.$$

Thus

$$\lambda^{\pm}_{n+1,m} = e^{\frac{2\pi i}{k}(x_m-\nu_m)}\lambda^{\pm}_{n,m}$$

which gives

$$\widetilde{\mathcal{O}}_w(q) = \sum_1 + \sum_2,$$

where

$$\begin{aligned}\sum_1 &= \frac{2\pi i}{k}(-1)^{\nu_m} e^{\frac{2\pi i h\nu_m^2}{k}} \sum_{\substack{m\geq 0\\ \pm}} \frac{\lambda^{\pm}_{2m+1,m}}{1-e^{\frac{2\pi i}{k}(x_m-\nu_m)}},\\ \sum_2 &= \frac{1}{k}\sum_{\nu \pmod k}(-1)^{\nu}\, e^{\frac{2\pi i h\nu^2}{k}} \sum_{\substack{n\in\mathbb{N}_0\\ \pm}} \int_{\mathbb{R}+\omega_n} \frac{e^{-\frac{2\pi w x^2}{k}+\frac{\pi i}{k}(2n+1)(x-\nu)}}{1+e^{2\pi i r\pm\frac{2\pi i h\nu}{k}\mp\frac{2\pi w}{k}x}}\,dx.\end{aligned}$$

A lengthy calculation gives that

$$\sum_1 = \frac{i}{w}\, e^{\frac{2\pi k r^2}{w}-\frac{\pi r}{w}} \sum_{\substack{m\in\mathbb{Z}\\ m \text{ odd}}} \frac{q_1^{\frac{1}{4}(m^2+m)}}{1-e^{-\frac{2\pi r}{w}}q_1^{\frac{1}{2}m}}.$$

To compute $\sum_2$, we change the sum over n back into a sum over $\mathbb{Z}$ and change $x\mapsto x+\omega_n$. This gives

$$\sum_2 = \frac{1}{k}\sum_{\nu \pmod k}(-1)^{\nu}\, e^{\frac{2\pi i h\nu^2}{k}} \sum_{n\in\mathbb{Z}} e^{-\frac{\pi}{8kw}(2n+1)^2-\frac{\pi i}{k}(2n+1)\nu}\int_{\mathbb{R}} \frac{e^{-\frac{2\pi w x^2}{k}}}{1+e^{-\frac{\pi i}{2k}(2n+1)+\frac{2\pi i h\nu}{k}-\frac{2\pi w x}{k}+2\pi i r}}\,dx.$$

Changing $n\mapsto 2p+\delta$ (with $p\in\mathbb{Z}$ and $\delta\in\{0,-1\}$) and $\nu\mapsto -h'(\nu+p)$ gives

$$\sum_2 = \frac{1}{k}\sum_{\nu,p,\delta} e^{\frac{2\pi i h'}{k}\left(-\nu^2-\frac{1}{16}(2\delta+1)^2+\frac{\nu}{2}(2\delta+1)\right)} q_1^{\frac{1}{16}(4p+2\delta+1)^2}\int_{\mathbb{R}} \frac{e^{-\frac{2\pi w x^2}{k}}}{1-e^{2\pi i r+\frac{2\pi i\nu}{k}-\frac{\pi i}{2k}(2\delta+1)-\frac{2\pi w x}{k}}}\,dx.$$

Now the sum over p equals $\frac{\eta^2\left(\frac{1}{k}\left(h'+\frac{i}{z}\right)\right)}{\eta\left(\frac{1}{2k}\left(h'+\frac{i}{z}\right)\right)}$ thus

$$\sum_2 = \frac{\eta^2\left(\frac{1}{k}\left(h'+\frac{i}{z}\right)\right)}{k\eta\left(\frac{1}{2k}\left(h'+\frac{i}{z}\right)\right)}\, e^{-\frac{\pi i h'}{8k}} \sum_{\substack{\nu \pmod k\\ \pm}} e^{\frac{2\pi i h'}{k}\left(-\nu^2+\frac{\nu}{2}\right)}\int_{\mathbb{R}} \frac{e^{-\frac{2\pi w x^2}{k}}}{1+e^{\pm\frac{2\pi i\nu}{k}\mp\frac{\pi i}{2k}+2\pi i r-\frac{2\pi w x}{k}}}\,dx$$

which gives the claim using (2.16), (2.17), and (2.18). □

¿From Theorem 4.1, we can conclude a transformation law for $\mathcal{O}(-1;q)$. For this define

$$\begin{aligned}\mathcal{U}(q) &:= \frac{4\eta(z)}{\eta^2(2z)}\sum_{m\in\mathbb{Z}}\frac{q^{\frac{1}{2}(m^2+m)}}{(1-q^m)^2},\\ I^{\pm}_{k,\nu}(w) &:= \int_{\mathbb{R}} \frac{e^{-\frac{2\pi w x^2}{k}}}{\left(1+e^{\pm\frac{2\pi i\nu}{k}\mp\frac{\pi i\tilde{k}}{2k}-\frac{2\pi w x}{k}}\right)^2}\,dx.\end{aligned}$$

Corollary 4.2. *Assume the notation above. We have.*

(1) *If k is odd, then*

$$\mathcal{O}(-1;q) = -\frac{1}{\sqrt{2}} w^{-\frac{3}{2}} \frac{\omega_{h,k}^2}{\omega_{2h,k}} e^{\frac{\pi i h'}{8k}} \,\mathcal{U}\left(q_1^{\frac{1}{2}}\right) + 2\sqrt{2} w^{\frac{1}{2}} \frac{\omega_{h,k}^2}{k\,\omega_{2h,k}} \sum_{\substack{\nu \pmod{k} \\ \pm}} e^{\frac{2\pi i h'}{k}\left(-\nu^2+\frac{\nu}{2}\right)} I_{k,\nu}^{\pm}(w).$$

(2) *If k is even, then*

$$\mathcal{O}(-1;q) = -w^{-\frac{3}{2}} \frac{\omega_{h,k}^2}{\omega_{h,k/2}} \mathcal{O}(-1;q_1) + \frac{4\omega_{h,k}^2}{k\omega_{h,k/2}} w^{\frac{1}{2}} \sum_{\nu \pmod{k}} (-1)^{\nu} e^{-\frac{2\pi i h' \nu^2}{k}} I_{k,\nu}^{+}(w).$$

Proof. We only prove (i), (ii) can be shown similarly. We have

$$\frac{1}{2\pi i}\left[\frac{\partial}{\partial r}\left(\frac{e^{\frac{2\pi k r^2}{w}-\frac{\pi r}{w}}}{1-e^{-\frac{2\pi r}{w}} q_1^{\frac{m}{2}}}\right)\right]_{r=0} = \frac{\left(3q_1^{\frac{m}{2}}-1\right)}{2iw\left(1-q_1^{\frac{m}{2}}\right)^2}.$$

Now

$$\begin{aligned}\sum_{m\in\mathbb{Z}} \frac{q_1^{\frac{1}{4}(m^2+m)}\left(3q_1^{\frac{m}{2}}-1\right)}{\left(1-q_1^{\frac{m}{2}}\right)^2} &= -3\sum_{m\in\mathbb{Z}} \frac{q_1^{\frac{1}{4}(m^2+m)}}{\left(1-q_1^{\frac{m}{2}}\right)} + 2\sum_{m\in\mathbb{Z}} \frac{q_1^{\frac{1}{4}(m^2+m)}}{\left(1-q_1^{\frac{m}{2}}\right)^2} \\ &= 2\sum_{m\in\mathbb{Z}} \frac{q_1^{\frac{1}{4}(m^2+m)}}{\left(1-q_1^{\frac{m}{2}}\right)},\end{aligned}$$

since in the first sum the mth and $-m$th term cancel. Now (i) can be easily concluded by using that

$$\frac{1}{2\pi i}\left[\frac{\partial}{\partial r}\left(\frac{1}{1+e^{\pm\frac{2\pi i\nu}{k}\mp\frac{\pi i}{2k}-\frac{2\pi w x}{k}+2\pi i r}}\right)\right]_{r=0} = -\frac{1}{\left(1+e^{\pm\frac{2\pi i\nu}{k}\mp\frac{\pi i}{2k}-\frac{2\pi w x}{k}}\right)^2}.$$

□

Remark. ¿From Corollary 4.2 we can obtain the transformation law for $\mathcal{O}(-1;z)$ under $\left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ by setting $h' = \alpha$, $k = \gamma$, $h = -\delta$, and $w = -i\,(\delta+\gamma\tau)$.

We next realize the integral occurring in Corollary 4.2 for $k=1$ as a theta integral. For this let

$$\begin{aligned} I_r^{\pm}(\tau) &:= \int_{\mathbb{R}} \frac{e^{-\frac{2\pi i x^2}{\tau}}}{1 \pm i e^{2\pi i r - \frac{2\pi i x}{\tau}}}, \\ I(\tau) &:= \frac{1}{2\pi i}\left[\frac{\partial}{\partial r}\left(I_r^{+}(\tau)+I_r^{-}(\tau)\right)\right]_{r=0}. \end{aligned}$$

Lemma 4.3. *We have*

$$I(\tau) = -\frac{(-i\tau)^2}{\sqrt{2}\pi}\int_0^{\infty} \frac{\eta^2(iu)}{\eta\left(\frac{iu}{2}\right)\left(-i(i\tau+iu)\right)^{\frac{3}{2}}}\,du.$$

Proof. Via analytic continuation it is sufficient to show the claim for $\tau = it$. Making the change of variables $x \mapsto -\frac{x}{t}$ gives

$$I_r^{\pm}(it) = t\int_{\mathbb{R}} \frac{e^{-2\pi t x^2}}{1 \pm i e^{2\pi i r}\, e^{2\pi x}}\,dx.$$

We use the theory of Mittag-Leffler to rewrite

$$\frac{1}{1+ie^{2\pi ir}\,e^{2\pi x}}+\frac{1}{1-ie^{2\pi ir}\,e^{2\pi x}}=-\frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\frac{1}{x-i\left(-m+\frac{1}{4}-r\right)}+\frac{1}{x-i\left(m-\frac{1}{4}-r\right)}.$$

This implies that

$$I_r^+(it)+I_r^-(it)=-\frac{t}{2\pi}\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}}\left(\frac{e^{-2\pi tx^2}}{x-i\left(-m+\frac{1}{4}-r\right)}+\frac{e^{-2\pi tx^2}}{x-i\left(m-\frac{1}{4}-r\right)}\right)dx.$$

From this we conclude

$$I(it)=\frac{t}{4\pi^2}\sum_{\substack{m\in\mathbb{Z}\\ \pm}}\int_{\mathbb{R}}\frac{e^{-2\pi tx^2}}{\left(x-i\left(m+\frac{1}{2}\mp\frac{1}{4}\right)\right)^2}\,dx=\frac{t}{4\pi^2}\sum_{\substack{m\in\mathbb{Z}\\ m\text{ odd}}}\int_{\mathbb{R}}\frac{e^{-2\pi tx^2}}{\left(x-\frac{im}{4}\right)^2}\,dx.$$

We have the integral identity

$$\int_{\mathbb{R}}e^{-2\pi u^2t}\frac{u}{u-is}\,du=\frac{1}{2\sqrt{2}}\int_0^\infty\frac{e^{-2\pi us^2}}{(u+t)^{\frac{3}{2}}}\,du\qquad(s\neq 0)$$

which follows since both sides are solutions of the differential equation $-\frac{\partial}{\partial t}+2\pi s^2=\frac{1}{2\sqrt{2}t^{\frac{3}{2}}}$ and have limit 0 as $t\to\infty$. Using integration by parts gives

$$\int_{\mathbb{R}}\frac{e^{-2\pi tx^2}}{(x-is)^2}\,dx=\sqrt{2}\pi t\int_0^\infty\frac{e^{-2\pi us^2}}{(u+t)^{\frac{3}{2}}}\,du.$$

Thus

$$I(it)=-\frac{t^2}{2\sqrt{2}\pi}\int_0^\infty\sum_{\substack{m\in\mathbb{Z}\\ m\text{ odd}}}\frac{e^{-\frac{\pi um^2}{8}}}{(u+t)^{\frac{3}{2}}}\,du$$

which easily gives the claim. □

Combining Theorem 4.1, Corollary 4.1, and Lemma 4.3 gives

Corollary 4.4. *For $z\in\mathbb{H}$, we have*

$$\mathcal{O}\left(-1;-\frac{1}{z}\right)=-\frac{i}{\sqrt{2}}(-iz)^{\frac{3}{2}}\cdot\mathcal{U}\left(-1;\frac{z}{2}\right)+2\frac{(-iz)^{\frac{3}{2}}}{\pi}\int_0^\infty\frac{\eta^2(iu)}{\eta(iu/2)(-i(iz+iu))^{\frac{3}{2}}}\,du.$$

We next give the transformation law of the non-holomorphic part of $\mathcal{M}(-1;z)$ which can be shown as in proof of Theorem 3.2.

Lemma 4.5. *For $z\in\mathbb{H}$, we have*

$$\begin{aligned}J(-1;z+1)&=J(-1;z),\\ \frac{1}{(-iz)^{\frac{3}{2}}}J\left(-1;-\frac{1}{z}\right)&=-\frac{2i}{\pi}\int_{-\bar{z}}^{i\infty}\frac{\eta^2(\tau)}{\eta(\tau/2)\,(-i(\tau+z))^{\frac{3}{2}}}\,d\tau+\frac{2i}{\pi}\int_0^\infty\frac{\eta^2(iu)}{\eta(iu/2)\,(-i(iu+z))^{\frac{3}{2}}}\,du.\end{aligned}$$

Using that $\frac{\eta^2(\tau)}{\eta(2\tau)}$ is a modular form on $\Gamma_0(16)$ gives the claim.

5. Congruences for overpartitions

Following the original strategy of Ono [27] and Ono and the first author [6, 8, 9] we can prove the congruences in Theorem 1.2. We limit ourselves to a sketch of the proof.

Sketch of Proof of Theorem 1.2. Throughout we assume the assumptions of Theorem 1.2. First one can show, using the results above that the function

$$\sum_{n=0}^{\infty}\left(\overline{N}(r,t;n)-\frac{\overline{p}(n)}{t}\right)q^n \tag{5.1}$$

is the holomorphic part of a weak Maass form of weight $\frac{1}{2}$ on $\Gamma_1\left(16c^2\right)$. In order to use results of Serre on p-adic modular forms, we next apply twists to the associated weak Maass forms, which "kill" the non-holomorphic part. This requires knowing on which arithmetic progressions it is supported. We prove

$$\begin{aligned}\mathcal{M}\left(\frac{a}{c};z\right) &= 1+\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\overline{N}(m,n)\zeta_c^{am}q^n\\ &+\sqrt{\pi}i\tan\left(\frac{\pi a}{c}\right)\sum_{k \pmod{2c}}(-1)^k e\left(\frac{2ka}{c}\right)\sum_{m\equiv k \pmod{2c}}\Gamma\left(\frac{1}{2};4\pi m^2y\right)q^{-m^2},\end{aligned} \tag{5.2}$$

where $e(x):=e^{2\pi i x}$, and where

$$\Gamma(a;x):=\int_x^{\infty}e^{-t}t^{a-1}\,dt \tag{5.3}$$

is the incomplete Gamma-function. Using this one can show that for a prime $p\nmid 6t$ the function

$$\sum_{\substack{n\geq 1\\ \left(\frac{n}{p}\right)=-\left(\frac{-1}{p}\right)}}\left(\overline{N}(r,t;n)-\frac{\overline{p}(n)}{t}\right)q^n \tag{5.4}$$

is a weight $\frac{1}{2}$ weakly holomorphic modular form on $\Gamma_1(16c^2p^4)$. Now the theorem can be concluded as in [9] using a geralization of Serre's results on p-adic modular forms (Theorem 4.2 of [9]). □

6. Proof of Theorem 1.3

Here we prove Theorem 1.3. While we follow the model of Ono [27] and the first author and Ono [6, 9], the proof of the modularity is rather delicate. For brevity we set $t:=\ell^m$. Define the function

$$g_r(z):=t\cdot\eta^{r_1}(2\ell z)\cdot\eta^{2\ell}(\ell z)\sum_{n=0}^{\infty}\overline{N}(r,t;n)q^n,$$

where $0<r_1<48$ is a solution of $r_1\ell\equiv -1 \pmod{48}$. We have

$$g_r(z)=\frac{\eta(2z)\cdot\eta^{r_1}(2\ell z)\cdot\eta^{2\ell}(\ell z)}{\eta^2(z)}+\sum_{j=1}^{t-1}\zeta_t^{-rj}\left(\mathcal{M}\left(\frac{j}{t};z\right)+J\left(\frac{j}{t};z\right)\right)\eta^{r_1}(2\ell z)\cdot\eta^{2\ell}(\ell z). \tag{6.1}$$

We denote the two summands by $f(z)$ and $f_r(z)$, respectively. Define for a function $h(z)=\sum_n a(n)q^n$ the twist

$$\widetilde{h}(z):=-\frac{1}{2}\left(\frac{-1}{\ell}\right)\left(h(z)-\left(\frac{-1}{\ell}\right)h(z)_\ell\right)_\ell, \tag{6.2}$$

where as before

$$h_\ell := \frac{g}{\ell} \sum_{\nu \pmod{\ell}} \left(\tfrac{\nu}{\ell}\right) h \mid \left(\begin{smallmatrix} 1 & -\frac{\nu}{\ell} \\ 0 & 1 \end{smallmatrix}\right). \tag{6.3}$$

Clearly

$$\widetilde{h}(z) = \sum_{\substack{n \\ \left(\frac{-n}{\ell}\right)=-1}} a(n)\, q^n. \tag{6.4}$$

The main step in the proof of Theorem 1.3 is the following theorem.

Theorem 6.1. *For every $u \geq 0$ there exist a character χ, integers λ, λ', N, N', and modular forms $h(z) \in S_{\lambda+\frac{1}{2}}\left(\Gamma_0\left(N\right), \chi\right)$ and $h_r(z) \in S_{\lambda'+\frac{1}{2}}\left(\Gamma_1\left(N'\right)\right)$ such that*

$$\frac{\widetilde{g_r}(z)}{\eta^{r_1}(2\ell z) \cdot \eta^{2\ell}(\ell z)} \equiv h(z) + h_r(z) \pmod{\ell^u}.$$

The proof of Theorem 6.1 is given later. We first show how Theorem 1.3 follows from Theorem 6.1.

Proof of Theorem 1.3. We easily see that

$$\frac{\widetilde{g_r}(z)}{\eta^{2\ell}(\ell z) \cdot \eta^{r_1}(2\ell z)} = t \sum_{\left(\frac{-n}{\ell}\right)=-1} \overline{N}(r,t;n)\, q^n.$$

Now we let $0 \leq \beta \leq \ell - 1$ with $\left(\frac{-\beta}{\ell}\right) = -1$ be given. Define

$$g_{r,\beta}(z) := t \sum_{n \equiv \beta \pmod{\ell}} \overline{N}\left(r,t;n\right)\, q^n.$$

Theorem 6.1 gives that

$$g_{r,\beta}(z) \equiv h_\beta(z) + h_{r,\beta}(z) \pmod{\ell^u},$$

where $h_\beta(z)$ and $h_{r,\beta}(z)$ denote the restrictions of the Fourier expansion of $h(z)$ resp. $h_r(z)$ to those coefficients n with $n \equiv \beta \pmod{\ell}$. Using the theory of Hecke operators, we can show that for all $n \equiv \beta \pmod{\ell}$ coprime to p we have

$$t\, \overline{N}\left(r,t;p^3 n\right) \equiv 0 \pmod{\ell^u}.$$

Dividing by t directly gives the theorem since u is arbitrary. $\square$

Proof of Theorem 6.1. If a is a positive integer, then define

$$E_{\ell,a}(z) := \frac{\eta^{\ell^a}(z)}{\eta(\ell^a z)} \in M_{\frac{\ell^a-1}{2}}\left(\Gamma_0(\ell^a), \chi_{\ell,a}\right),$$

where $\chi_{\ell,a}(d) := \left(\frac{(-1)^{(\ell^a-1)/2}\ell^a}{d}\right)$. It is well known that $E_{\ell,a}(z)$ vanishes at those cusps of $\Gamma_0(\ell^a)$ that are not equivalent to ∞ and that for all $u > 0$

$$E_{\ell,a}^{\ell^{u-1}}(z) \equiv 1 \pmod{\ell^u}. \tag{6.5}$$

We now treat the summands in (6.1) separately. We start with $f(z)$. Using Theorem 1.64 from [29], it is not hard to see that $f(z)$ is a modular form of weight $\frac{r_1+2\ell-1}{2}$ with some character on $\Gamma_0(2\ell)$. ¿From this we see that $\widetilde{f}(z) \in M_{\frac{r_1+2\ell-1}{2}}\left(\Gamma_0(2\ell^5), \widetilde{\chi}\right)$ for some character $\widetilde{\chi}$. For sufficiently large u' the function

$$h(z) := \frac{\widetilde{f}(z) \cdot E_{\ell,5}^{\ell^{u'}}(z)}{\eta^{r_1}(2\ell z) \cdot \eta^{2\ell}(\ell z)}$$

is a weakly holomorphic modular form on $\Gamma_0(8\ell^5)$ that vanishes at all cusps with the exception of ∞ and $\frac{1}{r\ell^5}$ with $r \in \{1,2,4\}$, and satisfies

$$h(z) \equiv \frac{\widetilde{f}(z)}{\eta^{r_1}(2\ell z)\eta^{2\ell}(\ell z)} \pmod{\ell^u}.$$

We show that $h(z)$ is a cusp form. It is easy to see that in the cusp ∞ the Fourier expansion of $\widetilde{f}(z)$ starts at least with $q^{\frac{\ell}{12}(r_1+\ell)+1}$. Since the Fourier expansion of $\eta^{r_1}(2\ell z)\cdot\eta^{2\ell}(\ell z)$ starts with $q^{\frac{\ell}{12}(r_1+\ell)}$, we see that $h(z)$ vanishes in ∞. Since $\widetilde{f}(z)$ is a modular form on $\Gamma_0\left(2\ell^5\right)$ and the Fourier expansion of $\eta^{r_1}(2\ell z)\cdot\eta^{2\ell}(\ell z)$ in $\frac{1}{r\ell^5}$ with r even starts with $q^{\frac{\ell}{12}(r_1+\ell)}$, $h(z)$ vanishes also in the cusps $\frac{1}{2\ell^5}$ and $\frac{1}{4\ell^5}$.

Next consider the cusp $\frac{1}{\ell^5}$. In (6.3), we can choose a set of respresentatives with ν even. Now for even ν it is not hard to see that $\left(\begin{smallmatrix}1 & -\frac{\nu}{\ell}\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & 0\\ \ell^5 & 1\end{smallmatrix}\right)$ is $\Gamma_0(2\ell^5)$- equivalent to $\left(\begin{smallmatrix}1 & 0\\ \ell^5 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & -\frac{\nu}{\ell}\\ 0 & 1\end{smallmatrix}\right)$. Thus

$$f(z)_\ell = \frac{g}{\ell}\sum_{\substack{\nu \pmod{\ell}\\ \nu \text{ even}}}\left(\frac{\nu}{\ell}\right) f|\left(\begin{smallmatrix}1 & 0\\ \ell^5 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & -\frac{\nu}{\ell}\\ 0 & 1\end{smallmatrix}\right).$$

It is not hard to see that $f(z)$ vanishes in $\frac{1}{\ell^5}$ of order $\frac{1}{24}(-3+r_1\ell+4\ell^2)$. Since the cusp width of $\frac{1}{\ell^5}$ in $\Gamma_0(2\ell^5)$ is 2, the Fourier expansion of $f|\left(\begin{smallmatrix}1 & 0\\ \ell^5 & 1\end{smallmatrix}\right)$ starts with q^{r_0}, where $r_0 := \frac{1}{48}(-3+r_1\ell+4\ell^2)$. Thus the Fourier expansion of $\left(\frac{-1}{\ell}\right)f_\ell$ starts with

$$\left(\frac{-1}{\ell}\right)\frac{g}{\ell}\sum_{\substack{\nu \pmod{\ell}\\ \nu \text{ even}}}\left(\frac{\nu}{\ell}\right)e^{-\frac{2\pi i r_0\nu}{\ell}} = \left(\frac{-r_0}{\ell}\right) = 1.$$

Since twisting doesn't decrease the order of vanishing, $\widetilde{f}(z)$ has in $\frac{1}{\ell^5}$ a Fourier expansion starting at least with $q^{r_0+\frac{1}{2}}$, whereas $\eta^{r_1}(2\ell z)\cdot\eta^{2\ell}(\ell z)$ has in $\frac{1}{\ell^5}$ a Fourier expansion starting with $q^{\frac{\ell(r_1+4\ell)}{48}}$. Thus $h(z)$ vanishes in all cusps and is therefore a cusp form.

We next turn to $f_r(z)$. Using Theorem 1.1, it is not hard to see that $\widetilde{f_r}(z)$ is the holomorphic part of a weak Maass form on $\Gamma_1(16t^2\ell^4)$. Moreover by (5.2) it is easy to see that the corresponding weak Maass form doesn't have a non-holomorphic part, and thus $\widetilde{f_r}(z)$ is a weakly holomorphic modular form. Since $E_{\ell,3m}(z)$ vanishes at each cusp $\frac{\alpha}{\gamma}$ with $t^3 \nmid \gamma$ for sufficiently large u', the function

$$f_r'(z) := E_{\ell,3m}^{\ell^{u'}}(z)\widetilde{f_r}(z)$$

is a weakly holomorphic modular form on $\Gamma_1\left(16t^2\ell^4\right)$ that vanishes at all cusps $\frac{\alpha}{\gamma}$ with $t^3 \nmid \gamma$ and satisfies

$$f_r'(z) \equiv f_r(z) \pmod{\ell^u}.$$

Therefore to finish the proof it remains to show that $\frac{\widetilde{f_r}(z)}{\eta^{r_1}(2\ell z)\eta^{2\ell}(\ell z)}$ vanishes also at those cusps $\frac{\alpha}{\gamma}$ with $t^3|\gamma$. Now let $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \Gamma_0(t^3)$. In the following we need the commutation relation for $\nu' \equiv \delta^2\nu \pmod{\ell}$

$$\begin{pmatrix}1 & -\frac{\nu}{\ell}\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix}\begin{pmatrix}1 & -\frac{\nu'}{\ell}\\ 0 & 1\end{pmatrix} \tag{6.6}$$

with

$$\begin{pmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix} = \begin{pmatrix}\alpha-\frac{\gamma\nu}{\ell} & \beta-\frac{\gamma\nu\nu'}{\ell^2}+\frac{\alpha\nu'-\delta\nu}{\ell}\\ \gamma & \delta+\frac{\gamma\nu'}{\ell}\end{pmatrix} \in \Gamma_0(t^3).$$

We distinguish the cases whether $2|\gamma$ or not.

If $2|\gamma$, then one can easily see that the Fourier expansion of $\eta^{r_1}(2\ell z)\eta^{2\ell}(\ell z)$ in $\frac{\alpha}{\gamma}$ starts with $\frac{\ell}{12}(r_1+\ell)=: n_0$. Clearly $\ell|n_0$. Thus we have to prove that the q-expansion of $\widetilde{f_r}(z)$ in $\frac{\alpha}{\gamma}$ starts with q^b with $b>n_0$. We may assume that $48|\nu,\nu'$. Then we can write

$$(f_r)_\ell|\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)=\sum_{\substack{\nu \pmod{\ell}\\ 48|\nu,\nu'}} f_r|\left(\begin{smallmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & -\frac{\nu'}{\ell}\\ 0 & 1\end{smallmatrix}\right).$$

Let

$$f_{r,j}(z):=\mathcal{M}\left(\frac{j}{t};z\right)\cdot\eta^{r_1}(2\ell z)\cdot\eta^{2\ell}(\ell z)$$

and define $\widetilde{f_{r,j}}(z)$ for weak Maass forms as for holomorphic forms. It is enough to show that $\widetilde{f_{r,j}}(z)$ starts with q^b with $b>n_0$. As before it follows that $\widetilde{f_{r,j}}(z)$ doesn't have a non-holomorphic part. We use twice (6.6), Corollary 2.3, and the transformation law of the η-function. We let $\widetilde{\alpha}:=(\alpha')'$, $\widetilde{\delta}:=(\delta')'$, and $\widetilde{\gamma}:=(\gamma')'$. The holomorphic part of $f_{r,j}|\left(\begin{smallmatrix}\widetilde{\alpha} & \widetilde{\beta}\\ \widetilde{\gamma} & \widetilde{\delta}\end{smallmatrix}\right)$ is given by

$$-ie^{\frac{\pi i\ell}{6\widetilde{\gamma}}(\widetilde{\alpha}+\widetilde{\delta})(r_1+\ell)}e^{\frac{2\pi ij^2\widetilde{\delta}\widetilde{\gamma}_1}{t}}\frac{\omega^2_{\widetilde{\alpha},\widetilde{\gamma}}}{\omega_{\widetilde{\alpha},\widetilde{\gamma}/2}\cdot\omega^{2\ell}_{\widetilde{\alpha},\widetilde{\gamma}/\ell}\cdot\omega^{r_1}_{\widetilde{\alpha},\widetilde{\gamma}/2\ell}}\tan\left(\frac{\pi j}{t}\right)\cot\left(\frac{\pi j\widetilde{\delta}}{t}\right)\left(-i\left(\widetilde{\gamma}z+\widetilde{\delta}\right)\right)^{\frac{1}{2}(1+r_1+2\ell)}f_{r,j\widetilde{\delta}}(z). \tag{6.7}$$

Using [26]

$$\omega^{-1}_{\alpha,\gamma}\cdot e^{\frac{\pi i}{12\gamma}(\alpha+\delta)}=\begin{cases}\left(\frac{\delta}{\gamma}\right)\cdot i^{\frac{1-\gamma}{2}}\cdot e^{\frac{\pi i}{12}(\beta\delta(1-\gamma^2)+\gamma(\alpha+\delta))} & \text{if } \gamma \text{ is odd},\\ \left(\frac{\gamma}{\delta}\right)e^{\frac{\pi i}{12}(\alpha\gamma(1-\delta^2)+\delta(\beta-\gamma+3))} & \text{if } \delta \text{ is odd},\end{cases} \tag{6.8}$$

we can show that in (6.7) we can change $\widetilde{\alpha}$, $\widetilde{\delta}$, and $\widetilde{\gamma}$ into α', δ', and γ', respectively if we change z into $z+\frac{\nu'}{\ell}$. The Fourier expansion of $f_{r,j\delta'}$ starts with q^{n_0} and $\ell|n_0$. Moreover

$$\sum_{\substack{\nu \pmod{\ell}\\ \nu\equiv 0 \pmod{48}}}\left(\frac{\nu}{\ell}\right)e^{-\frac{2\pi in_0\nu'}{\ell}}=\sum_{\nu \pmod{\ell}}\left(\frac{\nu}{\ell}\right)=0.$$

Thus the Fourier expansion of the holomorphic part of $(f_{r,j})_\ell|\left(\begin{smallmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{smallmatrix}\right)$ starts with at least q^{n_0+1} which implies that the Fourier expansion of the holomorphic part of

$$\left(f_{r,j}-\left(\frac{-1}{\ell}\right)(f_{r,j})_\ell\right)|\left(\begin{smallmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{smallmatrix}\right)$$

starts at least with q^{n_0}. Arguing in the same way, we obtain that the Fourier expansion of

$$\left(f_{r,j}-\left(\frac{-1}{\ell}\right)(f_{r,j})_\ell\right)_\ell|\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$$

starts with at least q^{n_0+1} as desired.

Next assume that γ is odd. It is not hard to see that the Fourier expansion of $\eta^{r_1}(2\ell z)\eta^{2\ell}(\ell z)$ in $\frac{\alpha}{\gamma}$ starts with $q^{\frac{\ell}{48}(r_1+4\ell)}=: q^{n_0}$. Since twisting does not decrease the order of vanishing, it is enough to show that the Fourier expansion of the holomorphic part of $\left(f_{r,j}-\left(\frac{-1}{\ell}\right)(f_{r,j})_\ell\right)|\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$

starts with q^b with $b > n_0$. This time we use (6.6) once. One can compute that the holomorphic part of $f_{r,j}|\left(\begin{smallmatrix}\alpha' & \beta' \\ \gamma' & \delta'\end{smallmatrix}\right)$ equals

$$\sqrt{2}ie^{\frac{\pi i\ell}{6\gamma'}\alpha'(r_1+\ell)+\frac{\pi i\delta'\ell}{24\gamma'}(r_1+4\ell)-\frac{\pi i\delta'}{8\gamma'}+\frac{2\pi ij^2\delta'\gamma_1'}{t}}\cdot\tan\left(\frac{\pi j}{t}\right)\frac{\omega^2_{\alpha',\gamma'}}{\omega_{2\alpha',\gamma'}\cdot\omega^{r_1}_{2\alpha',\gamma'/\ell}\cdot\omega^{2\ell}_{\alpha',\gamma'/\ell}}$$
$$(-i(\gamma'\tau+\delta'))^{\frac{r_1+2\ell+1}{2}}\cdot\eta^{2\ell}(\ell z)\cdot\eta^{r_1}\left(\frac{\ell z}{2}\right)\cdot\mathcal{U}\left(-\frac{j\delta}{t};z\right). \tag{6.9}$$

Again using (6.8), one can show that one can change α', δ', and γ' into α, δ, and γ, respectively if one changes z into $z+\frac{\nu'}{\ell}$. The expansion of (6.9) starts with $q^{\frac{1}{48}(\ell r_1+4\ell^2-3)}$. For $r_0 := \frac{1}{48}(\ell r_1+4\ell^2-3)$ one clearly has $(r_0,\ell)=1$. Moreover

$$\frac{g}{\ell}\left(\frac{-1}{\ell}\right)\sum_{\substack{\nu \pmod{\ell} \\ \nu,\nu'\equiv 0 \pmod{48}}}\left(\frac{\nu}{\ell}\right)e^{-\frac{2\pi i r_0\nu'}{\ell}}=\left(\frac{-r_0}{\ell}\right)=1.$$

Thus the expansion of $\left(f_{r,j}-\left(\frac{-1}{\ell}\right)(f_{r,j})_\ell\right)|\left(\begin{smallmatrix}\alpha' & \beta' \\ \gamma' & \delta'\end{smallmatrix}\right)$ starts at least with $q^{\frac{1}{48}(\ell r_1+4\ell^2+45)}$ which implies the claim. □

7. Proof of Theorem 1.5

Proof of Theorem 1.5. ¿From (5.1) it follows that

$$\sum_{n=0}^{\infty}\left(\overline{N}(s_i,\ell;n)-\frac{\overline{p}(n)}{\ell}\right)q^n$$

for $i\in\{1,2\}$ is the holomorphic part of a weak Maass form on $\Gamma_1(16\ell^2)$. Moreover the non-holomorphic part is supported on negative squares. The same is true for the function

$$\sum_{n=0}^{\infty}\left(\overline{N}(s_1,\ell;n)-\overline{N}(s_2,\ell;n)\right)q^n.$$

The restriction of the associated weak Maass form to those coefficients congruent to d modulo ℓ gives a weak Maass form on $\Gamma_1(16\ell^4)$. Since $\left(\frac{d}{\ell}\right)=-\left(\frac{-1}{\ell}\right)$ it does not have a non-holomorphic part which proves Theorem 1.5. □

Next we state some identities which may be deduced thanks to our theorem. Define for a positive integer N, g,h real numbers that are not simultaneously congruent to $0 \pmod{N}$, the generalized Dedekind eta-function

$$E_{g,h}(z):=q^{\frac{1}{2}B\left(\frac{g}{N}\right)}\cdot\prod_{m=1}^{\infty}\left(1-\zeta_N^h\cdot q^{m-1+\frac{g}{N}}\right)\left(1-\zeta_N^{-h}\cdot q^{m-\frac{g}{N}}\right),$$

where $B(x):=x^2-x+\frac{1}{6}$. We have the following identities.

$$\sum_{n=0}^{\infty}\left(\overline{N}(1,5,5n+2)-\overline{N}(2,5,5n+2)\right)q^{5n+2}=2\frac{\eta(50z)}{E_{1,0}(25z)}, \tag{7.1}$$

$$\sum_{n=0}^{\infty}\left(\overline{N}(1,5,5n+3)-\overline{N}(2,5,5n+3)\right)q^{5n+3}=-2\frac{\eta(50z)}{E_{2,0}(25z)}, \tag{7.2}$$

$$\sum_{n=0}^{\infty} \left(\overline{N}(0,5,5n+3) - \overline{N}(2,5,5n+3)\right) q^{5n+3} = 2\frac{\eta(50z)}{E_{2,0}(25z)}, \tag{7.3}$$

$$\sum_{n=0}^{\infty} \left(\overline{N}(0,5,5n+2) - \overline{N}(2,5,5n+2)\right) q^{5n+2} = 0, \tag{7.4}$$

$$\sum_{n=0}^{\infty} \left(\overline{N}(0,3,3n+1) - \overline{N}(1,3,3n+1)\right) q^{3n+1} = 2\frac{\eta(9z)\cdot\eta(18z)}{\eta(3z)}. \tag{7.5}$$

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School of Mathematics, University of Minnesota, Minneapolis, MN 55455, U.S.A.
E-mail address: bringman@math.umn.edu

CNRS, LIAFA, Université Denis Diderot, 2, Place Jussieu, Case 7014, F-75251 Paris Cedex 05, FRANCE
E-mail address: lovejoy@liafa.jussieu.fr