ARITHMETIC PROPERTIES OF COEFFICIENTS OF HALF-INTEGRAL WEIGHT MAASS-POINCARÉ SERIES

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1. Introduction and Statement of Results

Let $j(z)$ be the usual modular function for $\mathrm{SL}_2(\mathbb{Z})$

$$j(z) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots ,$$

where $q = e^{2\pi i z}$. The values of modular functions such as $j(z)$ at imaginary quadratic arguments in $\mathbb{H}$, the upper half of the complex plane, are known as *singular moduli*. Singular moduli are algebraic integers which play many roles in number theory. For example, they generate class fields of imaginary quadratic fields, and they parameterize isomorphism classes of elliptic curves with complex multiplication. We shall slightly abuse terminology by referring to the value of any modular invariant at an imaginary quadratic argument as a singular modulus.

In an important paper [23], Zagier gave a new proof of Borcherds' famous theorem on the infinite product expansions of integer weight modular forms on $\mathrm{SL}_2(\mathbb{Z})$ with Heegner divisor. This proof, as well as all of the results of [23], are connected to his beautiful observation that the generating functions for traces of singular moduli are essentially weight $3/2$ weakly holomorphic modular forms.

Remark. The observation that coefficients of certain automorphic forms can be expressed in terms of singular moduli is not entirely new. Earlier works by Maass [17], and Katok and Sarnak [13] contain such results in different contexts.

Zagier's results have inspired an astonishing number of recent works (for example, see [1, 3, 6, 7, 8, 9, 11, 12, 14, 15, 21, 22]). In view of the importance of his paper, combined with the fact that he only provides sketches of proofs for some of the key theorems (e.g. Theorems 6, 9, 10, 11), here we systematically revisit his work from the context of Maass-Poincaré series. Our uniform approach includes these key theorems as special cases of corollaries of a single theorem (i.e. Theorem 2.1), and, as an added bonus, gives exact formulas for traces of singular moduli.

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To make Zagier's results more precise, we first recall some definitions and fix notation. For integers λ, let $M^{!}_{\lambda+\frac{1}{2}}$ be the complex vector space of weight $\lambda+\frac{1}{2}$ *weakly holomorphic modular forms* on $\Gamma_0(4)$ satisfying the "Kohnen plus-space" condition. Recall that a meromorphic modular form is said to be weakly holomorphic if its poles (if there are any) are supported at the cusps, and it is said to satisfy Kohnen's plus-space condition if its Fourier expansion has the form

$$\sum_{(-1)^{\lambda}n\equiv 0,1 \pmod 4} a(n)q^n. \tag{1.1}$$

Let $M^{+}_{\lambda+\frac{1}{2}}$ (resp. $S^{+}_{\lambda+\frac{1}{2}}$) be the subspace of $M^{!}_{\lambda+\frac{1}{2}}$ consisting of those forms which are holomorphic modular forms (resp. cusp forms).

Throughout, let $d\equiv 0,3 \pmod 4$ be a positive integer (so that $-d$ is the discriminant of an order in an imaginary quadratic field), and let $H(d)$ be the Hurwitz-Kronecker class number for the discriminant $-d$. Let $\mathcal{Q}_d$ be the set of positive definite integral binary quadratic forms (note. including imprimitive forms, if there are any)

$$Q(x,y)=[a,b,c]=ax^2+bxy+cy^2$$

with discriminant $-d=b^2-4ac$. For each Q, let τ_Q be the unique complex number in the upper half-plane which is a root of $Q(x,1)=0$. The singular modulus $f(\tau_Q)$, for any modular invariant $f(z)$, depends only on the equivalence class of Q under the action of $\Gamma:=\mathrm{PSL}_2(\mathbb{Z})$.

If $\omega_Q\in\{1,2,3\}$ is given by

$$\omega_Q:=\begin{cases}2 & \text{if } Q\sim_{\Gamma}[a,0,a],\\ 3 & \text{if } Q\sim_{\Gamma}[a,a,a],\\ 1 & \text{otherwise},\end{cases}$$

then, for a modular invariant $f(z)$, define the trace $\mathrm{Tr}(f;d)$ by

$$\mathrm{Tr}(f;d):=\sum_{Q\in\mathcal{Q}_d/\Gamma}\frac{f(\tau_Q)}{\omega_Q}. \tag{1.2}$$

In this notation, Theorems 1 and 5 of [23] imply the following.

Theorem. (Zagier)
If $f(z)\in\mathbb{Z}[j(z)]$ has a Fourier expansion with constant term 0, then there is a finite principal part $A_f(z)=\sum_{n\leq 0}a_f(n)q^n$ for which

$$A_f(z)+\sum_{0<d\equiv 0,3 \pmod 4}\mathrm{Tr}(f;d)q^d\in M^{!}_{\frac{3}{2}}.$$

Remark. For brevity, we do not give a precise description of $A_f(z)$. It is easily determined using Theorems 1 and 5 of [23].

Example. If $\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$ is Dedekind's eta-function, then for $f(z) = j(z) - 744$ we have

$$-\frac{\eta(z)^2 \cdot \left(1 + 240 \sum_{n=1}^{\infty} \sum_{v|n} v^3 q^{4n}\right)}{\eta(2z)\eta(4z)^6} = -q^{-1} + 2 + \sum_{0<d\equiv 0,3 \pmod 4} \mathrm{Tr}(f;d)q^d \tag{1.3}$$
$$= -q^{-1} + 2 - 248q^3 + 492q^4 + \cdots \in M^!_{\frac{3}{2}}.$$

Remark. Using Poincaré series constructed in [7], facts about non-holomorphic modular forms due to Niebur [19], and facts about half-integral weight Kloosterman-Salié sums, Duke [8] and Jenkins [11] have provided new proofs of Theorems 1 and 5 of [23].

Remark. Kim [14, 15] has established the modularity for traces of singular moduli on certain genus zero congruence subgroups. Using theta lifts, Bruinier and Funke [6] have recently proven a more general theorem which holds for modular functions on modular curves of arbitrary genus. In all of these cases, the corresponding generating functions are weight 3/2 weakly holomorphic modular forms.

In [23], Zagier includes several generalizations of these results. Here we highlight two of these; the first concerns "twisted traces" of singular moduli. For fundamental discriminants D_1, let χ_{D_1} denote the associated genus character for positive definite binary quadratic forms whose discriminants are multiples of D_1 (see (3.2)). If λ is an integer and D_2 is a non-zero integer for which $(-1)^\lambda D_2 \equiv 0, 1 \pmod 4$ and $(-1)^\lambda D_1 D_2 < 0$, then define the twisted trace of a modular invariant $f(z)$, say $\mathrm{Tr}_{D_1}(f;D_2)$, by

$$\mathrm{Tr}_{D_1}(f;D_2) := \sum_{Q \in \mathcal{Q}_{|D_1D_2|}/\Gamma} \frac{\chi_{D_1}(Q) f(\tau_Q)}{\omega_Q}. \tag{1.4}$$

Zagier proved that these traces are also given by coefficients of weight 3/2 modular forms (see Theorem 6 of [23]).

The second generalization is discussed in the last section of [23]. There Zagier considers $\mathrm{Tr}(f;d)$ for special non-holomorphic modular functions $f(z)$. In these cases, the corresponding generating functions have weight $\lambda + \frac{1}{2}$, where $\lambda \in \{-6, -4, -3, -2, -1\}$ (see Theorems 10 and 11 of [23]).

We explicitly represent the coefficients of certain half-integral weight Maass-Poincaré series as traces of singular moduli. This result (see Theorem 1.2) includes the results of Zagier described above, and, as an added bonus, gives exact formulas for these traces. Apart from the calculations required to compute the Fourier expansions of these series, the proofs require little more than classical facts about Kloosterman and Salié sums.

To prove all of his results, Zagier instead depends on the existence of special bases for $M^!_{\lambda+\frac{1}{2}}$. These "good" bases occur in pairs, and they satisfy a striking "duality". A prominent example of this phenomenon pertains to the spaces $M^!_{\frac{1}{2}}$ and $M^!_{\frac{3}{2}}$. There is a natural infinite basis $\{F_1(-1;z), F_1(-4;z), F_1(-5;z), \dots\}$ for $M^!_{\frac{3}{2}}$, and the first few coefficients of these series are

$$\begin{array}{rcllllll} F_1(-1;z) & = & q^{-1} & -2 & +248q^3 & -492q^4 & +4119q^7 & -\cdots, \\ F_1(-4;z) & = & q^{-4} & -2 & -26752q^3 & -143376q^4 & -8288256q^7 & -\cdots, \\ F_1(-5;z) & = & q^{-5} & +0 & +85995q^3 & -565760q^4 & +52756480q^7 & -\cdots. \end{array} \tag{1.5}$$

Similarly, there is a basis $\{F_0(0;z), F_0(-3;z), F_0(-4;z), \dots\}$ for $M^!_{\frac{1}{2}}$, and the first few coefficients of these forms are

$$\begin{array}{rclllll} F_0(0;z) & = & 1 & +2q & +2q^4 & +0q^5 & +\cdots, \\ F_0(-3;z) & = & q^{-3} & -248q & +26752q^4 & -85995q^5 & +\cdots, \\ F_0(-4;z) & = & q^{-4} & +492q & +143376q^4 & +565760q^5 & +\cdots, \\ F_0(-7;z) & = & q^{-7} & -4119q & +8288256q^4 & -52756480q^5 & +\cdots. \end{array} \tag{1.6}$$

A brief inspection reveals a striking pattern relating the coefficients of the $F_1(-m;z)$, grouped by column, and the coefficients of the individual forms $F_0(-n;z)$. Zagier proved (see Theorem 4 of [23]), for non-negative integers n, that the nth coefficient of a form $F_1(-m;z)$ is the negative of the mth coefficient of $F_0(-n;z)$. Zagier's examples turn out to be special cases of generic duality which holds for the half-integral weight Maass-Poincaré series considered here (see Theorem 1.1).

To construct these series, let $k := \lambda + \frac{1}{2}$, where λ is an arbitrary integer, and let $M_{\nu,\mu}(z)$ be the usual M-Whittaker function (for example, see Chapter 4 of [2]) which is a solution to the differential equation

$$\frac{\partial^2 u}{\partial z^2} + \left(-\frac{1}{4} + \frac{\nu}{z} + \frac{\frac{1}{4}-\mu^2}{z^2}\right) u = 0.$$

Following Bruinier [4] (note. see [4] and [10] for background on Poincaré series) and [7], for $s \in \mathbb{C}$ and $y \in \mathbb{R} - \{0\}$ we define

$$\mathcal{M}_s(y) := |y|^{-\frac{k}{2}} M_{\frac{k}{2}\operatorname{sgn}(y),\, s-\frac{1}{2}}(|y|).$$

Suppose that $m \geq 1$ is an integer with $(-1)^{\lambda+1} m \equiv 0, 1 \pmod 4$. Define $\varphi_{-m,s}(z)$ by

$$\varphi_{-m,s}(z) := \mathcal{M}_s(-4\pi m y) e(-mx),$$

where $z = x + iy$, and $e(w) := e^{2\pi i w}$. Furthermore, let

$$\Gamma_\infty := \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \ : \ n \in \mathbb{Z} \right\}$$

denote the translations within $\mathrm{SL}_2(\mathbb{Z})$. Using this notation, define the Poincaré series

$$\mathcal{F}_\lambda(-m, s; z) := \sum_{A \in \Gamma_\infty \backslash \Gamma_0(4)} (\varphi_{-m,s} \mid_k A)(z) \tag{1.7}$$

for $\mathrm{Re}(s) > 1$ (see Section 2 for the definition of $|_k$).

Remark. The Fourier expansions of these series were previously computed in [7] for integers $\lambda \geq 1$.

If pr_λ is Kohnen's projection operator (see page 250 of [16]) to the weight $\lambda + \frac{1}{2}$ plus-space for $\Gamma_0(4)$, then for $\lambda \notin \{0,1\}$ define $F_\lambda(-m;z)$ by

$$F_\lambda(-m;z) := \begin{cases} \frac{3}{2}\mathcal{F}_\lambda\left(-m, \frac{k}{2}; z\right) \mid \mathrm{pr}_\lambda & \text{if } \lambda \geq 2, \\ \\ \frac{3}{2(1-k)\Gamma(1-k)}\mathcal{F}_\lambda\left(-m, 1-\frac{k}{2}; z\right) \mid \mathrm{pr}_\lambda & \text{if } \lambda \leq -1. \end{cases} \tag{1.8}$$

Remark. Strictly speaking, Kohnen defined pr_λ for holomorphic half-integral weight forms. It is easy to see that its definition applies for weakly holomorphic forms.

Remark. For $\lambda \leq -1$, the normalization above is chosen so that (1.9) and (1.10) below are valid.

If $\lambda = 0$ or 1, then we also have Poincaré series $F_\lambda(-m;z)$, but the construction requires more care (see Section 2). If $\lambda \geq -6$ with $\lambda \neq -5$, then $F_\lambda(-m;z) \in M^!_{\lambda+\frac{1}{2}}$. For such λ, it turns out $F_\lambda(-m;z)$ has an expansion of the form

$$F_\lambda(-m;z) = q^{-m} + \sum_{\substack{n\geq 0 \\ (-1)^\lambda n \equiv 0,1 \pmod 4}} b_\lambda(-m;n)q^n \in M^!_{\lambda+\frac{1}{2}}. \tag{1.9}$$

Remark. If $\lambda = 1$ and $-m = -1$, then

$$-F_1(-1;z) = -q^{-1} + 2 - 248q^3 + 492q^4 + \cdots \in M^!_{\frac{3}{2}}$$

is the modular form in (1.3). Furthermore, when $\lambda \in \{0,1\}$, these modular forms coincide with those given in (1.5) and (1.6).

If $\lambda = -5$ or $\lambda \leq -7$, then $F_\lambda(-m;z)$ is a weak Maass form of weight $\lambda + \frac{1}{2}$, and has a Fourier expansion of the form

$$F_\lambda(-m;z) = B_\lambda(-m;z) + q^{-m} + \sum_{\substack{n\geq 0 \\ (-1)^\lambda n \equiv 0,1 \pmod 4}} b_\lambda(-m;n)q^n, \tag{1.10}$$

where $B_\lambda(-m;z)$ is the "non-holomorphic" part of $F_\lambda(-m;z)$ described in Section 2.

The duality illustrated by (1.5) and (1.6) is a special case of the following general result which holds for all of the $F_\lambda(-m;z)$.

Theorem 1.1. *Assume the notation above. Suppose that $\lambda \geq 1$, and that m is a positive integer for which $(-1)^{\lambda+1}m \equiv 0,1 \pmod 4$. For every positive integer n with $(-1)^\lambda n \equiv 0,1 \pmod 4$ we have*

$$b_\lambda(-m;n) = -b_{1-\lambda}(-n;m).$$

Remark. Theorem 1.1 for $\lambda = 1$ (resp. $\lambda \in \{2,3,4,5,7\}$) is Theorem 4 of [23] (resp. Theorem 9 of [23]).

Remark. As an alternative approach, one can derive Theorem 1.1, for $\lambda \geq 2$, by applying Proposition 1.16 of [4], and by interpreting the relevant forms as vector valued forms on $\mathrm{Mp}_2(\mathbb{Z})$.

Motivated by Zagier's results, we now describe the coefficients of the $F_\lambda(-m;z)$ in terms of singular moduli. To state these results, we require the following Poincaré series defined by Niebur [19]. Throughout, let $I_s(x)$ denote the usual I-Bessel function, the so-called modified Bessel function of the first kind. If $\lambda > 1$, then let

$$\mathfrak{F}_\lambda(z) := \pi \sum_{A\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})} \mathrm{Im}(Az)^{\frac{1}{2}} I_{\lambda-\frac{1}{2}}(2\pi \mathrm{Im}(Az))e(-\mathrm{Re}(Az)). \tag{1.11}$$

Remark. For $\lambda = 1$, Niebur's [19] definition requires a careful argument involving analytic continuation. It turns out that $\mathfrak{F}_1(z) = \frac{1}{2}(j(z)-744)$, and this is the continuation, as $s \to 1$ from the right, of the expansion

$$-12 + \pi \sum_{A\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})} \mathrm{Im}(Az)^{\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi \mathrm{Im}(Az))e(-\mathrm{Re}(Az)).$$

The coefficients of $F_\lambda(-m;z)$ are traces and twisted traces of the singular moduli for $\mathfrak{F}_\lambda(z)$. In view of Theorem 1.1 on duality, to state this result we may without loss of generality assume that $\lambda \geq 1$.

Theorem 1.2. *If $\lambda, m \geq 1$ are integers for which $(-1)^{\lambda+1}m$ is a fundamental discriminant (note. which includes 1), then for each positive integer n with $(-1)^\lambda n \equiv 0, 1 \pmod 4$ we have*

$$b_\lambda(-m;n) = \frac{2(-1)^{[(\lambda+1)/2]}n^{\frac{\lambda}{2}-\frac{1}{2}}}{m^{\frac{\lambda}{2}}} \cdot \mathrm{Tr}_{(-1)^{\lambda+1}m}\left(\mathfrak{F}_\lambda; n\right).$$

Three remarks.
1) For $\lambda = 1$, Theorem 1.2 relates $b_1(-m;n)$ to traces and twisted traces of $\mathfrak{F}_1(z) = \frac{1}{2}(j(z)-744)$. Therefore, if $\lambda = 1$ and $m = 1$, then this is Theorem 1 of [23] (i.e. example (1.3)), and for general m is Theorem 6 of [23].
2) For $\lambda > 1$, the coefficients $b_\lambda(-m;n)$ are traces of singular moduli of non-harmonic Maass forms. This relates to Theorem 11 of [23] where these non-harmonic Maass forms are images of weakly holomorphic modular forms under differential operators which have the additional property that their singular moduli are algebraic. This phenomenon is explained in more detail in a recent preprint of Miller and Pixton [18].
3) Although we do not include the details here for brevity, we note that Theorem 1.2 has a straightforward generalization involving the action of the half-integral weight Hecke operators on $F_\lambda(-m;z)$. For $\lambda = 1$, this generalization is Theorem 5 of [23].

To prove Theorem 1.2, we first recall and derive the Fourier expansions of the half-integral weight series considered here. This is done in Section 2. These Fourier expansions are described in terms of the half-integral weight Kloosterman sums. In

Section 3, we give standard facts about Kloosterman sums and Salié sums. In particular, we recall how Kloosterman sums are related to certain Salié sums, and how such sums may be reformulated as Poincaré-type series over orbits of CM points. In Section 3 we prove Theorem 1.1 using the formulas obtained in Section 2. In Section 4 we describe Niebur's Poincaré series $\mathfrak{F}_\lambda(z)$, and explain their relation to weakly holomorphic modular forms. In the last section we use the results of Sections 2 and 3 to directly prove Theorem 1.2.

Remark. Although we have not carried out the details, it is not difficult to generalize Theorems 1.1 and 1.2 to arbitrary congruence subgroups $\Gamma_0(4N)$, where N is odd and square-free.

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2. Half-integral weight Maass-Poincaré series

Here we recall and derive the Fourier expansions of the series $F_\lambda(-m;z)$. The common feature of these series is that their Fourier coefficients for positive exponents are given as explicit infinite sums in half-integral weight Kloosterman sums weighted by Bessel functions. To define these Kloosterman sums, for odd δ let

$$\epsilon_\delta := \begin{cases} 1 & \text{if } \delta \equiv 1 \pmod 4, \\ i & \text{if } \delta \equiv 3 \pmod 4. \end{cases} \tag{2.1}$$

If λ is an integer, then we define the $\lambda + \frac{1}{2}$ weight Kloosterman sum $K_\lambda(m,n,c)$ by

$$K_\lambda(m,n,c) := \sum_{v \pmod{c}^*} \left(\frac{c}{v}\right) \epsilon_v^{2\lambda+1} e\left(\frac{m\bar{v} + nv}{c}\right). \tag{2.2}$$

In the sum, v runs through the primitive residue classes modulo c, and $\bar{v}$ denotes the multiplicative inverse of v modulo c.

Here we give the Fourier expansions of the Maass-Poincaré series $F_\lambda(-m;z)$. For $\lambda \geq 1$, these expansions are computed in [7]. For completeness, we give the following result giving formulas for the coefficients $b_\lambda(-m;n)$ for all λ. For convenience, we define $\delta_{\square,m} \in \{0,1\}$ by

$$\delta_{\square,m} := \begin{cases} 1 & \text{if } m \text{ is a square}, \\ 0 & \text{otherwise}, \end{cases} \tag{2.3}$$

and we let

$$\delta_{\text{odd}}(v) := \begin{cases} 1 & \text{if } v \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

Assuming the notation of (1.9) and (1.10), we have the following theorem.

Theorem 2.1. *Suppose that λ is an integer, and suppose that m is a positive integer for which $(-1)^{\lambda+1}m \equiv 0,1 \pmod 4$. Furthermore, suppose that n is a non-negative integer for which $(-1)^{\lambda}n \equiv 0,1 \pmod 4$.*

(1) *If $\lambda \geq 2$, then $b_\lambda(-m;0)=0$, and for positive n we have*

$$b_\lambda(-m;n) = (-1)^{[(\lambda+1)/2]}\pi\sqrt{2}(n/m)^{\frac{\lambda}{2}-\frac{1}{4}}(1-(-1)^{\lambda}i) \times \sum_{\substack{c>0 \\ c\equiv 0 \pmod 4}} (1+\delta_{\text{odd}}(c/4))\frac{K_\lambda(-m,n,c)}{c}\cdot I_{\lambda-\frac{1}{2}}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

(2) *If $\lambda \leq -1$, then*

$$b_\lambda(-m;0) = (-1)^{[(\lambda+1)/2]}\pi^{\frac{3}{2}-\lambda}2^{1-\lambda}m^{\frac{1}{2}-\lambda}(1-(-1)^{\lambda}i) \times \frac{1}{(\frac{1}{2}-\lambda)\Gamma(\frac{1}{2}-\lambda)} \sum_{\substack{c>0 \\ c\equiv 0 \pmod 4}} (1+\delta_{\text{odd}}(c/4))\frac{K_\lambda(-m,0,c)}{c^{\frac{3}{2}-\lambda}},$$

and for positive n we have

$$b_\lambda(-m;n) = (-1)^{[(\lambda+1)/2]}\pi\sqrt{2}(n/m)^{\frac{\lambda}{2}-\frac{1}{4}}(1-(-1)^{\lambda}i) \times \sum_{\substack{c>0 \\ c\equiv 0 \pmod 4}} (1+\delta_{\text{odd}}(c/4))\frac{K_\lambda(-m,n,c)}{c}\cdot I_{\frac{1}{2}-\lambda}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

(3) *If $\lambda = 1$, then $b_1(-m;0) = -2\delta_{\Box,m}$, and for positive n we have*

$$b_1(-m;n) = 24\delta_{\Box,m}H(n) - \pi\sqrt{2}(n/m)^{\frac{1}{4}}(1+i) \times \sum_{\substack{c>0 \\ c\equiv 0 \pmod 4}} (1+\delta_{\text{odd}}(c/4))\frac{K_1(-m,n,c)}{c}\cdot I_{\frac{1}{2}}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

(4) *If $\lambda = 0$, then $b_0(-m;0)=0$, and for positive n we have*

$$b_0(-m;n) = -24\delta_{\Box,n}H(m) + \pi\sqrt{2}(m/n)^{\frac{1}{4}}(1-i) \times \sum_{\substack{c>0 \\ c\equiv 0 \pmod 4}} (1+\delta_{\text{odd}}(c/4))\frac{K_0(-m,n,c)}{c}\cdot I_{\frac{1}{2}}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

Remark. For positive integers m and n, the formulas for $b_\lambda(-m;n)$ are nearly uniform in λ. In particular, the only difference between Theorem 2.1 (1) and (2) occurs in the I-Bessel function factor. For $\lambda \geq 2$ we have $I_{\lambda-\frac{1}{2}}$, and for $\lambda \leq -1$ we have $I_{\frac{1}{2}-\lambda}$ instead.

Before we prove this result, we first recall the construction of these forms. Suppose that λ is an integer, and that $k := \lambda + \frac{1}{2}$. For each $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(4)$, let

$$j(A,z) := \left(\frac{\gamma}{\delta}\right)\epsilon_\delta^{-1}(\gamma z+\delta)^{\frac{1}{2}}$$

be the usual factor of automorphy for half-integral weight modular forms. If $f : \mathbb{H} \to \mathbb{C}$ is a function, then for $A \in \Gamma_0(4)$ we let

$$(f \mid_k A)(z) := j(A,z)^{-2\lambda-1} f(Az). \tag{2.5}$$

As usual, let $z = x+iy$ be the standard variable on $\mathbb{H}$. For $s \in \mathbb{C}$ and $y \in \mathbb{R} - \{0\}$, we let

$$\mathcal{M}_s(y) := |y|^{-\frac{k}{2}} M_{\frac{k}{2}\,\mathrm{sgn}(y),\, s-\frac{1}{2}}(|y|), \tag{2.6}$$

where $M_{\nu,\mu}(z)$ is the standard M-Whittaker function which is a solution to the differential equation

$$\frac{\partial^2 u}{\partial z^2} + \left(-\frac{1}{4} + \frac{\nu}{z} + \frac{\frac{1}{4}-\mu^2}{z^2}\right) u = 0.$$

If m is a positive integer, then define $\varphi_{-m,s}(z)$ by

$$\varphi_{-m,s}(z) := \mathcal{M}_s(-4\pi m y)e(-mx),$$

and consider the Poincaré series

$$\mathcal{F}_\lambda(-m,s;z) := \sum_{A \in \Gamma_\infty \backslash \Gamma_0(4)} (\varphi_{-m,s} \mid_k A)(z). \tag{2.7}$$

It is easy to verify that $\varphi_{-m,s}(z)$ is an eigenfunction, with eigenvalue

$$s(1-s) + (k^2-2k)/4, \tag{2.8}$$

of the weight k hyperbolic Laplacian

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Since $\varphi_{-m,s}(z) = O\left(y^{\mathrm{Re}(s)-\frac{k}{2}}\right)$ as $y \to 0$, it follows that $\mathcal{F}_\lambda(-m,s;z)$ converges absolutely for $\mathrm{Re}(s) > 1$, is a $\Gamma_0(4)$-invariant eigenfunction of the Laplacian, and is real analytic.

These series provide examples of weak Maass forms of half-integral weight. Following Bruinier and Funke [5], we make the following definition.

Definition 2.2. *A weak Maass form of weight k for the group $\Gamma_0(4)$ is a smooth function $f : \mathbb{H} \to \mathbb{C}$ satisfying the following:*

(1) *For all $A \in \Gamma_0(4)$ we have*
$$(f \mid_k A)(z) = f(z).$$
(2) *We have $\Delta_k f = 0$.*
(3) *The function $f(z)$ has at most linear exponential growth at all the cusps.*

Remark. If a weak Maass form $f(z)$ is holomorphic on $\mathbb{H}$, then it is a weakly holomorphic modular form.

In view of (2.8), it follows that the special s-values at $k/2$ and $1-k/2$ of $\mathcal{F}_\lambda(-m, s; z)$ are weak Maass forms of weight $k = \lambda + \frac{1}{2}$ when the defining series is absolutely convergent.

If $\lambda \notin \{0, 1\}$ and $m \geq 1$ is an integer for which $(-1)^{\lambda+1} m \equiv 0, 1 \pmod 4$, then we recall the definition

$$F_\lambda(-m; z) := \begin{cases} \frac{3}{2}\mathcal{F}_\lambda\left(-m, \frac{k}{2}; z\right) \mid \mathrm{pr}_\lambda & \text{if } \lambda \geq 2, \\ \\ \frac{3}{2(1-k)\Gamma(1-k)}\mathcal{F}_\lambda\left(-m, 1-\frac{k}{2}; z\right) \mid \mathrm{pr}_\lambda & \text{if } \lambda \leq -1. \end{cases} \tag{2.9}$$

By the discussion above, it follows that $F_\lambda(-m; z)$ is a weak Maass form of weight $k = \lambda + \frac{1}{2}$ on $\Gamma_0(4)$. If $\lambda = 1$ and m is a positive integer for which $m \equiv 0, 1 \pmod 4$, then define $F_1(-m; z)$ by

$$F_1(-m; z) := \frac{3}{2}\mathcal{F}_1\left(-m, \frac{3}{4}; z\right) \mid \mathrm{pr}_1 + 24\delta_{\square, m} G(z). \tag{2.10}$$

The function $G(z)$ is given by the Fourier expansion

$$G(z) := \sum_{n=0}^{\infty} H(n) q^n + \frac{1}{16\pi\sqrt{y}} \sum_{n=-\infty}^{\infty} \beta(4\pi n^2 y) q^{-n^2},$$

where $H(0) = -1/12$ and

$$\beta(s) := \int_1^\infty t^{-\frac{3}{2}} e^{-st} dt.$$

Proposition 3.6 of [7] proves that each $F_1(-m; z)$ is in $M^!_{\frac{3}{2}}$. These series form the basis given in (1.5).

Remark. Note that the integral $\beta(s)$ is easily reformulated in terms of the incomplete Gamma-function. We make this observation since the non-holomorphic parts of the $F_\lambda(-m; z)$, for $\lambda \leq -7$ and $\lambda = -5$, will be described in such terms.

Remark. We may define the series $F_0(-m;z) \in M^!_{\frac{1}{2}}$ using an argument analogous to Proposition 3.6 of [7]. Instead, we simply note that the existence of the basis (1.6) of $M^!_{\frac{1}{2}}$, together with the duality of Theorem 4 [23] and an elementary property of Kloosterman sums (see Proposition 3.1), gives a direct realization of the Fourier expansions of $F_0(-m;z)$ in terms of the expansions of the $F_1(-n;z)$ described above.

To compute the Fourier expansions of these weak Maass forms, we require some further preliminaries. For $s \in \mathbb{C}$ and $y \in \mathbb{R} - \{0\}$, we let

$$\mathcal{W}_s(y) := |y|^{-\frac{k}{2}} W_{\frac{k}{2}\,\mathrm{sgn}(y),\, s-\frac{1}{2}}(|y|), \tag{2.11}$$

where $W_{\nu,\mu}$ denotes the usual W-Whittaker function. For $y > 0$, we shall require the following relations:

$$\mathcal{M}_{\frac{k}{2}}(-y) = e^{\frac{y}{2}}, \tag{2.12}$$

$$\mathcal{W}_{1-\frac{k}{2}}(y) = \mathcal{W}_{\frac{k}{2}}(y) = e^{-\frac{y}{2}}, \tag{2.13}$$

and

$$\mathcal{W}_{1-\frac{k}{2}}(-y) = \mathcal{W}_{\frac{k}{2}}(-y) = e^{\frac{y}{2}}\Gamma\left(1-k, y\right), \tag{2.14}$$

where

$$\Gamma(a,x) := \int_x^\infty e^{-t} t^a \frac{dt}{t}$$

is the incomplete Gamma function. For $z \in \mathbb{C}$, the functions $M_{\nu,\mu}(z)$ and $M_{\nu,-\mu}(z)$ are related by the identity

$$W_{\nu,\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\nu)} M_{\nu,\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\nu)} M_{\nu,-\mu}(z).$$

From these facts, we easily find, for $y > 0$, that

$$\mathcal{M}_{1-\frac{k}{2}}(-y) = (k-1)e^{\frac{y}{2}}\Gamma(1-k,y) + (1-k)\Gamma(1-k)e^{\frac{y}{2}}. \tag{2.15}$$

Proof of Theorem 2.1. Although the conclusions of Theorem 2.1 (1) and (3) were obtained previously by Bruinier, Jenkins and the second author in [7], for completeness we consider the calculation for general $\lambda \notin \{0,1\}$. In particular, for such λ, suppose that m is a positive integer for which $(-1)^{\lambda+1}m \equiv 0, 1 \pmod 4$.

Computing the Fourier expansion requires calculating the integral

$$\int_{-\infty}^{\infty} z^{-k} \mathcal{M}_s\left(-4\pi m \frac{y}{c^2|z|^2}\right) e\left(\frac{mx}{c^2|z|^2} - nx\right) dx.$$

This integral is computed on page 357 of [10] (see also page 33 of [4]), and it implies that $\mathcal{F}_\lambda\left(-m,s;z\right)$ has a Fourier expansion of the form

$$\mathcal{F}_\lambda\left(-m,s;z\right) = \mathcal{M}_s(-4\pi m y)e(-mx) + \sum_{n\in\mathbb{Z}} c(n,y,s)e(nx),$$

where the coefficients $c(n,y,s)$ are given by

$$\begin{cases} \dfrac{2\pi i^{-k}\Gamma(2s)}{\Gamma(s-\frac{k}{2})}\left|\dfrac{n}{m}\right|^{\frac{\lambda}{2}-\frac14} \displaystyle\sum_{\substack{c>0\\ c\equiv 0 \pmod 4}} \dfrac{K_\lambda(-m,n,c)}{c} J_{2s-1}\left(\dfrac{4\pi\sqrt{|mn|}}{c}\right)\mathcal{W}_s(4\pi n y), & n<0,\\ \dfrac{2\pi i^{-k}\Gamma(2s)}{\Gamma(s+\frac{k}{2})}(n/m)^{\frac{\lambda}{2}-\frac14} \displaystyle\sum_{\substack{c>0\\ c\equiv 0 \pmod 4}} \dfrac{K_\lambda(-m,n,c)}{c} I_{2s-1}\left(\dfrac{4\pi\sqrt{mn}}{c}\right)\mathcal{W}_s(4\pi n y), & n>0,\\ \dfrac{4^{\frac34-\frac{\lambda}{2}}\pi^{\frac34+s-\frac{\lambda}{2}}i^{-k}m^{s-\frac{\lambda}{2}-\frac14}y^{\frac34-s-\frac{\lambda}{2}}\Gamma(2s-1)}{\Gamma(s+\frac{k}{2})\Gamma(s-\frac{k}{2})} \displaystyle\sum_{\substack{c>0\\ c\equiv 0 \pmod 4}} \dfrac{K_\lambda(-m,0,c)}{c^{2s}}, & n=0. \end{cases}$$

Here $J_s(x)$ denotes the usual J-Bessel function, the so-called Bessel function of the first kind.

The Fourier expansion defines an analytic continuation of $\mathcal{F}_\lambda(-m,s;z)$ to $\mathrm{Re}(s)>3/4$. For $\lambda\geq 2$, we then find that $\mathcal{F}_\lambda(-m,\frac{k}{2};z)$ is a weak Maass form of weight $\lambda+\frac12$. Thanks to the Γ-factor, the Fourier coefficients $c(n,y,s)$ vanish for negative n, and so each $\mathcal{F}_\lambda(-m,\frac{k}{2};z)$ is a weakly holomorphic modular form on $\Gamma_0(4)$. Applying Kohnen's projection operator (see page 250 of [16]) to these series gives Theorem 2.1 (1).

The situation is a little more complicated if $\lambda\leq -1$. Arguing as above, by letting $s=1-k/2$ we obtain a weak Maass form $\mathcal{F}_\lambda(-m,1-\frac{k}{2};z)$ of weight $k=\lambda+\frac12$ on $\Gamma_0(4)$. Using (2.14) and (2.15), we find that its Fourier expansion has the form

$$\begin{aligned} &\mathcal{F}_\lambda\left(-m,1-\frac{k}{2};z\right)\\ &\qquad=(k-1)\left(\Gamma(1-k,4\pi m y)-\Gamma(1-k)\right)q^{-m}+\sum_{n\in\mathbb{Z}}c(n,y)e(nz), \end{aligned} \tag{2.16}$$

where the coefficients $c(n,y)$, for $n<0$, are given by

$$2\pi i^{-k}(1-k)\left|\frac{n}{m}\right|^{\frac{\lambda}{2}-\frac14}\Gamma(1-k,4\pi|n|y).\sum_{\substack{c>0\\ c\equiv 0 \pmod 4}}\frac{K_\lambda(-m,n,c)}{c}\cdot J_{\frac12-\lambda}\left(\frac{4\pi}{c}\sqrt{|mn|}\right).$$

For $n\geq 0$, (2.13) allows us to conclude that the $c(n,y)$ are given by

$$\begin{cases} 2\pi i^{-k}\Gamma(2-k)(n/m)^{\frac{\lambda}{2}-\frac14} \displaystyle\sum_{\substack{c>0\\ c\equiv 0 \pmod 4}} \dfrac{K_\lambda(-m,n,c)}{c}\cdot I_{\frac12-\lambda}\left(\dfrac{4\pi}{c}\sqrt{mn}\right), & n>0,\\ 4^{\frac34-\frac{\lambda}{2}}\pi^{\frac32-\lambda}i^{-k}m^{\frac12-\lambda} \displaystyle\sum_{\substack{c>0\\ c\equiv 0 \pmod 4}} \dfrac{K_\lambda(-m,0,c)}{c^{\frac32-\lambda}}. & n=0. \end{cases}$$

The numbers $b_\lambda(-m;n)$ are the Fourier coefficients of the images of the forms above under Kohnen's projection operator pr_λ. One readily checks that this returns the desired formulas. $\square$

Remark. If $\lambda \in \{-6,-4,-3,-2,-1\}$, then the functions $F_\lambda(-m;z)$ are in $M^!_{\lambda+\frac{1}{2}}$, and their q-expansions are of the form

$$F_\lambda(-m;z) = q^{-m} + \sum_{\substack{n\geq 0 \\ (-1)^\lambda n \equiv 0,1 \pmod 4}} b_\lambda(-m;n)q^n. \tag{2.17}$$

This claim is equivalent to the vanishing of the non-holomorphic terms appearing in the proof of Theorem 2.1 for these λ.

To justify this vanishing, recall that there is an anti-linear differential operator ξ_k that takes weak Maass forms of weight k to weakly holomorphic modular forms of weight $2-k$ (see Proposition 3.2 of [5]). It is defined by

$$\xi_k(f)(z) := 2iy^k \overline{\tfrac{\partial}{\partial \bar{z}} f(z)}. \tag{2.18}$$

This operator has the property that $\ker(\xi_k)$ is the subset of weight k weak Maass forms which are weakly holomorphic modular forms. To prove our claim, apply ξ_k to the Fourier expansion of $\mathcal{F}_\lambda\left(-m, 1-\frac{k}{2}; z\right)$ given by (2.16). Since $\xi_k(f) = 0$ for holomorphic f, and since ξ_k is anti-linear, the non-trivial contributions can only arise from $\xi_k\left(\Gamma\left(1-k, 4\pi|n|y\right)\right)$, where n is a negative integer. For negative n it is easy to show that

$$\xi_k\left(\Gamma(1-k, 4\pi|n|y)\right) = -(4\pi|n|)^{1-k} e^{4\pi n y}.$$

Therefore

$$\xi_k\left(\mathcal{F}_\lambda\left(-m, 1-\frac{k}{2}; z\right)\right) = \sum_{n>0} c(n)q^n,$$

where $c(n)$, for $n \neq m$, is

$$-2\pi i^{(\lambda+\frac{1}{2})}\left(\frac{1}{2}-\lambda\right)(n/m)^{\frac{\lambda}{2}-\frac{1}{4}}(4\pi n)^{\frac{1}{2}-\lambda} \sum_{\substack{c>0 \\ c\equiv 0 \pmod 4}} \overline{\frac{K_\lambda(-m,n,c)}{c}} \cdot \overline{J_{\frac{1}{2}-\lambda}\left(\frac{4\pi}{c}\sqrt{mn}\right)},$$

and for $n = m$ is

$$-\left(\frac{1}{2}-\lambda\right)\Gamma\left(\frac{1}{2}-\lambda\right)(4\pi m)^{\frac{1}{2}-\lambda} \\ -2\pi i^{(\lambda+\frac{1}{2})}\left(\frac{1}{2}-\lambda\right)(4\pi m)^{\frac{1}{2}-\lambda} \sum_{\substack{c>0 \\ c\equiv 0 \pmod 4}} \overline{\frac{K_\lambda(-m,m,c)}{c}} \cdot \overline{J_{\frac{1}{2}-\lambda}\left(\frac{4\pi}{c}m\right)}.$$

From these calculations it follows that $\xi_k\left(\mathcal{F}_\lambda\left(-m, 1-\frac{k}{2}; z\right)\right)$ is a weight $2-k$ cusp form on $\Gamma_0(4)$. Since $S^!_{2-k}$ is trivial for these λ, it follows that $c(n) = 0$ for all n.

Consequently, the coefficients for $n < 0$, with the exception of $n = -m$ must vanish. For $n = -m$, the normalization given by (1.8) now confirms (2.17).

3. Kloosterman and Salié sums and the proof of Theorem 1.1

Here we give classical facts concerning half-integral weight Kloosterman sums and Salié sums. We recall how such sums are related, and give a reformulation of certain Salié sums as Poincaré-type series over CM points. However, we begin by using the formulas from Theorem 2.1 to prove Theorem 1.1.

3.1. **Proof of Theorem 1.1.** Here we prove Theorem 1.1 using the explicit formulas contained in Theorem 2.1. Thanks to these formulas, duality follows from the following elementary proposition.

Proposition 3.1. *Suppose that $c > 0$ is a multiple of 4. If $\lambda, m,$ and n are integers, then*

$$K_\lambda(-m, n, c) = (-1)^\lambda i \cdot K_{1-\lambda}(-n, m, c).$$

Proof. For v coprime to c, we have that $\left(\frac{c}{v}\right) = \left(\frac{c}{-\overline{v}}\right)$, and $\overline{v} \equiv v \pmod 4$, and so

$$\begin{aligned} K_\lambda(-m, n, c) &= \sum_{v \pmod{c}^*} \left(\frac{c}{v}\right) \epsilon_v^{2\lambda+1} e\left(\frac{-m\overline{v} + nv}{c}\right) \\ &= \sum_{-\overline{v} \pmod{c}^*} \left(\frac{c}{v}\right) \epsilon_{-v}^{2\lambda+1} e\left(\frac{-n\overline{v} + mv}{c}\right). \end{aligned}$$

Since $\epsilon_{-v}^{2\lambda+1} = (-1)^\lambda i \epsilon_v^{2(1-\lambda)+1}$, it follows that $K_\lambda(-m, n, c) = (-1)^\lambda i \cdot K_{1-\lambda}(-n, m, c)$. □

Proof of Theorem 1.1. In Theorem 2.1, it is clear that the I-Bessel function factors exactly correspond for λ and $1-\lambda$. Furthermore, when $\lambda = 1$ and $1 - \lambda = 0$, Theorem 2.1 (3) and (4) shows that the class number summands obey the alleged duality. To complete the proof one simply observes that we may transform the formula for $b_\lambda(-m; n)$ into the formula for $-b_{1-\lambda}(-n; m)$. Thanks to Proposition 3.1, this is achieved by replacing, for each c, the Kloosterman sum $K_\lambda(-m, n, c)$ by $(-1)^\lambda i \cdot K_{1-\lambda}(-n, m, c)$. □

3.2. **Facts about Kloosterman sums and Salié sums.** The results in this paper are derived from the classical fact that the half-integral weight Kloosterman sums are easily described in terms of simpler sums, the Salié sums. Suppose that $D_1 \equiv 0, 1 \pmod 4$ is a fundamental discriminant. Recall that 1 is considered to be a fundamental discriminant. If λ is an integer, $D_2 \neq 0$ is an integer for which $(-1)^\lambda D_2 \equiv 0, 1 \pmod 4$, and N is a positive multiple of 4, then define the generalized Salié sum $S_\lambda(D_1, D_2, N)$ by

$$S_\lambda(D_1, D_2, N) := \sum_{\substack{x \pmod N \\ x^2 \equiv (-1)^\lambda D_1 D_2 \pmod N}} \chi_{D_1}\left(\frac{N}{4}, x, \frac{x^2 - (-1)^\lambda D_1 D_2}{N}\right) e\left(\frac{2x}{N}\right), \tag{3.1}$$

where $\chi_{D_1}(a, b, c)$, for a binary quadratic form $Q = [a, b, c]$, is given by

$$\chi_{D_1}(a, b, c) := \begin{cases} 0 & \text{if } (a, b, c, D_1) > 1, \\ \left(\frac{D_1}{r}\right) & \text{if } (a, b, c, D_1) = 1 \text{ and } Q \text{ represents } r \text{ with } (r, D_1) = 1. \end{cases} \tag{3.2}$$

Remark. If $D_1 = 1$, then χ_{D_1} is trivial. Therefore, if $(-1)^\lambda D_2 \equiv 0, 1 \pmod 4$, then

$$S_\lambda(1, D_2, N) = \sum_{\substack{x \pmod N \\ x^2 \equiv (-1)^\lambda D_2 \pmod N}} e\left(\frac{2x}{N}\right).$$

Half-integral weight Kloosterman sums are essentially equal to such Salié sums. Relations of this type are well known, and they play fundamental roles throughout the theory of half-integral weight modular forms (for example, in the work of Iwaniec, and later work of Duke, bounding coefficients of half-integral weight cusp forms). Here we recall a special case of such relations (for example, see Proposition 5 of [16]).

Proposition 3.2. *Suppose that N is a positive multiple of 4, and that D_1 is a fundamental discriminant. If λ is an integer, and D_2 is a non-zero integer for which $(-1)^\lambda D_2 \equiv 0, 1 \pmod 4$, then*

$$N^{-\frac{1}{2}}(1 - (-1)^\lambda i)(1 + \delta_{\mathrm{odd}}(N/4)) \cdot K_\lambda((-1)^\lambda D_1, D_2, N) = S_\lambda(D_1, D_2, N).$$

Using Proposition 3.2 and Theorem 2.1, we may rewrite the coefficients of the half-integral weight Maass-Poincaré series in terms of the simpler looking Salié sums. This simple reformulation, combined with the next proposition, underlies and explains the general phenomenon in which coefficients of half-integral weight modular forms are described as traces of singular moduli. The following proposition, well known to experts, describes these Salié sums themselves as Poincaré series over CM points.

Proposition 3.3. *Suppose that λ is an integer, and that D_1 is a fundamental discriminant. If D_2 is a non-zero integer for which $(-1)^\lambda D_2 \equiv 0, 1 \pmod 4$ and $(-1)^\lambda D_1 D_2 < 0$, then for every positive integer a we have*

$$S_\lambda(D_1, D_2, 4a) = 2 \sum_{Q \in \mathcal{Q}_{|D_1 D_2|}/\Gamma} \frac{\chi_{D_1}(Q)}{\omega_Q} \sum_{\substack{A \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z}) \\ \mathrm{Im}\left(A\tau_Q\right) = \frac{\sqrt{|D_1 D_2|}}{2a}}} e\left(-\mathrm{Re}\left(A\tau_Q\right)\right).$$

Proof. For every integral binary quadratic form

$$Q(x, y) = ax^2 + bxy + cy^2$$

of discriminant $(-1)^\lambda D_1 D_2$, there is a unique point τ_Q in the upper half of the complex plane that is a root of $Q(x, 1) = 0$. Clearly τ_Q is equal to

$$\tau_Q = \frac{-b + i\sqrt{|D_1 D_2|}}{2a}, \tag{3.3}$$

and the coefficient b of Q solves the congruence

$$b^2 \equiv (-1)^\lambda D_1 D_2 \pmod{4a}. \tag{3.4}$$

Conversely, every solution of (3.4) corresponds to a quadratic form with an associated CM point as above. There is a one-to-one correspondence between the solutions of

$$b^2 - 4ac = (-1)^\lambda D_1 D_2 \qquad (a, b, c \in \mathbb{Z}, a, c > 0)$$

and the points of the orbits

$$\bigcup_Q \left\{ A\tau_Q : \; A \in \mathrm{SL}_2(\mathbb{Z})/\Gamma_{\tau_Q} \right\},$$

where Γ_{τ_Q} denotes the isotropy subgroup of τ_Q in $\mathrm{SL}_2(\mathbb{Z})$, and where Q varies over the representatives of $\mathcal{Q}_{|D_1 D_2|}/\Gamma$. The group Γ_∞ preserves the imaginary part of such a CM point τ_Q, and preserves (3.4). However, it does not preserve the middle coefficient b of the corresponding quadratic forms modulo $4a$. It identifies the congruence classes $b, b + 2a \pmod{4a}$ appearing in the definition of $S_\lambda(D_1, D_2, 4a)$. Since $\chi_{D_1}(Q)$ is fixed under the action of Γ_∞, the corresponding summands for such pairs of congruence classes are equal. Proposition 3.3 follows easily since $\#\Gamma_{\tau_Q} = 2\omega_Q$, and since both Γ_{τ_Q} and Γ_∞ contain the negative identity matrix. ☐

4. Modular invariant Poincaré series

Here we recall the properties of the Poincaré series $\mathfrak{F}_\lambda(z)$ which are the modular invariants whose singular values determine the coefficients of the Maass-Poincaré series $F_\lambda(-m; z)$. These series were defined and investigated by Niebur in [19]. Recall that they are defined by

$$\mathfrak{F}_\lambda(z) := \begin{cases} \pi \sum_{A \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \mathrm{Im}(Az)^{\frac{1}{2}} I_{\lambda - \frac{1}{2}}(2\pi \mathrm{Im}(Az)) e(-\mathrm{Re}(Az)) & \text{if } \lambda > 1, \\ \\ -12 + \pi \sum_{A \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \mathrm{Im}(Az)^{\frac{1}{2}} I_{\frac{1}{2}}(2\pi \mathrm{Im}(Az)) e(-\mathrm{Re}(Az)) & \text{if } \lambda = 1. \end{cases} \tag{4.1}$$

Strictly speaking, this definition is not well defined when $\lambda = 1$ since the defining series is not absolutely convergent. A significant portion of Niebur's paper is devoted to defining $\mathfrak{F}_1(z)$ via analytic continuation, and it turns out that $\mathfrak{F}_1(z) = \frac{1}{2}(j(z) - 744)$.

For the remainder of this section suppose that $\lambda > 1$ is an integer. Since we have

$$y^{1/2} I_{\lambda-1/2}(y) = O\left(y^{\lambda}\right) \qquad (y \to 0),$$

it follows that the defining series for $\mathfrak{F}_\lambda(z)$ is absolutely uniformly convergent. Since the function

$$f_\lambda(z) := \pi \mathrm{Im}(z)^{\frac{1}{2}} I_{\lambda-\frac{1}{2}}(2\pi \mathrm{Im}(z)) e(-\mathrm{Re}(z))$$

satisfies

$$\Delta_0\left(f_\lambda(z)\right) = \lambda(1-\lambda) f_\lambda(z),$$

where

$$\Delta_0 := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right),$$

we have that

$$\Delta_0\left(\mathfrak{F}_\lambda(z)\right) = (1-\lambda)\lambda \mathfrak{F}_\lambda(z).$$

Therefore, $\mathfrak{F}_\lambda(z)$ is an eigenfunction of the weight 0 hyperbolic Laplacian Δ_0, and has eigenvalue $\lambda(1-\lambda)$. Consequently, $\mathfrak{F}_\lambda(z)$ is a weight 0 Maass form. However, it is not a weak Maass form since it is not harmonic.

Niebur [19] computed the Fourier expansion of $\mathfrak{F}_\lambda(z)$, and he found that

$$\mathfrak{F}_\lambda(z) = \pi I_{\lambda-\frac{1}{2}}(2\pi y) e^{-2\pi y} y^{\frac{1}{2}} q^{-1} + 2\sum_{n\in\mathbb{Z}} b_\lambda(n,y) q^n,$$

where

$$(4.2)\quad b_\lambda(n,y) := \begin{cases} \dfrac{y^{1-\lambda}\pi^{\lambda+1}}{(2\lambda-1)(\lambda-1)!} \displaystyle\sum_{c>0} \frac{\mathcal{S}(-1,0;c)}{c^{2\lambda}} & n=0, \\[2ex] \pi y^{1/2} e^{2\pi n y} K_{\lambda-\frac{1}{2}}(2\pi|n|y) \displaystyle\sum_{c>0} \frac{\mathcal{S}(n,-1;c)}{c} I_{2\lambda-1}\left(\frac{4\pi\sqrt{n}}{c}\right) & n>0, \\[2ex] \pi y^{1/2} e^{2\pi n y} K_{\lambda-\frac{1}{2}}(2\pi|n|y) \displaystyle\sum_{c>0} \frac{\mathcal{S}(n,-1;c)}{c} J_{2\lambda-1}\left(\frac{4\pi\sqrt{|n|}}{c}\right) & n<0. \end{cases}$$

Here $\mathcal{S}(n,m;c)$ denotes the integer weight Kloosterman sum

$$\mathcal{S}(n,m;c) := \sum_{\nu \pmod{c}^*} e\left(\frac{n\nu + m\overline{\nu}}{c}\right).$$

Remark. The reader is warned that there are typographical errors in Niebur's formulas.

In Theorems 10 and 11 of [23], Zagier proved that the coefficients of several half-integral weight modular forms are traces of non-holomorphic modular forms. Moreover, he notes that these forms are images of weakly holomorphic modular forms under standard differential operators. Theorem 1.2 includes these results thanks to Niebur's formulas (4.2). For these results, it turns out that the relevant weakly holomorphic modular forms (see [19]) $g_\lambda(z)$ of weight $2-2\lambda$ on $\mathrm{SL}_2(\mathbb{Z})$ have Fourier expansions of the form

$$\begin{aligned} g_\lambda(z) = q^{-1} &+ \frac{4(-1)^{\lambda+1}\pi^{2\lambda+\frac{1}{2}}}{\Gamma(\lambda-\frac{1}{2})(2\lambda-1)(\lambda-1)!}\cdot\sum_{c>0}\frac{\mathcal{S}(-1,0;c)}{c^{2\lambda}} \\ &+ 2\pi(-1)^{\lambda+1}\sum_{n\geq 1} n^{-\lambda+\frac{1}{2}}\sum_{c>0}\frac{\mathcal{S}(n,-1;c)}{c}\cdot I_{2\lambda-1}\left(\frac{4\pi\sqrt{n}}{c}\right)q^n. \end{aligned} \tag{4.3}$$

A brief inspection reveals that the same Kloosterman sum and I-Bessel factors appear in both (4.2) and (4.3). In a recent preprint [18], Miller and Pixton elaborate on these observations to obtain general theorems about "traces" of algebraic values of suitable non-holomorphic modular forms as coefficients of Maass-Poincaré series. These non-holomorphic modular forms, as in Theorem 11 of [23], are obtained by applying suitable differential operators to weakly holomorphic modular forms.

Example. For completeness, we note that the modular forms $g_\lambda(z)$ are simple to realize in terms of the classical modular forms on $\mathrm{SL}_2(\mathbb{Z})$. For example, we have that

$$\begin{aligned} g_1(z) &= j(z) - 720 = q^{-1} + 24 + 196884q + \cdots, \\ g_2(z) &= \frac{E_4(z)E_6(z)}{\Delta(z)} = q^{-1} - 240 - 141444q - \cdots, \\ g_3(z) &= \frac{E_4(z)^2}{\Delta(z)} = q^{-1} + 504 + 73764q + \cdots. \end{aligned}$$

Note that if $\lambda = 1$, then we have $2\mathfrak{F}_1(z) + 24 = g_1(z)$. If $\lambda = 2$, then

$$\mathfrak{F}_2(z) = -\frac{1}{2}\left(\frac{1}{2\pi i}\frac{\partial}{\partial z} + \frac{1}{2\pi \mathrm{Im}(z)}\right)g_2(z).$$

5. Proof of Theorem 1.2

Here we prove Theorem 1.2 using the definitions of the series $F_\lambda(-m;z)$ and $\mathfrak{F}_\lambda(z)$, and Propositions 3.2 and 3.3.

Proof of Theorem 1.2. Here we prove the cases where $\lambda \geq 2$. The argument when $\lambda = 1$ is identical. For $\lambda \geq 2$, Theorem 2.1 (1) implies that

$$\begin{aligned} b_\lambda(-m;n) = &(-1)^{[(\lambda+1)/2]}\pi\sqrt{2}(n/m)^{\frac{\lambda}{2}-\frac{1}{4}}(1-(-1)^\lambda i) \\ &\times \sum_{\substack{c>0 \\ c\equiv 0 \pmod 4}} (1+\delta_{\mathrm{odd}}(c/4))\frac{K_\lambda(-m,n,c)}{c}\cdot I_{\lambda-\frac{1}{2}}\left(\frac{4\pi\sqrt{mn}}{c}\right). \end{aligned}$$

Using Proposition 3.2, where $D_1 = (-1)^{\lambda+1}m$ and $D_2 = n$, for integers $N = c$ which are positive multiples of 4, we have

$$c^{-\frac{1}{2}}(1-(-1)^{\lambda}i)(1+\delta_{\mathrm{odd}}(c/4)) \cdot K_{\lambda}(-m,n,c) = S_{\lambda}((-1)^{\lambda+1}m,n,c).$$

These identities, combined with the change of variable $c = 4a$, give

$$b_{\lambda}(-m;n) = \frac{(-1)^{[(\lambda+1)/2]}\pi}{\sqrt{2}}(n/m)^{\frac{\lambda}{2}-\frac{1}{4}}\sum_{a=1}^{\infty}\frac{S_{\lambda}((-1)^{\lambda+1}m,n,4a)}{\sqrt{a}} \cdot I_{\lambda-\frac{1}{2}}\left(\frac{\pi\sqrt{mn}}{a}\right).$$

Using Proposition 3.3, this becomes

$$b_{\lambda}(-m;n) = \frac{2(-1)^{[(\lambda+1)/2]}\pi}{\sqrt{2}}(n/m)^{\frac{\lambda}{2}-\frac{1}{4}}\sum_{Q\in\mathcal{Q}_{nm}/\Gamma}\frac{\chi_{(-1)^{\lambda+1}m}(Q)}{\omega_Q}$$
$$\sum_{a=1}^{\infty}\sum_{\substack{A\in\Gamma_{\infty}\backslash\mathrm{SL}_2(\mathbb{Z})\\ \mathrm{Im}(A\tau_Q)=\frac{\sqrt{mn}}{2a}}}\frac{I_{\lambda-\frac{1}{2}}(2\pi\mathrm{Im}(A\tau_Q))}{\sqrt{a}} \cdot e(-\mathrm{Re}(A\tau_Q)).$$

The definition of $\mathfrak{F}_{\lambda}(z)$ in (1.11), combined with the obvious change of variable relating $1/\sqrt{a}$ to $\mathrm{Im}(A\tau_Q)^{\frac{1}{2}}$, gives

$$\begin{aligned} b_{\lambda}(-m;n) &= \frac{2(-1)^{[(\lambda+1)/2]}n^{\frac{\lambda}{2}-\frac{1}{2}}}{m^{\frac{\lambda}{2}}} \cdot \pi \sum_{Q\in\mathcal{Q}_{nm}/\Gamma}\frac{\chi_{(-1)^{\lambda+1}m}(Q)}{\omega_Q} \\ &\qquad \sum_{A\in\Gamma_{\infty}\backslash\mathrm{SL}_2(\mathbb{Z})}\mathrm{Im}(A\tau_Q)^{\frac{1}{2}} \cdot I_{\lambda-\frac{1}{2}}(2\pi\mathrm{Im}(A\tau_Q))e(-\mathrm{Re}(A\tau_Q)) \\ &= \frac{2(-1)^{[(\lambda+1)/2]}n^{\frac{\lambda}{2}-\frac{1}{2}}}{m^{\frac{\lambda}{2}}} \cdot \mathrm{Tr}_{(-1)^{\lambda+1}m}(\mathfrak{F}_{\lambda};n). \end{aligned}$$

□

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