CONGRUENCES FOR DYSON'S RANKS

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1. Introduction and Statement of Results

A *partition* of a positive integer n is any non-increasing sequence of positive integers whose sum is n. Let $p(n)$ denote the number of partitions of n (with the usual convention that $p(0) := 1$, and $p(n) := 0$ for $n \notin \mathbb{N}_0$).

Ramanujan proved that for every positive integer n, we have:

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned} \tag{1.1}$$

Using theta functions and q-series identities, Atkin [7] and Watson [17] later on showed generalisations of (1.1) for powers of 5, 7, and 11. In a celebrated paper Ono [14] treated these kinds of congruences systematically. Combining Shimura's theory of modular forms of half-integral weight with results of Serre on modular forms modulo ℓ he showed that for any prime $\ell \geq 5$ there exist infinitely many non-nested arithmetic progressions of the form $An+B$ such that

$$p(An+B) \equiv 0 \pmod{\ell}.$$

Ahlgren and Ono [1] and [2] extended this phenomenon to prime powers.

In order to explain the congruences in (1.1) with moduli 5 and 7 combinatorically, Dyson [11] introduced the rank of a partition. The *rank* of a partition is defined to be its largest part minus the number of its parts. Dyson conjectured that the partitions of $5n+4$ (resp. $7n+5$) form 5 (resp. 7) groups of equal size when sorted by their ranks modulo 5 (resp. 7). This conjecture was proven in 1954 by Atkin and Swinnerton Dyer [6]. Thus, if for integers r and t, we denote by $N(r,t;n)$ the number of partitions of n whose rank is r modulo t, then Dyson's conjecture reads as

$$\begin{aligned} N(r,5;5n+4) &= \frac{p(5n+4)}{5}, \\ N(r,7;7n+5) &= \frac{p(7n+5)}{7}. \end{aligned}$$

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In [10], Ono and the author showed that Dyson's rank partition function satisfies congruences of Ramanujan type.

Theorem 1.1. *Let t be a positive odd integer, and let $\ell \nmid 6t$ be a prime. If u is a positive integer, then there are infinitely many non-nested arithmetic progressions $An+B$ such that for every $0 \leq r < t$ we have*

$$N(r,t;An+B) \equiv 0 \pmod{\ell^u}.$$

In this paper, we consider the case $t=\ell^m$ ($m \in \mathbb{N}$) and show that a similar result as in Theorem 1.1 holds. For this we define for a prime $\ell \geq 5$ the integer $\delta_\ell := \frac{\ell^2-1}{24}$ and $\epsilon_\ell := \left(\frac{-6}{\ell}\right)$. Moreover we let

$$\mathcal{S}_\ell := \left\{0 \leq \beta \leq \ell-1;\ \left(\frac{\beta+\delta_\ell}{\ell}\right) = -\epsilon_\ell\right\}.$$

We show the following Theorem.

Theorem 1.2. *Suppose that $\ell \geq 5$ is a prime, $m,u \in \mathbb{N}$, and $\beta \in \mathcal{S}_\ell$. Then a positive proportion of primes $p \equiv -1 \pmod{24\ell}$ have the property that for every $0 \leq r \leq \ell^m - 1$*

$$N\left(r,\ell^m;\frac{p^3n+1}{24}\right) \equiv 0 \pmod{\ell^u}$$

for all $n \equiv 1-24\beta \pmod{24\ell}$ that are not divisible by p.

This directly implies the following corollary.

Corollary 1.3. *If $\ell \geq 5$ is a prime, $m,u \in \mathbb{N}$, then there are infinitely many non-nested arithmetic progressions $An+B$ such that*

$$N\left(r,\ell^m;An+B\right) \equiv 0 \pmod{\ell^u}$$

for all $0 \leq r \leq \ell^m - 1$.

Two remarks.
1) The congruences in Theorem 1.2 may be viewed as a combinatorial decomposition of the partition function congruence

$$p(An+B) \equiv 0 \pmod{\ell^u}.$$

2) Corollary 1.3 was conjectured in [10].

The paper is organized as follows: In Section 2 we recall facts about modular forms and weak Maass forms, in Section 3 we show certain properties of rank generating functions, and in Section 4 we prove Theorem 1.2.

2. Modular forms and weak Maass forms

Let us recall some basic facts about modular forms of half-integral weight. For this, we let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, $\lambda, N \in \mathbb{N}$, and χ a Dirichlet character modulo N.

Denote by $M_{\lambda+\frac{1}{2}}(\Gamma,\chi)$ (resp. $S_{\lambda+\frac{1}{2}}(\Gamma,\chi)$) the vector spaces of modular forms (resp. cusp forms) of weight $\lambda+\frac{1}{2}$ with Nebentypus character χ for Γ. In particular, we let

$$\Gamma_0(N) := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, \gamma \equiv 0 \pmod{N} \right\},$$
$$\Gamma_1(N) := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

For brevity, let $M_{\lambda+\frac{1}{2}}(\Gamma_1(N)) := M_{\lambda+\frac{1}{2}}(\Gamma_1(N),\chi_0)$ and $S_{\lambda+\frac{1}{2}}(\Gamma_1(N)) := S_{\lambda+\frac{1}{2}}(\Gamma_1(N),\chi_0)$, where χ_0 is the trivial character modulo N. Moreover we denote by $M^!_{\lambda+\frac{1}{2}}(\Gamma,\chi)$ the vector space of weakly holomorphic modular forms of weight $\lambda+\frac{1}{2}$ with Nebentypus character χ for Γ. We call a meromorphic modular form *weakly holomorphic* if its poles, if there are any, are supported at the cusps of Γ. Moreover we denote by $|_k$ the usual weight k slash operator.

Next we recall the definition of Hecke operators for modular forms of half-integral weight for $\Gamma_1(N)$. If p is a prime, then the Hecke operator $T(p^2)$ for $f=\sum_{n=1}^{\infty} a(n)q^n$ in $S_{\lambda+\frac{1}{2}}(\Gamma_1(N))$ is defined as

$$f \mid T(p^2) := \sum_{n=1}^{\infty} \left(a\left(p^2 n\right) + p^{\lambda-1}\left(\frac{(-1)^{\lambda} n}{p}\right) a(n) + p^{2\lambda-1} a\left(\frac{n}{p^2}\right)\right) q^n. \tag{2.1}$$

It is well known that $f|T\left(p^2\right) \in S_{\lambda+\frac{1}{2}}(\Gamma_1(N))$.

Moreover we consider certain quadratic twists of modular forms. For this we let for a prime p the Gauss sum with respect to p be given as $g := \sum_{v=1}^{p-1}\left(\frac{v}{p}\right)e^{\frac{2\pi i v}{p}}$, and define for $f(z)=\sum_n a(n)q^n$

$$f(z)_p := \frac{g}{p}\sum_{v=1}^{p-1}\left(\frac{v}{p}\right) f(z) \left| \begin{pmatrix} 1 & -\frac{v}{p} \\ 0 & 1 \end{pmatrix} \right. . \tag{2.2}$$

This definition is independent of the weight of the slash operator. Moreover it is not hard to see that $f(z)_p$ is the $p-$quadratic twist of f, i.e., the n-th Fourier coefficient in the $q-$expansion of f is multiplied by $\left(\frac{n}{p}\right)$. It is well known that if $f \in M_{\lambda+\frac{1}{2}}(\Gamma_0(N),\chi)$, then $f(z)_p \in M_{\lambda+\frac{1}{2}}\left(\Gamma_0(Np^2),\chi\chi_0\right)$, where χ_0 is the trivial character modulo p.

Moreover we need the fact that if $f(z)=\sum_{n=1}^{\infty} a(n)q^n \in S_{\lambda+\frac{1}{2}}(\Gamma_1(N))$, then for $m\in\mathbb{N}$ and $0 \le r \le m-1$, we have

$$\sum_{n\equiv r \pmod{m}} a(n)q^n \quad \in S_{\lambda+\frac{1}{2}}\left(\Gamma_1\left(Nm^2\right)\right). \tag{2.3}$$

The following theorem, which is shown in [10], follows from Serre's theory of modular forms modulo M.

Theorem 2.1. *Suppose that $f_1(z), f_2(z), \ldots, f_s(z)$ are half-integral weight cusp forms, where*

$$f_i(z) \in S_{\lambda_i+\frac{1}{2}}(\Gamma_1(N_i)) \cap \mathcal{O}_K[[q]],$$

and where $\mathcal{O}_K$ is the ring of integers of a fixed number field K. If ℓ is prime and $j \geq 1$ is an integer, then the set of primes p for which

$$f_i(z) \mid T\left(p^2\right) \equiv 0 \pmod{\ell^j},$$

for each $1 \leq i \leq s$, has positive Frobenius density.

Remark. The primes p in Theorem 2.1 can be chosen to satisfy $p \equiv -1 \pmod{N_1 \dots N_s \ell^j}$

Next we recall the definition of a weak Maass form of weight $k := \lambda + \frac{1}{2}$. For this we define for an odd integer v

$$e_v := \begin{cases} 1 & \text{if } v \equiv 1 \pmod 4, \\ i & \text{if } v \equiv 3 \pmod 4. \end{cases} \tag{2.4}$$

Moreover if $z = x + iy$ with $x, y \in \mathbb{R}$, then the *weight k hyperbolic Laplacian* is given by

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right). \tag{2.5}$$

A *weak Maass form* of weight k on $\Gamma_0(N)$ with Nebentypus character χ is any smooth function $f : \mathbb{H} \to \mathbb{C}$ satisfying the following:

(1) For all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and all $z \in \mathbb{H}$, we have
$$f(Az) = \chi(d) \left(\frac{c}{d} \right)^{2k} e_d^{-2k} \, (cz + d)^k \, f(z).$$
(2) We have that $\Delta_k f = 0$.
(3) The function $f(z)$ has at most linear exponential growth at all the cusps of $\Gamma_0(N)$.

It is not hard to see that we can define twists for weak Maass forms as in (2.2) and that the twist changes the level and the Nebentypus character in exactly the same way as for modular forms.

3. Properties of rank generating functions

If $N(m, n)$ denotes the number of partitions of n with rank m, then it is well known that

$$R(w; q) := 1 + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} N(m, n) w^m q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(wq; q)_n (w^{-1} q; q)_n}, \tag{3.1}$$

where

$$(a; q)_n := (1 - a)(1 - aq) \cdots (1 - aq^{n-1}).$$

In [10], Ono and the author showed that if $w \neq 1$ is a root of unity, then $R(w; q)$ is the holomorphic part of a weak Maass form (see [10] for a precise version). Here we need its transformation law under all elements of $\mathrm{SL}_2(\mathbb{Z})$. For this we define the theta function $\Theta\left(\frac{a}{c}; \tau\right)$ by

$$\Theta\left(\frac{a}{c}; \tau\right) := \sum_{m \pmod{f_c}} (-1)^m \sin\left(\frac{\pi a (6m + 1)}{c}\right) \cdot \theta\left(6m + 1, 6f_c; \frac{\tau}{24}\right), \tag{3.2}$$

where $f_c := \frac{2c}{\gcd(c,6)}$, and where

$$\theta(\alpha,\beta;\tau) := \sum_{n\equiv\alpha \pmod{\beta}} n e^{2\pi i\tau n^2}. \tag{3.3}$$

Moreover we define the period integral

$$S\left(\frac{a}{c};z\right) := -\frac{i\sin\left(\frac{\pi a}{c}\right)}{\sqrt{3}} \int_{-\bar{z}}^{i\infty} \frac{\Theta\left(\frac{a}{c};\tau\right)}{\sqrt{-i(\tau+z)}}\, d\tau. \tag{3.4}$$

Letting $\zeta_c := e^{\frac{2\pi i}{c}}$ and $q := e^{2\pi i z}$, define $D\left(\frac{a}{c};z\right)$ by

$$D\left(\frac{a}{c};z\right) := -S\left(\frac{a}{c};z\right) + q^{-\frac{1}{24}} R(\zeta_c^a;q). \tag{3.5}$$

If $\frac{b}{c} \in (0,1)\setminus\{\frac{1}{2},\frac{1}{6},\frac{5}{6}\}$, then define the integer $s(b,c)$ by

$$s(b,c) := \begin{cases} 0 & \text{if } 0 < \frac{b}{c} < \frac{1}{6}, \\ 1 & \text{if } \frac{1}{6} < \frac{b}{c} < \frac{1}{2}, \\ 2 & \text{if } \frac{1}{2} < \frac{b}{c} < \frac{5}{6}, \\ 3 & \text{if } \frac{5}{6} < \frac{b}{c} < 1. \end{cases}$$

Moreover let

$$N(a,b,c;q) := \frac{i}{2(q;q)_\infty}\left(\sum_{m=0}^{\infty} \frac{(-1)^m e^{-\frac{\pi i a}{c}} \cdot q^{\frac{m}{2}(3m+1)+ms(b,c)+\frac{b}{2c}}}{1-e^{-\frac{2\pi i a}{c}}\cdot q^{m+\frac{b}{c}}} - \sum_{m=1}^{\infty} \frac{(-1)^m e^{\frac{\pi i a}{c}} \cdot q^{\frac{m}{2}(3m+1)-ms(b,c)-\frac{b}{2c}}}{1-e^{\frac{2\pi i a}{c}}\cdot q^{m-\frac{b}{c}}}\right),$$

where

$$(a;q)_\infty := \lim_{n\to\infty} (a;q)_n.$$

Define

$$\mathcal{N}(a,b,c;q) := 4\sin\left(\frac{\pi a}{c}\right) e^{-2\pi i\frac{a\cdot s(b,c)}{c}+3\pi i\frac{b}{c}\left(\frac{2a}{c}-1\right)} \cdot \zeta_c^{-b} \cdot q^{\frac{b}{c}s(b,c)-\frac{3b^2}{2c^2}-\frac{1}{24}} \cdot N(a,b,c;q)$$

and

$$S(a,b,c;z) := -\frac{\sin\left(\frac{\pi a}{c}\right)\zeta_{2c}^{-5b}}{2\sqrt{3}} \int_{-\bar{z}}^{i\infty} \frac{\Theta(a,b,c;\tau)}{\sqrt{-i(\tau+z)}}\, d\tau,$$

where for $t_c := \operatorname{lcm}(c,6)$ the theta series $\Theta(a,b,c;\tau)$ is defined as

$$\Theta(a,b,c;\tau) := \sum_{m \pmod{t_c}} (-1)^m \sin\left(\frac{\pi}{3}(2m+1)\right) e^{2\pi i m\frac{a}{c}} \cdot \theta\left(2cm+6b+c, 2ct_c; \frac{\tau}{24c^2}\right).$$

Now define

$$D(a,b,c;z) := \mathcal{N}(a,b,c;z) - S(a,b,c;z).$$

Moreover let $\omega_{h,k}$ be given by

$$\omega_{h,k} := \exp\left(\pi i t(h,k)\right),$$

where

$$t(h,k) := \sum_{\mu \pmod{k}} \left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{h\mu}{k}\right)\right).$$

Here

$$((x)) := \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \in \mathbb{R}\setminus\mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

Combining the results of [8] and [10] gives the following transformation law for the function $D\left(\frac{a}{c};z\right)$.

Proposition 3.1. *Assume that* $\left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$*,* $a, c \in \mathbb{N}$ *with* $0 < a < c$ *and* c *odd. Moreover let* $c_1 = \frac{c}{\gcd(c,\gamma)}$*,* $0 < l < c_1$*, such that* $l \equiv a\gamma_1 \pmod{c_1}$ *where* $\gamma_1 = \frac{\gamma}{\gcd(c,\gamma)}$*,* $s := s(l, c_1)$*.*

(1) *If* $c|\gamma$*, then*

$$D\left(\frac{a}{c};\frac{\alpha z+\beta}{\gamma z+\delta}\right) = \frac{(-1)^{a\gamma} i^{\frac{1}{2}} \cdot \sin\left(\frac{\pi a}{c}\right)\cdot \omega_{\delta,\gamma}}{\sin\left(\frac{\pi a\delta}{c}\right)} \cdot e^{\frac{3\pi i a^2 \gamma_1 \delta}{c}} \cdot e^{-\frac{\pi}{12\gamma}(\alpha+\delta)} \cdot (\gamma z+\delta)^{\frac{1}{2}} \cdot D\left(\frac{a\delta}{c};z\right).$$

(2) *If* $c \nmid \gamma$*, then*

$$D\left(\frac{a}{c};\frac{\alpha z+\beta}{\gamma z+\delta}\right) = \frac{(-1)^{a\gamma+l} i^{\frac{1}{2}} \sin\left(\frac{\pi a}{c}\right)\omega_{\delta,\gamma}}{\sin\left(\frac{\pi a\delta}{c}\right)} e^{\frac{5\pi i l}{c_1}+\frac{3\pi i a^2\delta\gamma_1}{cc_1}} e^{-\frac{\pi}{12\gamma}(\alpha+\delta)} (\gamma z+\delta)^{\frac{1}{2}} D\left(-\alpha\delta, \frac{lc}{c_1}, c; z\right).$$

For the proof of Theorem 1.2, we apply a certain twist to "kill" the non-holomorphic part of $D\left(\frac{a}{c};z\right)$. For this we need to know on which arithmetic progressions its non-holomorphic part is supported. Similarly as in [10], we obtain

Proposition 3.2. *For integers* $0 < a < c$*, we have*

$$D\left(\frac{a}{c};z\right) = q^{-\frac{1}{24}} + \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} N(m,n)\zeta_c^{am} q^{n-\frac{1}{24}}$$
$$-\frac{2\sin\left(\frac{\pi a}{c}\right)}{\sqrt{\pi}} \sum_{m \pmod{f_c}} (-1)^m \sin\left(\frac{\pi a(6m+1)}{c}\right) \sum_{n\equiv 6m+1 \pmod{6f_c}} \Gamma\left(\frac{1}{2};\frac{\pi n^2 y}{6}\right) q^{-\frac{n^2}{24}},$$

where

$$\Gamma(a;x) := \int_x^{\infty} e^{-t} t^{a-1}\, dt.$$

4. Proof of Theorem 1.2

For brevity we set $t := \ell^m$. Using the orthogonality of roots of unity easily gives

$$\sum_{n=0}^{\infty} N(r,t;n)q^n = \frac{1}{t}\sum_{n=0}^{\infty} p(n)q^n + \frac{1}{t}\sum_{j=1}^{t-1} \zeta_t^{-rj} R\left(\zeta_t^j; q\right). \tag{4.1}$$

Define the function

$$g_r(z) := t\sum_{n=0}^{\infty} N(r,t;n)q^{n+\delta_\ell} \prod_{n=1}^{\infty}\left(1-q^{\ell n}\right)^{\ell}.$$

Then (3.5) and (4.1) imply

$$g_r(z) = \frac{\eta^\ell(\ell z)}{\eta(z)} + \sum_{j=1}^{t-1} \zeta_t^{-rj}\left(D\left(\frac{j}{t};z\right) + S\left(\frac{j}{t};z\right)\right)\eta^\ell(\ell z), \tag{4.2}$$

where $\eta(z) := q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$ is Dedekind's eta function. We denote the two summands by $f(z)$ and $f_r(z)$, respectively. We now define for a function $h(z) := \sum_n a(n)q^n$

$$\widetilde{h}(z) := \epsilon_\ell\left(h(z) - \epsilon_\ell h(z)_\ell\right)_\ell.$$

Clearly

$$\widetilde{h}(z) = \epsilon_\ell \sum_n a(n)\left(1 - \epsilon_\ell\left(\frac{n}{\ell}\right)\right)\left(\frac{n}{\ell}\right)q^n. \tag{4.3}$$

The main step in the proof of Theorem 1.2 is the following Theorem.

Theorem 4.1. *For every $u \geq 0$ there exists a character χ, $\lambda, \lambda' \in \mathbb{N}$, and modular forms $h(z) \in S_{\lambda+\frac{1}{2}}\left(\Gamma_0\left(576\ell^5\right),\chi\right)$ and $h_r(z) \in S_{\lambda'+\frac{1}{2}}\left(\Gamma_1\left(576\ell^4t^2\right)\right)$ such that*

$$\frac{\widetilde{g_r}(24z)}{\eta^\ell(24\ell z)} \equiv h(z) + h_r(z) \pmod{\ell^u}.$$

The proof of Theorem 4.1 is given in Section 4.1. We first show how Theorem 1.2 follows from Theorem 4.1.

Proof of Theorem 1.2. We easily see that

$$\frac{\widetilde{g_r}(24z)}{\eta^\ell(24\ell z)} = -2t\sum_{\left(\frac{n}{\ell}\right)=-\epsilon_\ell} N(r,t;n-\delta_\ell)\cdot q^{24n-\ell^2}.$$

Now we consider for $\beta \in \mathcal{S}_\ell$ the series

$$g_{r,\beta}(z) := -2t\sum_{\substack{n=\beta+\delta_\ell+\ell m\\ m\in\mathbb{Z}}} N(r,t;n-\delta_\ell)\cdot q^{24n-\ell^2} = -2t\sum_{n\equiv 24\beta-1 \pmod{24\ell}} N\left(r,t;\frac{n+1}{24}\right)\cdot q^n.$$

Theorem 4.1 gives that

$$g_{r,\beta}(z) \equiv h_\beta(z) + h_{r,\beta}(z) \pmod{\ell^u},$$

where $h_\beta(z)$ and $h_{r,\beta}(z)$ denote the restrictions of the Fourier expansion of $h(z)$ resp. $h_r(z)$ to those coefficients n with $n \equiv 24\beta - 1 \pmod{24\ell}$. By (2.3), we have that $h_\beta(z) \in S_{\lambda+\frac{1}{2}}\left(\Gamma_1\left(2^{12}3^4\ell^7\right)\right)$ and $h_{r,\beta}(z) \in S_{\lambda+\frac{1}{2}}\left(\Gamma_1\left(2^{12}3^4\ell^6t^2\right)\right)$. Due to Theorem 2.1 a positive proportion of all primes $p \equiv -1 \pmod{24\ell}$ satisfy for all r

$$h_{r,\beta}(z)|T(p^2) \equiv h_\beta(z)|T(p^2) \equiv 0 \pmod{\ell^u}.$$

Thus we have for all r

$$g_{r,\beta}(z)|T(p^2) \equiv 0 \pmod{\ell^u}.$$

Replacing n by pn in (2.1) gives that for all $n \equiv 1 - 24\beta \pmod{24\ell}$ that are not divisible by p, we have

$$-2tN\left(r, t; \frac{p^3n+1}{24}\right) \equiv 0 \pmod{\ell^u}.$$

Dividing by $-2t$ directly gives the Theorem since u is arbitrary. □

4.1. **Proof of Theorem 4.1.** If a is a positive integer, then define

$$E_{\ell,a}(z) := \frac{\eta^{\ell^a}(z)}{\eta(\ell^a z)} \in M_{\frac{\ell^a-1}{2}}\left(\Gamma_0(\ell^a), \chi_{\ell,a}\right),$$

where $\chi_{\ell,a}(d) := \left(\frac{(-1)^{(\ell^a-1)/2}\ell^a}{d}\right)$. It is well known that $E_{\ell,a}(z)$ vanishes at those cusps of $\Gamma_0(\ell^a)$ that are not equivalent to ∞ and that for all $u > 0$

$$E_{\ell,a}^{\ell^{u-1}}(z) \equiv 1 \pmod{\ell^u}. \tag{4.4}$$

We now treat the summands in (4.2) separately. Since $f(z) \in M_{\frac{\ell-1}{2}}\left(\Gamma_0(\ell), \left(\frac{\bullet}{\ell}\right)\right)$ we have that $\widetilde{f}(z) \in M_{\frac{\ell-1}{2}}\left(\Gamma_0(\ell^5), \left(\frac{\bullet}{\ell}\right)\right)$. For sufficiently large u' the function

$$f'(z) := \widetilde{f}(z) \cdot E_{\ell,5}^{\ell^{u'}}(z)$$

is a cusp form on $\Gamma_0(\ell^5)$ with character $\chi_{\ell,5} \cdot \left(\frac{\bullet}{\ell}\right)$ satisfying

$$f'(z) \equiv \widetilde{f}(z) \pmod{\ell^u}$$
$$\mathrm{ord}_\infty(f'(z)) \geq \delta_\ell + 1.$$

Thus for u' sufficiently large $\left(\frac{f'(z)}{\eta^\ell(\ell z)}\right)^{24}$ vanishes at ∞ which implies that $\frac{f'(24z)}{\eta^\ell(24\ell z)}$ is a cusp form for $\Gamma_0\left(576\ell^5\right)$ with some character χ.

We next turn to $f_r(z)$. Let us recall the transformation law for Dedekind's eta function. For $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ we have

$$\eta\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right) = \omega_{-\delta,\gamma} \cdot (-i)^{\frac{1}{2}} \cdot (\gamma z+\delta)^{\frac{1}{2}} \cdot e^{\frac{\pi i}{12\gamma}(\alpha+\delta)} \cdot \eta(z). \tag{4.5}$$

Combining the fact that $f(z) \in M_{\frac{\ell-1}{2}}\left(\Gamma_0(\ell), \left(\frac{\cdot}{\ell}\right)\right)$ with Proposition 3.1 and (4.5) directly gives that

$$D\left(\frac{j}{t}; z\right)\eta^{\ell}(\ell z) = D\left(\frac{j}{t}; z\right)\eta(z) \cdot \frac{\eta^{\ell}(\ell z)}{\eta(z)}$$

is a weak Maass form of weight $\frac{\ell+1}{2}$ on $\Gamma_1(2t^2)$. From Propositon 3.2 we see that its non holomorphic part is supported on exponents of the form $q^{-\frac{n^2}{24}+\frac{\ell^2}{24}+m\ell}$, where $n \in \mathbb{Z}$ with $n \equiv 1 \pmod 6$ and $m \in \mathbb{N}_0$. Now (4.3) easily gives that the non holomorphic part of $\widetilde{f}_r(z)$ is vanishing, thus $\widetilde{f}_r(z)$ is a weakly holomorphic modular form of weight $\frac{\ell+1}{2}$ on $\Gamma_1\left(2t^2\ell^4\right)$. Since $E_{\ell,2m}(z)$ vanishes at each cusp $\frac{\alpha}{\gamma}$ with $t^2 \nmid \gamma$ if we take u' sufficiently large, the function

$$f_r'(z) := E_{\ell,2m}^{\ell^{u'}}(z) f_r(z)$$

is a weakly holomorphic modular form on $\Gamma_1\left(2t^2\right)$ that vanishes at all cusps $\frac{\alpha}{\gamma}$ with $t^2 \nmid \gamma$ and satisfies

$$f_r'(z) \equiv f_r(z) \pmod{\ell^u}.$$

Therefore to finish the proof it remains to show that $\frac{\tilde{f}_r(z)}{\eta^{\ell}(\ell z)}$ vanishes also at those cusps $\frac{\alpha}{\gamma}$ with $t^2|c$. If $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \Gamma_0(t^2)$, then the q-expansion of $\eta^{\ell}(\ell z)|_{\ell/2}\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ starts with $q^{\frac{\ell^2}{24}}$ (times a non zero constant). Thus we have to prove that the $q-$expansion of $\widetilde{f}_r(z)$ starts with q^b with $b > \frac{\ell^2}{24}$. In the following we need the commutation relation for $\nu' \equiv \delta^2\nu \pmod{\ell}$

$$\begin{pmatrix}1 & -\frac{\nu}{\ell}\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix}\begin{pmatrix}1 & -\frac{\nu'}{\ell}\\ 0 & 1\end{pmatrix} \tag{4.6}$$

with

$$\begin{pmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix} = \begin{pmatrix}\alpha - \frac{\gamma\nu}{\ell} & \beta - \frac{\gamma\nu\nu'}{\ell^2} + \frac{\alpha\nu' - \delta\nu}{\ell}\\ \gamma & \delta + \frac{\gamma\nu'}{\ell}\end{pmatrix} \in \Gamma_0(t^2).$$

Combining this with the fact that $|_{\frac{\ell+1}{2}}\left(\begin{smallmatrix}1 & -\frac{\nu'}{\ell}\\ 0 & 1\end{smallmatrix}\right)$ doesn't decrease the order of vanishing we show that the order of vanishing of the holomorphic part of

$$\left(f_r(z) - \epsilon_\ell f_r(z)_\ell\right)|_{\frac{\ell+1}{2}}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$$

is bigger than $\frac{\ell^2}{24}$. This follows if we show that the order of vanishing of the holomorphic part of

$$\left(f_{r,j}(z) - \epsilon_\ell f_{r,j}(z)_\ell\right)|_{\frac{\ell+1}{2}}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$$

is bigger than $\frac{\ell^2}{24}$. Here

$$f_{r,j}(z) := D\left(\frac{j}{t}; z\right) \cdot \eta^{\ell}(\ell z).$$

From Propostion 3.1 easily gives that the holomorphic part of $f_{r,j}(z)|_{\frac{\ell+1}{2}}\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ is given by

$$(-1)^{j\gamma}\cdot\frac{\sin\left(\frac{\pi j}{t}\right)}{\sin\left(\frac{\pi j\delta}{t}\right)}\cdot e^{\frac{3\pi i j^2\gamma_1\delta}{t}}\cdot\left(\frac{\delta}{t}\right)\cdot f_{r,j\delta}(z). \tag{4.7}$$

To compute the holomorphic part of $\epsilon_\ell f_{r,j}(z)_\ell|\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ we use again (4.6). Using Propostion 3.1 it is not hard to see that the holomorphic part of $\epsilon_\ell f_{r,j}(z)_\ell|_{\frac{\ell+1}{2}}\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ is given by

$$\epsilon_\ell\cdot(-1)^{j\gamma}\cdot\frac{\sin\left(\frac{\pi j}{t}\right)}{\sin\left(\frac{\pi j\delta}{t}\right)}\cdot e^{\frac{3\pi i j^2\gamma_1\delta}{t}}\cdot\left(\frac{\delta}{t}\right)\cdot\frac{g}{\ell}\sum_{\nu=1}^{\ell-1}\left(\frac{\nu}{\ell}\right)f_{r,j\delta}(z)|\begin{pmatrix}1 & -\frac{\nu'}{\ell}\\ 0 & 1\end{pmatrix}. \tag{4.8}$$

Since the holomorphic parts of the q-expansions of (4.7) and (4.8) start with $q^{\frac{\ell^2-1}{24}}$ and have integer exponents, it is enough to show that the $\frac{\ell^2-1}{24}$-th coefficients coincide. But this follows directly from

$$\frac{\epsilon_\ell\cdot g}{\ell}\sum_{\nu\pmod{\ell}^*}\left(\frac{\nu}{\ell}\right)e^{-\frac{2\pi i\nu'(\ell^2-1)/24}{\ell}}=\frac{\epsilon_\ell\cdot g}{\ell}\sum_{\nu\pmod{\ell}^*}\left(\frac{\nu'\frac{\ell^2-1}{24}}{\ell}\right)e^{-\frac{2\pi i\nu}{\ell}}=\frac{g\bar{g}}{\ell}=1.$$

This proves Theorem 4.1.

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