RANK AND CONGRUENCES FOR OVERPARTITION PAIRS

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Abstract. The rank of an overpartition pair is a generalization of Dyson's rank of a partition. The purpose of this paper is to investigate the role that this statistic plays in the congruence properties of $\overline{pp}(n)$, the number of overpartition pairs of n. Some generating functions and identities involving this rank are also presented.

1. Introduction

An overpartition λ of n is a partition of n in which the first occurrence of a number may be overlined. An overpartition pair (λ, μ) of n is a pair of overpartitions where the sum of all of the parts is n. For example, there are 12 overpartition pairs of 2,

$$((2), \emptyset), ((\overline{2}), \emptyset), ((1,1), \emptyset), ((\overline{1}, 1), \emptyset), ((1), (\overline{1})), ((1), (1)), ((\overline{1}), (\overline{1})), ((\overline{1}), (1)),$$
$$(\emptyset, (\overline{2})), (\emptyset, (2)), (\emptyset, (1,1)), (\emptyset, (\overline{1}, 1)).$$

Since the overlined parts of an overpartition form a partition into distinct parts and the non-overlined parts of an overpartition form an unrestricted partition, we have the generating function

$$\sum_{n \geq 0} \overline{pp}(n) q^n = \prod_{n \geq 1} \frac{(1+q^n)^2}{(1-q^n)^2} = \frac{\eta^2(2z)}{\eta^4(z)} = 1 + 4q + 12q^2 + 32q^3 + 76q^4 + \cdots .$$

Here $\overline{pp}(n)$ denotes the number of overpartition pairs of n, and $\eta(z)$ is the eta-function

$$\eta(z) := q^{1/24} \prod_{n \geq 1} (1 - q^n), \tag{1.1}$$

where $q := e^{2\pi i z}$.

It has recently become clear that overpartition pairs play an important role in the theory of q-series and partitions. They provide a natural and general setting for the study of q-series identities [8, 13, 14, 39] and q-difference equations [25, 26, 28]. One of the important statistics that has arisen in the study of overpartition pairs is the *rank*. To define the rank of an overpartition pair we use the notations $\ell(\cdot)$ and $n(\cdot)$ for the largest part and the number of parts of an object. Overlining these functions indicates that we are only considering the overlined parts. We order the parts of (λ, μ) by stipulating that for a number k,

$$\overline{k}_\lambda > k_\lambda > \overline{k}_\mu > k_\mu, \tag{1.2}$$

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where the subscript indicates to which of the two overpartitions the part belongs.

Definition 1.1. *The rank of an an overpartition pair* (λ, μ) *is*

$$\ell((\lambda, \mu)) - n(\lambda) - \overline{n}(\mu) - \chi((\lambda, \mu)), \tag{1.3}$$

where $\chi((\lambda, \mu))$ *is defined to be* 1 *if the largest part of* (λ, μ) *is non-overlined and in* μ, *and* 0 *otherwise.*

For example, the rank of the overpartition pair $((\overline{6}, 6, 5, 4, 4, 4, \overline{3}, \overline{1}), (7, 7, \overline{5}, 2, 2, 2))$ is $7 - 8 - 1 - 1 = -3$, while the rank of the overpartition pair $((4, \overline{3}, 3, \overline{2}, 1), (4, 4, 4, \overline{1}))$ is $4 - 5 - 1 - 0 = -2$. When the overpartition μ is empty and λ has no overlined parts, the overpartition pair is just a single partition λ, and the rank reduces to the largest part minus the number of parts, which is Dyson's original rank for partitions [15].

Dyson defined his rank in order to understand certain congruence properties for the partition function $p(n)$. Ramanujan had proven that

$$p(5n + 4) \equiv 0 \pmod{5}, \tag{1.4}$$

$$p(7n + 5) \equiv 0 \pmod{7}, \tag{1.5}$$

$$p(11n + 6) \equiv 0 \pmod{11}. \tag{1.6}$$

Using theta functions and q-series identities, Atkin [6] and Watson [38] later on showed generalisations for powers of 5, 7, and 11. In a celebrated paper Ono [32] treated these kinds of congruences systematically. Combining Shimura's theory of modular forms of half-integral weight with results of Serre on modular forms modulo ℓ he showed that for any prime $\ell \geq 5$ there exist infinitely many non-nested arithmetic progressions of the form $An + B$ such that

$$p(An + B) \equiv 0 \pmod{\ell}.$$

Ahlgren and Ono [1] and [2] extended this phenomenon to prime powers.

Dyson [15] conjectured that

$$N(r, 5, 5n + 4) = \frac{p(5n + 4)}{5} \quad \text{for } 0 \leq r < 5, \tag{1.7}$$

$$N(r, 7, 7n + 5) = \frac{p(7n + 5)}{5} \quad \text{for } 0 \leq r < 7. \tag{1.8}$$

Here $N(r, t, n)$ denotes the number of partitions of n whose rank is congruent to r modulo t. Dyson's conjectures were proven by Atkin and Swinnerton-Dyer [7]. It is interesting to note that the congruence modulo 11 may not be established in the same way using the rank. However, there is a statistic called the crank, predicted by Dyson and found by Andrews and Garvan [4], which provides combinatorial decompositions like (1.7) and (1.8) for all three of Ramanujan's congruences [18].

In this paper we investigate the role played by the rank of an overpartition pair in congruence properties of $\overline{pp}(n)$. Our first result shows that this statistic implies a simple Ramanujan-type congruence in the same way as the rank for ordinary partitions implies (1.4) and (1.5).

Theorem 1.2. *Let* $\overline{NN}(r, t, n)$ *denote the number of overpartition pairs of* n *whose rank is congruent to* r *modulo* t. *For all* $0 \leq r < 3$ *we have*

$$\overline{NN}(r, 3, 3n + 2) = \frac{\overline{pp}(3n + 2)}{3}. \tag{1.9}$$

Corollary 1.3. *For all natural numbers n, we have*

$$\overline{pp}(3n+2) \equiv 0 \pmod 3.$$

The interested reader may check, for example, that of the 12 overpartition pairs of 2 listed on the previous page, four have rank 1, four have rank 0, and four have rank -1.

When there is no congruence in the arithmetic progression $An+B$, it is still natural to seek information about the *rank differences* $\overline{NN}(r,t,An+B) - \overline{NN}(s,t,An+B)$. In the case of the partition function and Dyson's rank, such rank differences have been studied by many authors (see [5, 7, 9, 33], for instance) who prove a variety of results. For example, some rank differences are 0, some have nice infinite product generating functions, and some are always positive or always negative. In the case of overpartition pairs, there are several cases where the rank differences can be expressed as nice infinite products, the following being just one example. More of these identities are given in Section 3.

Theorem 1.4. *We have*

$$4 + \sum_{n\geq 1} \left(\overline{NN}(0,3,n) - \overline{NN}(1,3,n)\right) q^n = \frac{4\eta^2(2z)}{\eta(z)\eta(3z)} \tag{1.10}$$

$$= \frac{4\eta(6z)\eta^2(9z)}{\eta^2(3z)\eta(18z)} + \frac{4\eta^2(18z)}{\eta(3z)\eta(9z)}. \tag{1.11}$$

Perhaps the most interesting feature of such identities is that many of the infinite products can be interpreted in a natural way as generating functions for overpartitions or overpartition pairs. Moreover, some of these products have shown up in other studies of overpartitions. We shall record some of the resulting combinatorial identities, the following corollary of Theorem 1.4 being one example:

Corollary 1.5. *Let $S_1(n)$ denote the number of overpartitions of n, where parts differ by at least 3 if the smaller part is overlined OR both parts are divisible by 3, and parts differ by at least 6 if the smaller is overlined AND both parts are divisible by 3. Let $S_2(n)$ denote the number of overpartitions $\lambda = \lambda_1 + \lambda_2 + \cdots$ of n, where parts occur at most twice and $\lambda_i - \lambda_{i+2}$ is at least 2 if λ_{i+2} is non-overlined and at least 1 if λ_{i+2} is overlined. Let $S_3(n)$ denote the number of overpartitions of n into parts not divisible by 3. Then $4S_1(n) = 4S_2(n) = 4S_3(n) = \overline{NN}(0,3,3n) - \overline{NN}(1,3,3n)$.*

Instead of asking if the rank implies congruences in the sense of Theorem 1.2, we may ask if congruences for $\overline{pp}(n)$ are implied by the fact that the counting functions $\overline{NN}(r,t,An+B)$ themselves satisfy congruences. This turns out to be true, much like it is for partitions [10, 12] and overpartitions [11], but with two notable differences. First, for partitions and overpartitions one obtains weak Maass forms, while for overpartition pairs we use classical modular forms. Second, in the case of overpartition pairs we are dealing with objects of integral weight instead of half-integral weight. Hence we obtain strong density results in addition to congruences in arithmetic progressions.

Theorem 1.6. *Let ℓ be an odd prime. For any positive integers j and u, almost all natural numbers n satisfying $\left(\frac{n}{\ell}\right) = -\left(\frac{-2}{\ell}\right)$ have the property that*

$$\overline{NN}(r,\ell^u,n) \equiv 0 \pmod{\ell^j} \tag{1.12}$$

for all r with $0 \leq r \leq \ell^u - 1$.

Theorem 1.7. *Let ℓ be an odd prime and let t be an odd number which is a power of ℓ or relatively prime to ℓ. For any positive integer j, there are infinitely many non-nested arithmetic progressions $An+B$ such that*

$$\overline{NN}(r,t,An+B) \equiv 0 \pmod{\ell^j} \tag{1.13}$$

for all r with $0 \leq r \leq t-1$.

We note that since

$$\overline{pp}(n) = \sum_{r=0}^{\ell^u-1} \overline{NN}(r,\ell^u,n),$$

Theorem 1.6 is also true for $\overline{pp}(n)$. For completeness, we shall also study the case $\ell = 2$ and thus establish the following.

Theorem 1.8. *For any prime ℓ and natural number j, we have*

$$\liminf_{X\to\infty} \frac{\#\{n \leq X : \overline{pp}(n) \equiv 0 \pmod{\ell^j}\}}{X} \geq \begin{cases} \frac{1}{2} - \frac{1}{\ell}, & \ell \text{ odd,} \\ 1, & \ell = 2. \end{cases}$$

The paper is organized as follows. In the next section, we give some fundamental generating functions for the rank. In Section 3 we prove Theorems 1.2 and 1.4, Corollary 1.5, and related results. In Section 4, we use the theory of modular forms in the spirit of Ahlgren-Ono [2], Mahlburg [29], and Serre [35, 36] to deduce Theorems 1.6 and 1.7.

2. Generating functions

In this section we give some basic generating functions required to prove the main theorems as well as some that will be of independent interest. To begin, let $\overline{NN}(m,n)$ denote the number of overpartition pairs of n with rank m. We assume that the empty overpartition pair of 0 has rank 0. We shall establish a two-variable generating function for $\overline{NN}(m,n)$. Our argument will depend on the following, which is equation (1.11) in [22]:

$$\frac{4}{(1+z)(1+z^{-1})} + \sum_{n\geq 1} \frac{(-1)_n^2 q^n}{(zq,q/z)_n} = \frac{4(-q)_\infty^2}{(1+z)(1+z^{-1})(zq,q/z)_\infty}. \tag{2.1}$$

Here we have employed the standard q-series notation [17],

$$(a_1,a_2,\ldots,a_k)_n := (a_1,a_2,\ldots,a_k;q)_n := \prod_{j=0}^{n-1}(1-a_1q^j)(1-a_2q^j)\cdots(1-a_kq^j).$$

We shall also require the q-Gauss summation [17],

$$\sum_{n\geq 0} \frac{(a,b)_n (c/ab)^n}{(c,q)_n} = \frac{(c/a,c/b)_\infty}{(c,c/ab)_\infty}. \tag{2.2}$$

Proposition 2.1. *We have*

$$\sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}} \overline{NN}(m,n) z^m q^n = \sum_{n\geq 0} \frac{(-1)_n^2 q^n}{(zq,q/z)_n} \tag{2.3}$$

Proof. We begin by recalling from the elementary theory of partitions [3, p. 16] that $1/(zq)_n$ (resp., $(-zq)_n$) is the generating function for partitions (resp., partitions into distinct parts), where the exponent of q is the number being partitioned, the exponent of z is the number of parts, and all parts are less than or equal to n. Using these generating functions together with the definition of the rank (1.3), and splitting the overpartition pairs with a positive number of parts into four cases, depending on whether the largest part is overlined and whether it is in λ or μ, gives us four series. For example, the series

$$\sum_{n\geq 1} \frac{(-q/z)_{n-1}^2 q^n z^{n-1}}{(q/z)_{n-1}(q)_n}$$

is the generating function for overpartition pairs whose largest part n is overlined and in μ, where the exponent of q is the number being partitioned and the exponent of z is the rank. Combining this with the three other cases gives

$$\begin{aligned}
\sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}} \overline{NN}(m,n)z^m q^n &= 1+\sum_{n\geq 1}\frac{(-q/z)_{n-1}(-q/z)_n q^n z^{n-1}}{(q/z,q)_n}+\sum_{n\geq 1}\frac{(-q/z)_{n-1}(-q/z)_n q^n z^{n-1}}{(q/z,q)_n}\\
&+ \sum_{n\geq 1}\frac{(-q/z)_{n-1}^2 q^n z^{n-1}}{(q/z)_{n-1}(q)_n}+\sum_{n\geq 1}\frac{(-q/z)_{n-1}^2 q^n z^{n-1}}{(q/z)_{n-1}(q)_n}\\
&= 1+2\sum_{n\geq 1}\frac{(-q/z)_{n-1}(-q/z)_n q^n z^{n-1}}{(q/z,q)_n}+2\sum_{n\geq 1}\frac{(-q/z)_{n-1}^2 q^n z^{n-1}}{(q/z)_{n-1}(q)_n}\\
&= 1+2\sum_{n\geq 1}\frac{(-q/z)_{n-1}^2 q^n z^{n-1}}{(q/z)_{n-1}(q)_n}\left(1+\frac{1+\frac{q}{z}}{1-\frac{q}{z}}\right)\\
&= 1+\frac{4z^{-1}}{(1+\frac{1}{z})^2}\sum_{n\geq 1}\frac{(-1/z)_n^2 (zq)^n}{(q/z,q)_n}\\
&= 1+\frac{4}{(1+z)(1+\frac{1}{z})}\left(\sum_{n\geq 0}\frac{(-1/z)_n^2 (zq)^n}{(q/z,q)_n}-1\right)\\
&= 1+\frac{4}{(1+z)(1+\frac{1}{z})}\left(\frac{(-q)_\infty^2}{(zq,q/z)_\infty^2}-1\right)\\
&= 1+\sum_{n\geq 1}\frac{(-1)_n^2 q^n}{(q/z,zq)_n},
\end{aligned}$$

where the last two inequalities follow from (2.2) and (2.1). □

We would like to make two remarks here. First, the above proposition implies the symmetry $\overline{NN}(m,n) = \overline{NN}(-m,n)$. Second, one can use the "Frobenius representation of an overpartition pair" to give an alternative definition of the rank. This is the approach taken in [28], and it immediately gives the symmetry as well as the generating function (2.3). Using the above argument or the bijection in [13, 27], one can then recover the definition for the rank that we employ in the present paper.

There are some nice generating functions which follow from (2.3), and we record these here, even though they are not required in the sequel.

Proposition 2.2. *For any integer m we have*

$$\sum_{n\geq 1}\overline{NN}(m,n)q^n = \frac{(-1)_\infty^2}{(q)_\infty^2}\sum_{n\geq 1}\frac{(-1)^{n-1}q^{n(n+1)/2+|m|n}(1-q^n)}{(1+q^n)^2}. \tag{2.4}$$

Proof. We employ a limiting case of Watson's transformation [17],

$$\sum_{n=0}^{\infty}\frac{(aq/bc,d,e)_n(\frac{aq}{de})^n}{(q,aq/b,aq/c)_n} = \frac{(aq/d,aq/e)_\infty}{(aq,aq/de)_\infty}\sum_{n=0}^{\infty}\frac{(a,b,c,d,e)_n(1-aq^{2n})(aq)^{2n}(-1)^nq^{n(n-1)/2}}{(q,aq/b,aq/c,aq/d,aq/e)_n(1-a)(bcde)^n}. \tag{2.5}$$

Taking $a=1$, $b=z$, $c=1/z$, $d=-1$, and $e=-1$, we obtain

$$\sum_{n=0}^{\infty}\frac{(-1)_n^2q^n}{(zq,q/z)_n} = \frac{(-q)_\infty^2}{(q)_\infty^2}\left(1+4\sum_{n=1}^{\infty}\frac{(1-z)(1-1/z)(-1)^nq^{n(n+1)/2+n}}{(1-zq^n)(1-q^n/z)(1+q^n)}\right).$$

Now it is easily verified that

$$\frac{(1-z)(1-1/z)q^n}{(1-zq^n)(1-q^n/z)} = 1-\frac{(1-q^n)}{(1+q^n)}\sum_{m=0}^{\infty}z^mq^{mn}-\frac{(1-q^n)}{(1+q^n)}\sum_{m=1}^{\infty}z^{-m}q^{mn}.$$

Substituting this into the above equation and picking off the coefficient of z^m immediately gives the desired result for $m\neq 0$. By Proposition 2.1 with $m=0$, the generating function

$$\sum_{n\geq 1}\overline{NN}(0,n)q^n$$

is the coefficient of z^0 in

$$-1+\sum_{n\geq 0}\frac{(-1)_n^2q^n}{(zq,q/z)_n}.$$

Arguing as above, this constant term is

$$-1+\frac{(-q)_\infty^2}{(q)_\infty^2}\left(1+4\sum_{n\geq 1}\frac{(-1)^nq^{n(n+1)/2}}{1+q^n}+4\sum_{n\geq 1}\frac{(-1)^{n-1}q^{n(n+1)/2}(1-q^n)}{(1+q^n)^2}\right).$$

Now it is known (see [17, 5.2, p.134], for example) that the first two terms in parentheses sum to

$$\frac{(q)_\infty^2}{(-q)_\infty^2}.$$

Hence we end up with

$$\sum_{n\geq 1}\overline{NN}(0,n)q^n = \frac{(-1)_\infty^2}{(q)_\infty^2}\sum_{n\geq 1}\frac{(-1)^{n-1}q^{n(n+1)/2}(1-q^n)}{(1+q^n)^2},$$

as desired. □

The following is deduced immediately from Proposition 2.2

Corollary 2.3. *For $0 \leq r \leq t$, we have*

$$\sum_{n\geq 1} \overline{NN}(r,t,n)q^n = \frac{(-1)^2_\infty}{(q)^2_\infty} \sum_{n\geq 1} \frac{(-1)^{n-1}q^{n(n+1)/2}(q^{rn}+q^{(t-r)n})(1-q^n)}{(1+q^n)^2(1-q^{tn})}. \tag{2.6}$$

Proof. Use Proposition 2.2 plus the fact that

$$\overline{NN}(r,t,n) = \sum_{k\in\mathbb{Z}} \overline{NN}(r+tk,n).$$

□

We close this section with a useful decomposition of the generating function for $\overline{NN}(r,t,n)$ into a linear combination of certain modular forms.

Proposition 2.4. *For t an odd integer and $0 \leq r \leq t-1$, let $C(r,t)$ be the constant*

$$C(r,t) := \frac{4}{t}\sum_{s=0}^{t-1} \frac{\zeta_t^{-rs}}{(1+\zeta_t^s)(1+\zeta_t^{-s})}.$$

Then we have

$$C(r,t) + \sum_{n\geq 1} \overline{NN}(r,t,n)q^n = \frac{1}{t}\sum_{n\geq 0}\overline{pp}(n)q^n + \frac{1}{t}\sum_{s=1}^{t-1}\zeta_t^{-rs}R(\zeta_t^s;q). \tag{2.7}$$

Here

$$R(z;q) := \frac{4(-q)^2_\infty}{(1+z)(1+1/z)(zq,q/z)_\infty}.$$

Proof. Since $\sum \overline{pp}(n)q^n = R(1;q)$, by (2.1) and (2.3), we see that for $n \geq 1$ the coefficient of q^n on the right hand side of (2.7) is

$$\frac{1}{t}\sum_{s=0}^{t-1}\zeta_t^{-rs}\sum_{m\in\mathbb{Z}}\overline{NN}(m,n)\zeta_t^{sm} = \frac{1}{t}\sum_{m\in\mathbb{Z}}\overline{NN}(m,n)\sum_{s=0}^{t-1}\zeta_t^{(m-r)s}.$$

Now the sum on s is equal to t if $m \equiv r \pmod t$, and is 0 otherwise. This establishes the theorem up to the constant term, which is easily calculated from (2.1). □

Notice that because of the denominator in $R(z;q)$, we can only take t odd in order to avoid the case $z=-1$. This restricts the values of t for which we can obtain information about the counting functions $\overline{NN}(r,t,n)$. It would be interesting to know what can be done in the case t even.

3. Generating functions for rank differences

In this section we prove Theorem 1.4 and Corollary 1.5, along with some similar results when $t=2$ or 4. Theorem 1.2 will also follow from this work, as we shall see shortly.

Proof of Theorems 1.2 and 1.4. Using (2.3) and (2.1) with $z=\zeta_3$, a third root of unity, together with the fact that $\overline{NN}(1,3,n) = \overline{NN}(2,3,n)$, we may conclude that

$$4 + \sum_{n\geq 1}\left(\overline{NN}(0,3,n) - \overline{NN}(1,3,n)\right)q^n = 4\frac{(-q)^2_\infty (q)_\infty}{(q^3;q^3)_\infty}. \tag{3.1}$$

This is (1.10). By Jacobi's triple product identity [17],

$$\sum_{n\in\mathbb{Z}} z^n q^{n^2} = (-zq, -q/z, q^2; q^2)_\infty \tag{3.2}$$

we have

$$2(-q)_\infty^2 (q)_\infty = \sum_{n\in\mathbb{Z}} q^{\binom{n+1}{2}}. \tag{3.3}$$

Since there are no triangular numbers congruent to 2 mod 3, we have that

$$\overline{NN}(0,3,3n+2) = \overline{NN}(1,3,3n+2).$$

This establishes that for $r = 0, 1$, or 2,

$$\overline{NN}(r,3,3n+2) = \frac{\overline{pp}(3n+2)}{3},$$

which completes the proof of Theorem 1.2. For (1.11), we note that the triangular numbers congruent to 1 modulo 3 are of the form $(3n+1)(3n+2)/2$, and so by (3.2) we have

$$\sum_{n\in\mathbb{Z}} q^{(3n+1)(3n+2)/2} = 2q(-q^9; q^9)_\infty (q^{18}; q^{18})_\infty.$$

Similarly, the triangular numbers divisible by 3 are of the form $(3n)(3n+1)/2$ (each of these contributing twice), and we have

$$2\sum_{n\in\mathbb{Z}} q^{(3n)(3n+1)/2} = 2\frac{(q^9;q^9)_\infty(-q^3;q^3)_\infty}{(-q^9;q^9)_\infty}.$$

Putting these last two equations together with (3.1) and (3.3) above, we have Theorem 1.4. □

Theorem 3.1. *We have*

$$\sum_{n\geq 0} \left(\overline{NN}(0,2,2n+1) - \overline{NN}(1,2,2n+1)\right) q^n = \frac{4\eta^8(2z)}{\eta^4(z)} \tag{3.4}$$

Proof. From (2.3), we have

$$\sum_{n\geq 1} \left(\overline{NN}(0,2,n) - \overline{NN}(1,2,n)\right) q^n = \sum_{n\geq 1} \frac{(-1)_n^2 q^n}{(-q)_n^2}.$$

This final series may be written as

$$4\sum_{n\geq 1} \frac{q^n}{(1+q^n)^2} = 4\sum_{n\geq 1}(q^n - 2q^{2n} + 3q^{3n} - 4q^{4n} + \cdots),$$

and so we see that

$$\overline{NN}(0,2,2n+1) - \overline{NN}(1,2,2n+1) = 4\sigma(2n+1),$$

where $\sigma(n)$ is the sum of the divisors of n. Now it is well known (see [16, p. 482-483], for example) that

$$\sum_{n\geq 0} \sigma(2n+1) q^n = \frac{\eta^8(2z)}{\eta^4(z)},$$

which completes the proof. □

Theorem 3.2. *We have*

$$2+\sum_{n\geq 1}\left(\overline{NN}(0,4,n)-\overline{NN}(2,4,n)\right)q^n$$

$$=\frac{2\eta^3(2z)}{\eta^2(z)\eta(4z)} \tag{3.5}$$

$$=\frac{2\eta^5(8z)}{\eta^2(2z)\eta(4z)\eta^2(16z)}+\frac{4\eta(4z)\eta^2(16z)}{\eta^2(2z)\eta(8z)} \tag{3.6}$$

$$=\frac{2\eta^5(8z)\eta^3(16z)}{\eta^6(4z)\eta^2(32z)}+\frac{4\eta^7(16z)}{\eta^4(4z)\eta(8z)\eta^2(32z)}+\frac{4\eta^7(8z)\eta^2(32z)}{\eta^6(4z)\eta^3(16z)}+\frac{8\eta(8z)\eta(16z)\eta^2(32z)}{\eta^4(4z)}. \tag{3.7}$$

Proof of Theorem 3.2. Using $z=i$ in (2.3) and (2.1), together with the fact that $\overline{NN}(1,4,n)=\overline{NN}(3,4,n)$, we have

$$2+\sum_{n\geq 1}\left(\overline{NN}(0,4,n)-\overline{NN}(2,4,n)\right)=2\frac{(-q)_\infty^2}{(-q^2;q^2)_\infty}=\frac{2(-q;q^2)_\infty}{(q;q^2)_\infty}=2\frac{(q^2;q^2)_\infty^3}{(q)_\infty^2(q^4;q^4)_\infty}.$$

This is (3.5). For (3.6), we observe that

$$\frac{(-q^2;q^2)_\infty(-q,-q,q^2;q^2)_\infty}{(q^2;q^2)_\infty}=\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\times\sum_{n\in\mathbb{Z}}q^{n^2}.$$

Now, (3.6) follows from (3.2) and the fact that n^2 is even if and only if n is even. The proof of (3.7) is similar. In this case, we write

$$\begin{aligned}\frac{(-q;q^2)_\infty}{(q;q^2)_\infty} &= \frac{(-q,-q,-q^2,-q^2,q^2;q^2)_\infty}{(q^4;q^4)_\infty}\\ &= (-q,-q,q^2;q^2)_\infty(-q^2,-q^2,q^4;q^4)_\infty\times\frac{(-q^4;q^4)_\infty^2}{(q^4;q^4)_\infty^2}\\ &= \sum_{n\in\mathbb{Z}}q^{n^2}\sum_{m\in\mathbb{Z}}q^{2m^2}\times\frac{(-q^4;q^4)_\infty^2}{(q^4;q^4)_\infty^2},\end{aligned}$$

the last equality following from (3.2). Now n^2+2m^2 is divisible by 4 if and only if n and m are even. Hence the powers of q divisible by 4 in $(-q;q^2)_\infty/(q;q^2)_\infty$ come from

$$\sum_{n\in\mathbb{Z}}q^{4n^2}\sum_{m\in\mathbb{Z}}q^{8m^2}\times\frac{(-q^4;q^4)_\infty^2}{(q^4;q^4)_\infty^2},$$

which by (3.2) is

$$\frac{(-q^4;q^4)_\infty^2(-q^4,-q^4,q^8;q^8)_\infty(-q^8,-q^8,q^{16};q^{16})_\infty}{(q^4;q^4)_\infty^2}.$$

This is the first term in (3.7), and the other three follow in a similar way. This completes the proof of Theorem 3.2. □

We now highlight a few combinatorial identities that follow from the rank difference generating functions above, beginning with Corollary 1.5.

Proof of Corollary 1.5. The generating function for $S_3(n)$ is

$$\sum_{n\geq 0} S_3(n)q^n = \frac{(-q)_\infty(q^3;q^3)_\infty}{(q)_\infty(-q^3;q^3)_\infty}.$$

Hence from (1.11), it is clear that for $n \geq 1$ we have

$$\overline{NN}(0,3,3n) - \overline{NN}(1,3,3n) = 4S_3(n).$$

The fact that $S_3(n) = S_2(n)$ is the case $k = 3$ of Theorem 1.1 in [21], while the fact that $S_3(n) = S_1(n)$ follows from Corollary 1.4 of [24] □

Corollary 3.3. *Let $T_1(n)$ denote the number of overpartitions of n into distinct parts, where parts differ by at least two if the smaller is overlined. Let $T_2(n)$ denote the number of overpartitions of n into odd parts. Then for all natural numbers n we have*

$$\overline{NN}(0,4,n) - \overline{NN}(2,4,n) = 2T_1(n) = 2T_2(n).$$

Proof. By (3.5), $\overline{NN}(0,4,n) - \overline{NN}(2,4,n)$ is equal to twice the number of overpartitions of n into odd parts. But this latter quantity is also equal to twice $T_1(n)$ (see Theorem 1.1 of [21], for example). □

As two more examples we record the following, which is easily deduced from (3.6).

Corollary 3.4. *Let $U(n)$ denote the number of overpartition pairs (λ,μ) of n, where the parts of λ are not divisible by 4 and the parts of μ are congruent to 2 modulo 4. Let $V(n)$ denote the number of overpartition pairs (λ,μ) of n, where the non-overlined parts of λ are not divisible by 8 and μ contains only overlined parts, these being divisible by 4. Then for all natural numbers n we have*

(*i*) $\overline{NN}(0,4,2n) - \overline{NN}(2,4,2n) = 2U(n)$,

(*ii*) $\overline{NN}(0,4,2n+1) - \overline{NN}(0,4,2n+1) = 4V(n)$.

Proof. The right side of (3.6) may be written as

$$\frac{2(-q^2;q^2)_\infty(-q^4;q^8)_\infty}{(q^2;q^2)_\infty(q^4;q^8)_\infty} + \frac{4q(-q^2;q^2)_\infty(-q^8;q^8)_\infty(q^{16};q^{16})_\infty}{(q^2;q^2)_\infty},$$

which is clearly equal to

$$2\sum_{n\geq 0} U(n)q^{2n} + 4\sum_{n\geq 0} V(n)q^{2n+1}.$$

□

It is intriguing to wonder if the equality between the rank differences and the overpartition-theoretic functions in Corollaries 1.5, 3.3, and 3.4 could be proven using combinatorial mappings. Corollaries 1.5 and 3.3 are particularly interesting, as they relate overpartition pairs to overpartitions.

4. Congruences for the rank

In this section we establish Theorems 1.6 and 1.7. These results depend heavily on the theory of modular forms [19], particularly the properties of eta-functions [31] and Klein forms [20, 34]. Before we prove the main result of this section, we review a few definitions.

Let $\Gamma = \Gamma_0(N)$ or $\Gamma_1(N)$ denote the congruence subgroups of $SL_2(\mathbb{Z})$ consisting of matrices equivalent modulo N to $\begin{pmatrix} \bullet & \bullet \\ 0 & \bullet \end{pmatrix}$ or $\begin{pmatrix} 1 & \bullet \\ 0 & 1 \end{pmatrix}$, respectively. Let $M_k(\Gamma_0(N), \chi)$ (resp. $S_k(\Gamma_0(N)), \chi)$ denote the usual space of weight k modular forms (resp. cusp forms) on $\Gamma_0(N)$ with character χ, and let $M_k(\Gamma_1(N))$ (resp. $S_k(\Gamma_1(N)))$ denote the space of weight k modular forms (resp. cusp forms) on $\Gamma_1(N)$. For a matrix in $SL_2(\mathbb{Z})$ and function $f(z)$ on the upper half plane, define the weight k slash operator by

$$f(z)\bigg|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} := (cz+d)^k f\left(\frac{az+b}{cz+d}\right).$$

Recall that the $(0,s)$-Klein form is defined relative to a modulus N, for all s with $0 \leq s \leq N-1$, by

$$K_{0,s}(z) := \frac{-i\omega_s q^{1/12}}{2\pi} \times \frac{1}{\eta^2(z)} \prod_{n\geq 1} (1-\zeta^s q^n)(1-\zeta^{-s}q^n). \tag{4.1}$$

Here we have $\zeta := e^{2\pi i/N}$ and $\omega_s := \zeta^{s/2}(1-\zeta^{-s})$. This function is a weakly holomorphic modular form of weight -1 on $\Gamma_1(2N^2)$. (Recall that weakly holomorphic means that the poles, if there are any, are supported at the cusps.) More specifically, if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, then

$$K_{0,s}(z)\bigg|_{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \beta_s K_{0,\overline{ds}}(z). \tag{4.2}$$

Here $\overline{d}$ denotes d modulo N and

$$\beta_s := \exp\left(\frac{cs + (ds - \overline{ds})}{2N} + \frac{cs \cdot (ds - \overline{ds}) - cs \cdot \overline{ds}}{2N^2}\right),$$

with $\exp(z) := e^{2\pi i z}$.

Theorem 4.1. *For an odd prime ℓ and natural numbers j and u, there exists a weight k and a level N so that modulo ℓ^j we have*

$$\sum_{\left(\frac{n}{\ell}\right) = -\left(\frac{-2}{\ell}\right)} \overline{NN}(r, \ell^u, n) q^n \in S_k(\Gamma_1(N)) \tag{4.3}$$

for all r with $0 \leq r \leq \ell^u - 1$. Moreover, this cusp form has coefficients of the form $a(n)/\ell^u$, with $a(n)$ an integer.

Proof. Let ℓ be an odd prime, let $t = \ell^u$, and suppose that $0 \leq r \leq \ell^u - 1$. With ζ an ℓ^uth root of unity, define the function

$$f(z) := \frac{-2i}{\pi} \sum_{s=0}^{t-1} \frac{\omega_s \zeta^{-rs} \eta^2(2z) \eta^{6\ell-8}(2\ell z) \eta^4(\ell z)}{(1+\zeta^s)(1+\zeta^{-s}) \eta^4(z) K_{0,s}(z)}. \tag{4.4}$$

Notice that from Proposition 2.4 and equation (4.1), $f(z)$ has a q-expansion with integer coefficients. Basic properties of eta-functions and of Klein forms tell us that

$$\frac{\eta^2(2z)\eta^{6\ell-8}(2\ell z)\eta^4(\ell z)}{\eta^4(z)}$$

is a holomorphic modular form in the space $M_{3\ell-3}(\Gamma_0(2\ell))$ and $K_{0,s}(z)$ is a weakly holomorphic form of weight -1 on $\Gamma_1(2t^2)$. Hence the function $f(z)$ is a weakly holomorphic modular form in the space $M_{3\ell-2}(\Gamma_1(2t^2))$.

For any modular form $F = \sum a(n)q^n$, define $\widetilde{F}(z)$ by

$$\widetilde{F}(z) := \left(F(z) - \left(\tfrac{-2}{\ell}\right) F(z) \otimes \left(\tfrac{\bullet}{\ell}\right)\right) \otimes \left(\tfrac{\bullet}{\ell}\right).$$

Recall that the twist of a modular form $F \otimes \left(\frac{\bullet}{\ell}\right)(z)$ is defined by

$$F \otimes \left(\tfrac{\bullet}{\ell}\right)(z) := \sum_n \left(\tfrac{n}{\ell}\right) a(n)\, q^n,$$

which may also be written

$$F \otimes \left(\tfrac{\bullet}{\ell}\right)(z) = \frac{g}{\ell} \sum_{v \pmod{\ell}} \left(\tfrac{v}{\ell}\right) F(z) \Big| \begin{pmatrix} 1 & -v/\ell \\ 0 & 1 \end{pmatrix},$$

where g is the usual Gauss sum

$$g := \sum_{v \pmod{\ell}} \left(\tfrac{v}{\ell}\right) e^{2\pi i v/\ell}.$$

By the theory of twists of modular forms, the function $\widetilde{f}(z)$ is a weakly holomorphic modular form in the space $M_{3\ell-2}(\Gamma_1(2\ell^4t^2))$.

Now let $E_{\ell,x}(z)$ be defined by

$$E_{\ell,x}(z) := \frac{\eta^{\ell^x}(z)}{\eta(\ell^x z)}. \tag{4.5}$$

It is known that $E_{\ell,x}(z)$ is a modular form in the space $M_{(\ell^x-1)/2}(\Gamma_0(\ell^x), \chi)$ for a certain character χ, that $E_{\ell,x}(z)^{\ell^y} \equiv 1 \pmod{\ell^{y+1}}$, and that $E_{\ell,x}(z)$ vanishes at every cusp a/c with ℓ^x not dividing c.

Define $G(z)$ by

$$G(z) := \frac{\widetilde{f}(z)}{\eta^{6\ell-8}(2\ell z)\eta^4(\ell z)} \times E_{\ell,u+1}(z)^T,$$

where T is a large enough power of ℓ so that

$$G(z) \equiv \frac{\widetilde{f}(z)}{\eta^{6\ell-8}(2\ell z)\eta^4(\ell z)} \pmod{\ell^j}$$

and $G(z)$ vanishes at all cusps a/c with ℓt $(= \ell^{u+1})$ not dividing c. We now want to show that $G(z)$ vanishes at the cusps a/c with $\ell t | c$. We shall consider two cases, depending on the parity of c.

If c is even, then let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2\ell t)$ be the matrix for such a cusp. The expansion for the denominator of $G(z)$ at such a cusp up to a constant is

$$\eta^{6\ell-8}(2\ell z)\eta^4(\ell z)| \begin{pmatrix} a & b \\ c & d \end{pmatrix} = q^{(\ell^2-\ell)/2} + \cdots .$$

Now we treat the function $f(z) \otimes \left(\frac{\bullet}{\ell}\right)$. First recall the commutation relation, that for any $v' \equiv d^2 v \pmod{\ell}$ we have

$$\begin{pmatrix} 1 & -v/\ell \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} 1 & -v'/\ell \\ 0 & 1 \end{pmatrix}, \tag{4.6}$$

where

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a - cv/\ell & b - cvv'/\ell^2 + (av' - dv)/\ell \\ c & d + cv'/\ell \end{pmatrix}. \tag{4.7}$$

Since we are in the case where c is even, we have $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma_0(2\ell t)$.

So, we have

$$\begin{aligned}
& f(z) \otimes \left(\tfrac{\bullet}{\ell}\right) \Big| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
= \; & \frac{g}{\ell} \sum_{v=1}^{\ell-1} \left(\tfrac{v}{\ell}\right) f(z) \Big| \begin{pmatrix} 1 & -v/\ell \\ 0 & 1 \end{pmatrix} \Big| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
= \; & \frac{g}{\ell} \sum_{v=1}^{\ell-1} \left(\tfrac{v}{\ell}\right) f(z) \Big| \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \Big| \begin{pmatrix} 1 & -v'/\ell \\ 0 & 1 \end{pmatrix} \\
= \; & \frac{-2ig}{\pi\ell} \sum_{v=1}^{\ell-1} \left(\tfrac{v}{\ell}\right) \left(\sum_{s=0}^{t-1} \frac{\omega_s \zeta^{-rs} \eta^2(2z)\eta^{6\ell-8}(2\ell z)\eta^4(\ell z)}{(1+\zeta^s)(1+\zeta^{-s})\eta^4(z) K_{0,s}(z)} \right) \Big| \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \Big| \begin{pmatrix} 1 & -v'/\ell \\ 0 & 1 \end{pmatrix} \\
= \; & \frac{-2ig}{\pi\ell} \sum_{v=1}^{\ell-1} \left(\tfrac{v}{\ell}\right) \left(\sum_{s=0}^{t-1} \frac{\omega_s \zeta^{-rs} \eta^2(2z)\eta^{6\ell-8}(2\ell z)\eta^4(\ell z)}{(1+\zeta^s)(1+\zeta^{-s})\eta^4(z) \beta'_s K_{0,\overline{d's}}(z)} \right) \Big| \begin{pmatrix} 1 & -v'/\ell \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

Here we have used the transformation for the Klein forms (4.2) together with the fact that

$$\frac{\eta^2(2z)\eta^{6\ell-8}(2\ell z)\eta^4(\ell z)}{\eta^4(z)} \in M_{3\ell-3}(\Gamma_0(2\ell, \chi_{triv})).$$

Now, the first term in this q-expansion is

$$\frac{4g}{\ell} q^{(\ell^2-\ell)/2} \left(\sum_{s=0}^{t-1} \frac{\omega_s \zeta^{-rs}}{\omega_{\overline{d's}} \beta'_s (1+\zeta^s)(1+\zeta^{-s})} \sum_{v=1}^{\ell-1} \left(\tfrac{v}{\ell}\right) e^{-2\pi i v'(\ell^2-\ell)/2\ell} \right).$$

Since $2\ell | (\ell^2 - \ell)$, the sum on v is equal to 0. This means the expansion of $f(z) \otimes \left(\frac{\bullet}{\ell}\right)$ at the cusp a/c starts with (at least) $q^{(\ell^2-\ell)/2+1}$. We can also see that twisting again will not decrease the order of vanishing at this cusp (since the matrix $\begin{pmatrix} 1 & -v'/\ell \\ 0 & 1 \end{pmatrix}$ does not decrease the order of vanishing). Hence $f(z) \otimes \left(\frac{\bullet}{\ell}\right) \otimes \left(\frac{\bullet}{\ell}\right)$ also vanishes to order at least $(\ell^2 - \ell)/2 + 1$, and so $G(z)$ vanishes at the cusp a/c.

Now suppose that c is odd, and let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\ell t)$ be the matrix for such a cusp. The expansion for the denominator of $G(z)$ at such a cusp is in terms of $q_2 := q^{1/2}$, and up to a constant this expansion begins

$$\eta^{6\ell-8}(2\ell z)\eta^4(\ell z)|\begin{pmatrix} a & b \\ c & d \end{pmatrix} = q_2^{\ell^2/4} + \cdots .$$

One may also compute that the expansion of $\eta^2(2z)\eta^{6\ell-8}(2\ell z)\eta^4(\ell z)/\eta^4(z)$ at such a cusp a/c begins with $\star q_2^{(\ell^2-1)/4} + \cdots$.

Next we calculate the order of vanishing of the numerator of $G(z)$ at the cusp a/c. First, for $f(z)$, we have

$$\begin{aligned}
f(z)\Big|\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \frac{-2i}{\pi}\left(\sum_{s=0}^{t-1} \frac{\omega_s \zeta^{-rs}\eta^2(2z)\eta^{6\ell-8}(2\ell z)\eta^4(\ell z)}{(1+\zeta^s)(1+\zeta^{-s})\eta^4(z)K_{0,s}(z)}\right)\Big|\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
&= \frac{-2i}{\pi}\left(\star q_2^{(\ell^2-1)/4} + \cdots\right)\sum_{s=0}^{t-1} \frac{\omega_s \zeta^{-rs}}{(1+\zeta^s)(1+\zeta^{-s})K_{0,s}(z)}\Big|\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
&= \frac{-2i}{\pi}\left(\star q_2^{(\ell^2-1)/4} + \cdots\right)\sum_{s=0}^{t-1} \frac{\omega_s \zeta^{-rs}}{(1+\zeta^s)(1+\zeta^{-s})\beta_s K_{0,\overline{ds}}(z)}.
\end{aligned}$$

The first term in this expansion is

$$4 \star q^{(\ell^2-1)/8} \sum_{s=1}^{t-1} \frac{\omega_s \zeta^{-rs}}{(1+\zeta^s)(1+\zeta^{-s})\beta_s \omega_{\overline{ds}}}. \tag{4.8}$$

Now, for $f \otimes \left(\frac{\bullet}{\ell}\right)(z)$, we make the observation that in the commutation relation (4.6) above we may select v' even. This ensures that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}^{-1} \in \Gamma_0(2\ell t),$$

and in particular,

$$\frac{\eta^2(2z)\eta^{6\ell-8}(2\ell z)\eta^4(\ell z)}{\eta^4(z)}\Big|\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{\eta^2(2z)\eta^{6\ell-8}(2\ell z)\eta^4(\ell z)}{\eta^4(z)}\Big|\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}.$$

Hence we have

$$\begin{aligned}
&f \otimes \left(\tfrac{\bullet}{\ell}\right)(z)\Bigg| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
=\quad & \frac{g}{\ell}\sum_{v=1}^{\ell-1}\left(\tfrac{v}{\ell}\right) f(z)\Bigg|\begin{pmatrix} 1 & -v/\ell \\ 0 & 1 \end{pmatrix}\Bigg|\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
=\quad & \frac{g}{\ell}\sum_{v=1}^{\ell-1}\left(\tfrac{v}{\ell}\right) f(z)\Bigg|\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\Bigg|\begin{pmatrix} 1 & -v'/\ell \\ 0 & 1 \end{pmatrix} \\
=\quad & \frac{-2ig}{\pi\ell}\sum_{v=1}^{\ell-1}\left(\tfrac{v}{\ell}\right)\left(\sum_{s=0}^{t-1}\frac{\omega_s\zeta^{-rs}\eta^2(2z)\eta^{6\ell-8}(2\ell z)\eta^4(\ell z)}{(1+\zeta^s)(1+\zeta^{-s})\eta^4(z)K_{0,s}(z)}\right)\Bigg|\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\Bigg|\begin{pmatrix} 1 & -v'/\ell \\ 0 & 1 \end{pmatrix} \\
=\quad & \frac{-2ig}{\pi\ell}\sum_{v=1}^{\ell-1}\left(\tfrac{v}{\ell}\right)\left(\sum_{s=0}^{t-1}\frac{\omega_s\zeta^{-rs}\left(\star q_2^{(\ell^2-1)/4}+\cdots\right)}{(1+\zeta^s)(1+\zeta^{-s})\beta_s' K_{0,\overline{d's}}(z)}\right)\Bigg|\begin{pmatrix} 1 & -v'/\ell \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

We have $\beta_s' = \beta_s$ and $d' \equiv d \pmod t$, and so the above expansion begins as

$$\frac{4g}{\ell}\sum_{s=0}^{t-1}\frac{\omega_s\zeta^{-rs}}{(1+\zeta^s)(1+\zeta^{-s})\beta_s\omega_{\overline{ds}}}\sum_{v=1}^{\ell-1}\left(\tfrac{v}{\ell}\right)\star\left(q_2^{(\ell^2-1)/4}\Bigg|\begin{pmatrix} 1 & -v'/\ell \\ 0 & 1 \end{pmatrix}+\cdots\right),$$

which begins with

$$4\star q^{(\ell^2-1)/8}\frac{g}{\ell}\sum_{s=1}^{t-1}\frac{\omega_s\zeta^{-rs}}{(1+\zeta^s)(1+\zeta^{-s})\beta_s\omega_{\overline{ds}}}\sum_{v=1}^{\ell-1}\left(\tfrac{v}{\ell}\right)e^{\frac{-2\pi i v'(\ell^2-1)/8}{\ell}}.$$

If we multiply this by $\left(\frac{-2}{\ell}\right)$ and simplify using $g^2 = \ell\left(\frac{-1}{\ell}\right)$, then we get the same term as in (4.8). This shows that

$$f(z) - \left(\tfrac{-2}{\ell}\right) f(z) \otimes \left(\tfrac{\bullet}{\ell}\right)$$

has an expansion at a/c which begins at least with $q^{(\ell^2+7)/8}$. Twisting again, as before, does not lower this. Since the denominator $\eta^{6\ell-8}(2\ell z)\eta^4(\ell z)$ had an expansion at a/c that starts with $q^{\ell^2/8}$, we may conclude that $G(z)$ vanishes at a/c, and is therefore a cusp form.

To finish the proof, observe that for a series of the form

$$\sum a(n)q^n \sum b(\ell n)q^{\ell n},$$

twisting by $\left(\frac{\bullet}{\ell}\right)$ gives

$$\left(\sum a(n)q^n \sum b(\ell n)q^{\ell n}\right)\otimes\left(\tfrac{\bullet}{\ell}\right) = \left(\sum a(n)q^n \otimes \left(\tfrac{\bullet}{\ell}\right)\right)\left(\sum b(\ell n)q^{\ell n}\right).$$

This fact, combined with the definition of $G(z)$ and the generating function (2.7), shows that

$$\sum_{\left(\frac{n}{\ell}\right) = -\left(\frac{-2}{\ell}\right)} \ell^u \overline{NN}(r, \ell^u, n)q^n$$

is congruent to a cusp form with integer coefficients modulo ℓ^j. Since j is arbitrary and u is fixed, dividing by ℓ^u completes the proof of the theorem. □

Using a result of Serre on modular forms, Theorem 1.6 is an easy corollary of the previous theorem:

Proof of Theorem 1.6. Serre [35] has shown that for any integer M, any holomorphic modular form of integer weight on any congruence subgroup of $SL_2(\mathbb{Z})$ with integer coefficients has almost all of its Fourier coefficients divisible by M. From Theorem 4.1, the series

$$\sum_{\left(\frac{n}{\ell}\right)=-\left(\frac{-2}{\ell}\right)} \overline{NN}(r,\ell^u,n)q^n \tag{4.9}$$

is congruent to a cusp form modulo ℓ^j for any j. By construction, that cusp form has coefficients of the type $a(n)/\ell^u$, with $a(n) \in \mathbb{Z}$. Since (4.9) contains $1/2 - 1/\ell$ of the values of $\overline{NN}(r,\ell^u,n)$, the theorem follows from Serre's result. □

As mentioned in the introduction, Theorem 1.6 is also true for $\overline{pp}(n)$. We now complete the proof of Theorem 1.8 by considering the case $\ell = 2$.

Proof of Theorem 1.8. The function

$$\left(\frac{\eta^2(z)}{\eta(2z)}\right)^{2^j}$$

is congruent to 1 modulo 2^{j+1}, and so the function

$$\frac{\eta^2(2z)}{\eta^4(z)} \times \left(\frac{\eta^2(z)}{\eta(2z)}\right)^{2^j} = \frac{\eta^{2^{j+1}-4}(z)}{\eta^{2^j-2}(2z)} \tag{4.10}$$

is congruent modulo 2^{j+1} to

$$\sum_{n\geq 0} \overline{pp}(n)q^n.$$

It is easily verified using properties of eta functions that (4.10) is a holomorphic modular form, and hence Serre's result implies that almost all n have $\overline{pp}(n)$ divisible by 2^j. □

Proof of Theorem 1.7. Fix an odd prime ℓ and a power j. It follows from work of Serre [36] that for any cusp form $f(z) = \sum a(n)q^n$ on $S_k(\Gamma_0(N),\chi)$ with algebraic integer coefficients, and for any modulus M, a positive proportion of the primes p satisfy

$$f(z)|T(p) \equiv 0 \pmod{M}.$$

Here $T(p)$ is the integral weight Hecke operator which acts on such forms by

$$f(z)|T(p) = \sum_{n\geq 0} \left(a(pn) + \chi(p)p^{k-1}a(n/p)\right) q^n.$$

From work of Ono [30, Theorem 2.2] and the classical decomposition of $S_k(\Gamma_1(N))$, it follows that for any finite set of holomorphic integer weight cusp forms $f_i(z)$ on $\Gamma_1(N_i)$ with algebraic integer coefficients, a positive proportion of the primes p satisfy

$$f_i(z)|T(p) \equiv 0 \pmod{M}$$

for all i. For the case of $t = \ell^u$, let f_i $(0 \leq i \leq \ell^u - 1)$ be the form

$$\sum_{\left(\frac{n}{\ell}\right)=-\left(\frac{-2}{\ell}\right)} \overline{NN}(i,\ell^u,n)q^n.$$

Then for a positive proportion of the primes p, if $(p,n)=1$ and $\left(\frac{n}{\ell}\right)=-\left(\frac{-2}{\ell}\right)$, $\overline{NN}(i,\ell^u,pn)\equiv 0 \pmod{\ell^j}$ for all i. The congruences in arithmetic progressions follow immediately.

For the case $(t,\ell)=1$ we use Definition 4.1 and Corollary 4.3 of [29] to see that each $R(\zeta_t^s;q)$ is a weakly holomorphic modular form in $\Gamma_1(8t^2)$. In particular, we have that up to a constant multiple,

$$R(\zeta_t^s;q)=\frac{\eta^2(2z)}{\eta^4(z)K_{0,s}(z)}.$$

Now, by a result of Treneer [37, Theorem 3.1] (and the remarks at the end of [12]), we may conclude that there is a natural number m so that

$$\sum_{\ell\nmid n}\overline{NN}(i,t,\ell^m n)q^n$$

is congruent modulo ℓ^j to a cusp form. Again applying the work of Serre and Ono, there are infinitely many integer-weight Hecke operators that annihilate all of these cusp forms modulo ℓ^j. As in the case $t=\ell^j$, this gives Theorem 1.7 for $(t,\ell)=1$. $\square$

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