SECORD-ORDER CUSP FORMS AND MIXED MOCK MODULAR FORMS

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Dedicated to Mourad Ismail and Dennis Stanton

ABSTRACT. In this paper, we consider the space of second order cusp forms. We determine that this space is precisely the same as a certain subspace of mixed mock modular forms. Based upon Poincaré series of Diamantis and O'Sullivan [21] which span the space of second order cusp forms, we construct Poincaré series which span a natural (more general) subspace of mixed mock modular forms.

1. INTRODUCTION

In his last letter to Hardy (see pages 57–61 of [34]), Ramanujan described 17 q-hypergeometric series which he called *mock theta functions*. For example, denoting $q := e^{2\pi i\tau}$ for $\tau \in \mathbb{H}$, one such function is

$$f(\tau) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2}.$$

Ramanujan mysteriously said that these q-series "enter mathematics as beautifully as ordinary theta-functions" (Ramanujan referred to all modular forms as "theta functions"), but only gave a vague definition. While many of his claims were proven throughout the years, the exact role of his functions within the theory of automorphic forms remained a mystery until Zwegers [39] showed that each of the mock theta functions is the "holomorphic part" of a weight $\frac{1}{2}$ harmonic weak Maass form. A *harmonic weak Maass form* of weight κ is a certain non-holomorphic automorphic form which is annihilated by the weight κ hyperbolic Laplacian (see Section 2 for a definition). Following Zagier [35], we call the holomorphic part of a harmonic weak Maass form a *mock modular form*. The non-holomorphic part of a harmonic weak Maass form is related to a non-holomorphic Eichler integral g^* (see Section 2) arising from a weakly holomorphic modular form (i.e., a meromorphic modular form whose only possible poles occur at cusps) g of weight $2-\kappa$. One refers to g as the *shadow* of the mock modular form.

As evidence of their influence, mock modular forms and harmonic weak Maass forms naturally appear in partition theory (for example [2, 4, 7, 11, 13]), Zagier's duality [36] (for example [12]), and derivatives of L-functions (for example [16, 17]). Extending work of Conway and Norton [19] and Borcherds [3] on classical Monstrous Moonshine, Eguchi, Ooguri, and Tachikawa [25] have recently observed a connection between mock modular forms and Monstrous Moonshine of the largest Mathieu group M_{24}. To expound upon another application, an exact formula for the Fourier coefficients $\alpha(n)$ of $f(\tau)$ (as a sum involving Kloosterman sums and Bessel functions) was

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conjectured by Andrews and Dragonette [1, 23]. Although this formula resembled those proven by Rademacher and Zuckerman [32, 33, 37] for coefficients of weakly holomorphic modular forms, no further progress was made for 40 years. The Andrews–Dragonette conjecture was finally resolved [11] by realizing $\alpha(n)$ as the coefficients of a certain non-holomorphic Poincaré series. Such Poincaré series span the space $H_\ell(N)$ of harmonic weak Maass forms of weight ℓ for $\Gamma_0(N)$ and in special cases can be shown to be the classical Poincaré series $P_{m,\ell} \in M_\ell^!(N)$ (the space of weakly holomorphic modular forms of weight ℓ for $\Gamma_0(N)$) defined in (4.2).

The notion of mock modular forms has recently been generalized to mixed mock modular forms. We call a holomorphic function h a *mixed mock modular form* of weight $(2-k,\ell)$ if there exist $g_1,\dots,g_n \in M_k^!(N)$ and $f_1,\dots,f_n \in M_\ell^!(N)$ so that

$$\mathcal{M}_h := h + \sum_{i=1}^{n} g_i^* f_i \tag{1.1}$$

transforms like a weight $2-k+\ell$ modular form on $\Gamma_0(N)$. As before, g^* is the non-holomorphic Eichler integral of g defined in (2.2). We denote the space comprised of weight $(2-k,\ell)$ mixed mock modular forms by $\mathscr{M}_{2-k,\ell}(N)$ and the space of their completions (1.1) by $\mathbb{M}_{2-k,\ell}(N)$. We define $\mathscr{S}_{2-k,\ell}(N)$ to be the subspace of $\mathscr{M}_{2-k,\ell}(N)$ consisting of those mixed mock modular forms which "vanish" at each cusp of $\Gamma_0(N)$. More precisely, we say that $h \in \mathscr{M}_{2-k,\ell}(N)$ vanishes at a cusp ρ if the holomorphic part of the Fourier expansion of $\mathcal{M}_h$ at ρ decays exponentially. The subspace $\mathscr{M}^+_{2-k,\ell}(N)$ (resp. $\mathscr{S}^+_{2-k,\ell}(N)$) of $\mathscr{M}_{2-k,\ell}(N)$ (resp. $\mathscr{S}_{2-k,\ell}(N)$) is characterized by the restriction that, in (1.1), every $g_i \in S_k(N)$. Define $S_k^!(N)$ to be the subspace of $M_k^!(N)$ consisting of those elements for which the constant term of the Fourier expansion at each cusp is zero. Let $\mathscr{S}^{\mathrm{cusp}}_{2-k,\ell}(N) \subseteq \mathscr{S}_{2-k,\ell}(N)$ be the subspace precisely containing those mixed mock modular forms for which every $g_i \in S_k^!(N)$ and every $f_i \in S_\ell(N)$.

Similarly to mock modular forms, mixed mock modular forms have appeared in a variety of fields. Answering a question of Kac, the first author and Ono (Theorem 1.1 of [14]) have proven that certain characters arising in affine Lie superalgebras are mixed mock modular forms. Exploiting the "modularity" of the Kac–Wakimoto characters, the first author and Folsom [5] have proven an asymptotic expansion near the cusp 0, improving upon the main term shown by Kac and Wakimoto [28].

Zwegers [38] observed that mock theta functions appear as coefficients of meromorphic Jacobi forms. More generally, motivated by their appearance in the quantum theory of black holes, Dabholkar, Murthy, and Zagier [20] have recently shown that the coefficients of meromorphic Jacobi forms with poles of order at most 2 are indeed related to mixed mock modular forms (see also [6]).

Analogous to the case of mock modular forms, the coefficients of mixed mock modular forms encode important arithmetic information. However, as proven by the first author and Mahlburg [9], the shape of the Fourier expansions of mixed mock modular forms differ from those of mock modular forms. The first author and Manschot [10] then used the method developed in [9] to prove exact formulas for the Euler numbers of certain moduli spaces. In addition to terms which resemble those given for mock modular forms, there are extra terms which do not appear as coefficients of known Poincaré series.

In hope of a better understanding of these coefficients, one would like to construct Poincaré series for the completions (1.1) of mixed mock modular forms. In this note, we set out to construct Poincaré series of weight $(2-k,\ell)$ mock modular forms and their completions for some special cases of k and ℓ.

Our motivation comes from a link between mixed mock modular forms and so-called weight ℓ second-order cusp forms. Second-order modular forms, initially studied by Kleban and Zagier [29] while investigating crossing probabilities in percolation models, and by Chinta, Diamantis, and O'Sullivan [18] while studying Eisenstein series with modular symbols, are essentially holomorphic functions on $\mathbb{H}$ which, instead of satisfying modularity, "fail to be modular" by a usual cusp form (see (2.6)). A precise definition of second-order cusp forms is given in Definition 2.2 and the space of second-order cusp forms is denoted $S_\ell^{(2)}(N)$. Weight ℓ second-order cusp forms turn out to be weight $(0,\ell)$ mixed mock modular forms.

Theorem 1.1. *For $\ell \geq 4$ and $N \in \mathbb{N}$, one has*

$$\mathscr{S}_{0,\ell}^{cusp}(N) = S_\ell^{(2)}(N).$$

Having established the relationship between second-order cusp forms and mixed mock modular forms, whenever $\ell > k \geq 2$, we are able to mimic the construction of Poincaré series for second-order modular forms by Diamantis and O'Sullivan [21] in order to define Poincaré series for mixed mock modular forms of weight $(2-k,\ell)$. For notational simplicity, we only address the case $N=1$ in this note and, for brevity, omit the level in our notation (e.g., $M_k^!$ instead of $M_k^!(1)$). For $m \in \mathbb{Z}$ and $g \in S_k$ (the subspace of $M_k^!$ consisting of cusp forms) the shadow of a mock modular form with completion $M \in H_{2-k}^+$ (the subspace of H_{2-k} consisting of those harmonic weak Maass forms which map to cusp forms under the anti-holomorphic differential operator $\xi_{2-k} := 2iy^{2-k}\overline{\frac{\partial}{\partial\overline{\tau}}}$) we construct a mixed mock modular form Poincaré series $Q_{m,2-k,\ell}^M$ in (4.5) and a Poincaré series $P_{m,2-k,\ell}^M$ for its completion in (4.1).

Theorem 1.2. *Suppose that $\ell > k \geq 12$ are integers and $m \in \mathbb{Z}$. If $M \in H_{2-k}^+$, then*

$$P_{m,2-k,\ell}^M - M^- P_{m,\ell} \in \mathscr{S}_{2-k,\ell}^+$$

(so that $P_{m,2-k,\ell}^M \in \mathbb{M}_{2-k,\ell}$) and

$$P_{m,2-k,\ell}^M - MP_{m,\ell} \in M_{2-k+\ell}^!.$$

Here M^- is the non-holomorphic part (see Section 2 for the definition). Furthermore, the space $\mathscr{S}_{2-k,\ell}^+$ (resp. $\mathscr{M}_{2-k,\ell}^+$) is spanned by the holomorphic functions $P_{m,2-k,\ell}^M - M^- P_{m,\ell}$ and the classical Poincaré series $P_{r,2-k+\ell}$ with $r \in \mathbb{N}$ (resp. $r \in \mathbb{Z}$).

Remark. Although we only address Poincaré series of level 1, one can construct analogous such functions for $k \geq 2$ and arbitrary level N which vanish in all but one cusp. However, the generalization to higher level does not display any interesting new behavior, so we choose $N=1$ to avoid obfuscating technical details while still demonstrating the essense of the construction. For $k=2$, the arbitrary level N version of the Poincaré series $Q_{m,2-k,\ell}^M$ was defined by Diamantis and O'Sullivan [21] in the context of second-order modular forms.

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2. Background on Harmonic weak Maass forms and second-order cusp forms

In this section we recall the definitions of harmonic weak Maass forms and second-order cusp forms as well as their properties which are necessary for our purposes. Good background references for harmonic weak Maass forms and second-order cusp forms are [15] and [18], respectively.

As in the case of usual modular forms, for a function $h : \mathbb{H} \to \mathbb{C}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, one defines the weight $\ell \in \mathbb{Z}$ *slash-operator* $|_\ell$ by

$$h\Big|_\ell \gamma(\tau) := j(\gamma,\tau)^{-\ell} h\left(\frac{a\tau+b}{c\tau+d}\right), \tag{2.1}$$

where $j(\gamma,\tau) := c\tau+d$. As usual, we write $\tau = x+iy \in \mathbb{H}$ with $x, y \in \mathbb{R}$. For $k \in \mathbb{Z}$, the weight $2-k$ *hyperbolic Laplacian* is defined by

$$\Delta_{2-k} := -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) + i(2-k)y\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right).$$

Definition 2.1. *For $k \in 2\mathbb{N}$, a harmonic weak Maass form $M : \mathbb{H} \to \mathbb{C}$ of weight $2-k \in \mathbb{Z}$ for $\Gamma_0(N) \subseteq \mathrm{SL}_2(\mathbb{Z})$ is a real analytic function satisfying:*

1. *$M|_{2-k}\gamma(\tau) = M(\tau)$ for every $\gamma \in \Gamma_0(N)$.*
2. *$\Delta_{2-k}(M) = 0$.*
3. *M has at most linear exponential growth at each cusp of $\Gamma_0(N)$.*

The weight $2-k$ hyperbolic Laplacian is related to the anti-holomorphic differential operator ξ_{2-k} through

$$\Delta_{2-k} = -\xi_k \xi_{2-k}.$$

In particular, holomorphic functions are annihilated by Δ_{2-k}, since they are annihilated by ξ_{2-k}. We define $H_{2-k}(N)$ to be the space of harmonic weak Maass forms of weight $2-k$ and level N and $H^+_{2-k}(N) \subseteq H_{2-k}(N)$ to be the subspace of those $M \in H_{2-k}(N)$ for which $\xi_{2-k}(M) \in S_k(N)$. Each $M \in H_{2-k}(N)$ naturally splits into a holomorphic part and a non-holomorphic part. To describe this decomposition, we define the *non-holomorphic Eichler integral* of $g \in S_k(N)$ by

$$g^*(\tau) := \left(\frac{1}{2i}\right)^{k-1} \int_{-\overline{\tau}}^{i\infty} (z+\tau)^{k-2}\,\overline{g(-\overline{z})}dz. \tag{2.2}$$

The operator defined in (2.2) is clearly antilinear. More generally, suppose that

$$g(\tau) = \sum_{n\neq 0} a(n)q^n \in S_k^!(N).$$

In this case one defines

$$g^*(\tau) := -(4\pi)^{1-k} \sum_{n\neq 0} \overline{a(-n)}\, n^{1-k}\,\Gamma(k-1; -4\pi n y)\, q^n,$$

where for $z \in \mathbb{C}$, $\Gamma(k-1; z) := \int_z^\infty e^{-t}t^{k-2}dt$ is the *incomplete gamma function*. For each $M \in H_{2-k}(N)$ there exists a unique $g \in M_k^!(N)$ such that $M - g^*$ is holomorphic. Conversely, for every $g \in M_k^!(N)$, there exists $M \in H_{2-k}(N)$ with $\xi_{2-k}(M) = g$ (see [15]). One refers to $M^- := g^*$ as the *non-holomorphic part* of M and $M^+ := M - g^*$ as the *holomorphic part* of M. Since ξ_{2-k} annihilates holomorphic functions, the function g may be determined by

$$\xi_{2-k}(M) = \xi_{2-k}(g^*) = g.$$

Similarly, $M \in H_{2-k}(N)$ has a Fourier expansion at the other cusps of $\Gamma_0(N)$ from which one may define a holomorphic and non-holomorphic part at each cusp. The operator ξ_{2-k} is complemented by the differential operator D^{k-1}, where $D := \frac{1}{2\pi i}\frac{\partial}{\partial\tau}$. Bol's identity ([31], see also [24]) states that

$$D^{k-1}\left(H_{2-k}(N)\right) \subseteq M_k^!(N).$$

Furthermore, a direct calculation shows that

$$D^{k-1}\left(g^*\right) = 0. \tag{2.3}$$

Hence for every $M \in H_{2-k}(N)$,

$$D^{k-1}(M) = D^{k-1}\left(M^+\right) \in M_k^!(N). \tag{2.4}$$

The other automorphic objects which we relate in this note to harmonic weak Maass forms are n-th order (specifically, second-order) cusp forms, which we now define inductively. Denote the space of n-th order cusp forms of weight k and level N by $S_k^{(n)}(N)$.

Definition 2.2. *Define $S_k^{(0)}(N) := \{0\}$ and for $n \geq 1$ a holomorphic function $h : \mathbb{H} \to \mathbb{C}$ is an n-th order cusp form of weight k and level N if the following hold:*

(1) *The function h satisfies the transformation property*

$$h\Big|_k(\gamma - 1) \in S_k^{(n-1)}(N) \tag{2.5}$$

for every $\gamma \in \Gamma_0(N)$.

(2) *For each cusp ρ of $\Gamma_0(N)$ there exists $c \in \mathbb{R}^+$ such that*

$$f\Big|_k \sigma_\rho(\tau) \ll e^{-cy}$$

as $y \to \infty$ uniformly in x. Here $\sigma_\rho \in \mathrm{SL}_2(\mathbb{Z})$ is the scaling matrix which maps $i\infty$ to ρ.

(3) *For every parabolic element $\gamma \in \Gamma_0(N)$, one has*

$$h\Big|_k(\gamma - 1) = 0.$$

For $n = 1$, one has $S_k^{(1)}(N) = S_k(N)$ (for example, see Lemma 2 of [22]) so that, for every $\gamma \in \Gamma_0(N)$, the second-order cusp forms ($n = 2$) satisfy the transformation property

$$h\Big|_k(\gamma - 1) \in S_k(N). \tag{2.6}$$

In other words, for any $\gamma_1, \gamma_2 \in \Gamma_0(N)$,

$$h\Big|_k(\gamma_1 - 1)(\gamma_2 - 1) = 0,$$

from which second-order modular forms obtain their nomenclature.

To aid the reader, we now give a brief summary of the spaces of functions which are of interest in this paper. These are denoted as follows:

(1) $M_k^!$: Weakly holomorphic modular forms and the subspaces
- $S_k^!$: consisting of those $f \in M_k^!$ for which the constant term of the Fourier expansion at every cusp is zero,
- M_k: holomorphic modular forms,
- S_k: cusp forms.

(2) $H_{2-k}(N)$: The space of harmonic weak Maass forms of weight $2-k$ and level N together with the subspace
(3) $H_{2-k}^{+}(N)$: The space consisting of those $M\in H_{2-k}$ for which $\xi_{2-k}(M)\in S_k(N)$.
(4) $\mathscr{M}_{2-k,\ell}(N)$: Weight $(2-k,\ell)$ mixed mock modular forms of level N and the following subspaces, defined by the restrictions of g_i and f_i in (1.1) and growth conditions at the cusps of $\Gamma_0(N)$:

subspace	g_i	f_i	growth at cusps
$\mathscr{M}_{2-k,\ell}^{+}(N)$	$S_k(N)$	$M_\ell^!(N)$	at most linear exponential
$\mathscr{S}_{2-k,\ell}(N)$	$M_k^!(N)$	$M_\ell^!(N)$	vanishes
$\mathscr{S}_{2-k,\ell}^{+}(N)$	$S_k(N)$	$M_\ell^!(N)$	vanishes
$\mathscr{S}_{2-k,\ell}^{\mathrm{cusp}}(N)$	$S_k^!(N)$	$S_\ell(N)$	vanishes

(5) $\mathbb{M}_{2-k,\ell}(N)$: The space consisting of all completions $\mathcal{M}_h$ (from (1.1)) of every $h\in\mathscr{M}_{2-k,\ell}(N)$.
(6) $S_k^{(n)}$: The space of n-th order cusp forms.

3. Relating second-order cusp forms and mixed mock modular forms

This section is devoted to establishing the connection between second-order cusp forms and mixed mock modular forms. In order to prove Theorem 1.1, we require two lemmas. The following lemma may be known to the experts, but does not appear to be stated in this form in the literature. The proof we give here is based upon an operator defined in Section 1.1 of [8].

Lemma 3.1. *If $k\in 2\mathbb{N}$ and $g\in S_k^!(N)$, then there exists a constant $c\in\mathbb{C}$ (unique for $k>2$) such that $g^*+c\in H_{2-k}(N)$ if and only if $g\in D^{k-1}\left(M_{2-k}^!(N)\right)$.*

Remark. Suppose that $k>2$. For $g\in D^{k-1}\left(M_{2-k}^!(N)\right)$, the constant c satisfying $g^*+c\in H_{2-k}(N)$ may be given explicitly in terms of the *principal part of* g (the terms in the Fourier expansion which exhibit growth) at every cusp. Moreover, if $M\in M_{2-k}^!(N)$ has constant coefficient c_M and satisfies $g=D^{k-1}(M)$, then

$$c=\frac{(k-2)!}{(-4\pi)^{k-1}}c_M.$$

Proof. First suppose that there exists a constant c such that $g^*+c\in H_{2-k}(N)$. By (an extension, to include $k=2$ and arbitrary growth towards all cusps, of) Theorem 1.1 of [8], there exists an involution $\mathcal{F}$ on $H_{2-k}(N)$ such that for every $\mathcal{M}\in H_{2-k}(N)$, we have

$$\frac{(-4\pi)^{k-1}}{(k-2)!}D^{k-1}(\mathcal{M})=\xi_{2-k}(\mathcal{F}(\mathcal{M})). \tag{3.1}$$

Choosing $\mathcal{M}=\widetilde{M}:=\mathcal{F}(g^*+c)$, (3.1) (together with the fact that $\mathcal{F}$ is an involution) yields

$$g=\xi_{2-k}(g^*+c)=\frac{(-4\pi)^{k-1}}{(k-2)!}D^{k-1}\left(\widetilde{M}\right). \tag{3.2}$$

Moreover, after applying (3.1) with $\mathcal{M}=g^*+c$, (2.3) implies that

$$\xi_{2-k}\left(\widetilde{M}\right)=\frac{(-4\pi)^{k-1}}{(k-2)!}D^{k-1}(g^*+c)=0.$$

By Proposition 3.2 of [15], the kernel of ξ_{2-k} is $M_{2-k}^{!}(N)$, and hence one concludes that $\widetilde{M} \in M_{2-k}^{!}(N)$. Therefore $g \in D^{k-1}\left(M_{2-k}^{!}(N)\right)$ by (3.2).

Conversely, assume that $g \in D^{k-1}\left(M_{2-k}^{!}(N)\right)$ and choose $\widetilde{M} \in M_{2-k}^{!}(N)$ so that $g = D^{k-1}\left(\widetilde{M}\right)$. Then $M := \frac{(k-2)!}{(-4\pi)^{k-1}}\mathcal{F}\left(\widetilde{M}\right)$ satisfies $M^{-} = g^{*}$ by (3.1). Furthermore, (2.4) and (3.1) imply that

$$D^{k-1}\left(M^{+}\right) = D^{k-1}(M) = \xi_{2-k}\left(\widetilde{M}\right) = 0,$$

because ξ_{2-k} annihilates holomorphic functions. One concludes that M^{+} is a polynomial in τ of degree at most $k-2$ as well as a q-series, and hence a constant $c \in \mathbb{C}$. Thus $g^{*} + c \in H_{2-k}(N)$. This completes the proof of the lemma. □

Before stating our second lemma, recall that $D\left(M_0^{!}(N)\right) \subseteq S_2^{!}(N)$ by (2.4). Furthermore, Theorem 1 of [26] states that

$$\overline{S}_2^{!}(N) := S_2^{!}(N) \Big/ D\left(M_0^{!}(N)\right)$$

has a basis of Hecke eigenforms and $\dim\left(\overline{S}_2^{!}(N)\right) = 2d$, where $d := \dim\left(S_2(N)\right)$. As standard, we slightly abuse notation throughout by writing $g \in \overline{S}_2^{!}(N)$ to mean a representative $g \in S_2^{!}(N)$ of the coset.

Lemma 3.2. *Suppose that $N, \ell \in \mathbb{N}$ with $\ell \geq 4$, $\{g_1, \ldots, g_{2d}\}$ is a basis of $\overline{S}_2^{!}(N)$ and $\{f_1, \ldots, f_n\}$ is a basis of $S_\ell(N)$. Then for $c_{i,j} \in \mathbb{C}$ ($1 \leq i \leq n$, $1 \leq j \leq 2d$),*

$$\mathcal{M} := \sum_{i=1}^{n}\sum_{j=1}^{2d} c_{i,j} g_j^{*} f_i$$

is an element of $\mathbb{M}_{0,\ell}(N)$ if and only if $c_{i,j} = 0$ for every i, j.

Proof. If $c_{i,j} = 0$ for every i, j, then $0 = \mathcal{M} \in \mathbb{M}_{0,\ell}(N)$ trivially.

Conversely assume that $\mathcal{M} \in \mathbb{M}_{0,\ell}(N)$. Then for every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, the weight ℓ modularity of $\mathcal{M}$ and every f_i imply that

$$0 = \mathcal{M}\Big|_{\ell}(\gamma - 1)(\tau) = \sum_{i=1}^{n}\sum_{j=1}^{2d} c_{i,j}\left(g_j^{*}\left(\frac{a\tau + b}{c\tau + d}\right) - g_j^{*}(\tau)\right) f_i(\tau). \tag{3.3}$$

However, for each $g \in S_2(N)$, one can show that

$$g^{*}\left(\frac{a\tau + b}{c\tau + d}\right) = g^{*}(\tau) + c_{\gamma,g} \tag{3.4}$$

for some $c_{\gamma,g} \in \mathbb{C}$ (for example, see [30]). Therefore by (3.3), we have

$$\sum_{i=1}^{n}\left(\sum_{j=1}^{2d} c_{i,j} c_{\gamma,g_j}\right) f_i(\tau) = 0.$$

Since $f_1, \ldots, f_n$ are linearly independent, it follows that

$$\sum_{j=1}^{2d} c_{i,j} c_{\gamma, g_j} = 0.$$

One conclude that

$$\sum_{j=1}^{2d} c_{i,j} g_j^* \in H_0(N).$$

Therefore, Lemma 3.1 implies that

$$\sum_{j=1}^{2d} \overline{c_{i,j}} g_j \in D\left(M_0^!(N)\right).$$

Because $g_1, \ldots, g_{2d}$ form a basis for $\overline{S}_2^!(N)$, we deduce by linear independence that $c_{i,j} = 0$, completing the proof. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Suppose that $h \in \mathscr{S}_{0,\ell}^{\mathrm{cusp}}(N)$ and define its completion $\mathcal{M}_h$ as in (1.1), with $g_1, \ldots, g_n \in S_2^!(N)$ and $f_1, \ldots, f_n \in S_\ell(N)$. Combining (3.4) with the automorphicity of $\mathcal{M}_h$ and every f_i, one concludes that

$$h\Big|_\ell (\gamma - 1) = -\sum_{i=1}^{n} c_{\gamma, g_i} f_i \in S_\ell(N). \tag{3.5}$$

This is precisely the transformation law (2.6) for second-order modular forms. By considering every $M \in H_0(N)$ as a component of a vector-valued automorphic form of level 1, one can prove, analogously to (3.4), that for every $g \in S_2(N)$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ there exist $g_\gamma \in S_2(N)$ and $c_{\gamma,g} \in \mathbb{C}$ such that

$$g^*\left(\frac{a\tau + b}{c\tau + d}\right) = g_\gamma^*(\tau) + c_{\gamma,g}.$$

Similarly, considering $f \in S_\ell(N)$ as a component of a vector-valued modular form, one may prove that, for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, $f_\gamma := f|_\ell \gamma$ is a cusp form of weight ℓ. Since $c_{\gamma,g}$ is a constant, one can show that h has a Fourier expansion in every cusp (note that this argument does not work for $k > 2$). Furthermore, denoting $g_{i,\gamma} := (g_i)_\gamma$ and $f_{i,\gamma} := (f_i)_\gamma$, for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, one has

$$h\Big|_\ell \gamma = \mathcal{M}_h\Big|_\ell \gamma - \sum_{i=1}^{n} \left(g_{i,\gamma}^* + c_{\gamma, g_i}\right) f_{i,\gamma}. \tag{3.6}$$

Because $h \in \mathscr{S}_{0,\ell}^{\mathrm{cusp}}(N) \subseteq \mathscr{S}_{0,\ell}(N)$, the holomorphic part of $\mathcal{M}_h|_\ell \gamma$ decays exponentially by assumption. Furthermore, since each $f_{i,\gamma}$ vanishes towards every cusp and every $g_{i,\gamma}^* f_{i,\gamma}$ does not contribute to the holomorphic part, it follows that $h|_\ell \gamma$ exponentially decays. Hence condition (2) of Definition 2.2 is satisfied and h is a second-order cusp form.

In Theorem 2.2 of [21], Diamantis and O'Sullivan prove that

$$\dim\left(S_\ell^{(2)}(N)\right) = (2d+1)\dim\left(S_\ell(N)\right), \tag{3.7}$$

where we recall that $d = \dim\left(S_2\left(N\right)\right)$. It hence remains to prove that

$$\dim\left(\mathscr{S}_{0,\ell}^{\mathrm{cusp}}\left(N\right)\right) = \left(2d+1\right)\dim\left(S_\ell\left(N\right)\right). \tag{3.8}$$

We next prove that for every $g \in \overline{S}_2^!\left(N\right)$ and $f \in S_\ell\left(N\right)$ (possibly with $g = 0$),

$$S_{f,g} := \left\{h \in \mathscr{S}_{0,\ell}^{\mathrm{cusp}}\left(N\right) : h + g^* f \in \mathbb{M}_{0,\ell}\left(N\right)\right\}$$

has the same dimension as $S_\ell\left(N\right)$. By Lemma 3.2, we may then conclude (3.8) by letting g and f run through basis elements of $\overline{S}_2^!\left(N\right)$ and $S_\ell\left(N\right)$, respectively. Note that from (3.6) one may conclude that every $h \in S_{f,g}$ vanishes at each cusp. For every $h_1, h_2 \in S_{f,g}$, the (exponential) decay at each cusp together with the automorphicity of $h_1 - h_2$ implies that $h_1 - h_2 \in S_\ell\left(N\right)$. Conversely, for every $h \in S_{f,g}$,

$$\left\{h + \widetilde{f} : \widetilde{f} \in S_\ell\left(N\right)\right\} \subseteq S_{f,g}.$$

Thus, to determine the dimension of $S_{f,g}$, it is enough to show that $S_{f,g}$ is not empty. By the surjectivity of ξ_0 onto $S_2^!\left(N\right)$, proven by Bruinier and Funke [15], one may choose $M \in H_0\left(N\right)$ with $\xi_0\left(M\right) = g$ (so that the non-holomorphic part of M is $g^* = M^-$). Therefore $h_0 := M^+ f \in \mathscr{M}_{0,\ell}\left(N\right)$ because

$$h_0 + g^* f = M^+ f + M^- f = Mf \in \mathbb{M}_{0,\ell}\left(N\right).$$

In order to augment h_0 so that it vanishes at each cusp, one must subtract a weakly holomorphic modular form which cancels the growth at each cusp. Indeed (since $\ell > 2$), for each cusp ρ of $\Gamma_0\left(N\right)$ and $m \geq 0$, there is a distinguished element $f_{m,\rho} \in M_\ell^!\left(N\right)$ (for example, see Chapter 3 of [27]) which grows towards ρ like $q^{-\frac{m}{t_\rho}}$ and vanishes towards every other cusp, where t_ρ is the cusp width of ρ. Since h_0 has a Fourier expansion at each cusp, one may subtract an appropriate linear combination of these $f_{m,\rho}$ from h_0 to obtain an element $h \in \mathscr{S}_{0,\ell}^{\mathrm{cusp}}\left(N\right)$. ☐

4. Poincaré series of mixed weight $(2-k,\ell)$

In this section we prove Theorem 1.2. We first give the definition of $P_{m,2-k,\ell}^M$ and then show that the Poincaré series converges compactly on $\mathbb{H}$.

Set $h_m\left(\tau\right) := e^{2\pi i m\tau}$. For $M \in H_{2-k}^+$ with non-holomorphic part $M^- = g^*$ (for some $g \in S_k$) and $\ell, k \in \mathbb{Z}$, define the Poincaré series

$$P_{m,2-k,\ell}^M\left(\tau\right) := \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \left(M^- h_m\right)\Big|_{2-k+\ell}\gamma\left(\tau\right), \tag{4.1}$$

where $\Gamma_\infty := \{\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right) : n \in \mathbb{Z}\}$, which is the *stabilizer* in $\mathrm{SL}_2\left(\mathbb{Z}\right)$ of $M^- h_m$. For $\ell, m \in \mathbb{Z}$, we also define the classical (weakly) holomorphic Poincaré series (for example, see Chapter 3 of [27]),

$$P_{m,\ell}\left(\tau\right) := \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} h_m\Big|_\ell \gamma\left(\tau\right) \in M_\ell^!. \tag{4.2}$$

For $\ell > 2$, these series converge compactly on $\mathbb{H}$ and $\{P_{m,\ell} : m \in \mathbb{Z}\}$ spans the space $M_\ell^!$. We next prove compact convergence of $P_{m,2-k,\ell}^M$ whenever $\ell > k \geq 12$.

Lemma 4.1. *If $\ell > k \geq 12$ are even integers, then $P_{m,2-k,\ell}^M$ converges compactly on $\mathbb{H}$.*

Proof. We first prove the estimate

$$M^{-}(\tau)=g^{*}(\tau)\ll 1, \tag{4.3}$$

where the implied constant only depends on g. Note that since

$$g^{*}(\tau+1)=g^{*}(\tau),$$

we may assume that $0\leq x<1$. We rewrite (2.2) as

$$\frac{1}{2^{k-1}}\int_{y}^{\infty}(v+y)^{k-2}\,\overline{g(x+iv)}dv\ll\int_{y}^{\infty}v^{k-2}\left|g(x+iv)\right|dv. \tag{4.4}$$

Since $v^{\frac{k}{2}}\left|g(x+iv)\right|$ is bounded on $\mathbb{H}$ (by a constant depending on g) and $k>2$, (4.4) is bounded by

$$\int_{1}^{\infty}v^{k-2}\left|g(x+iv)\right|dv+\int_{y}^{1}v^{\frac{k}{2}-2}dv\ll 1.$$

Thus (4.3) follows. Similarly, $h_m(\tau)\ll 1$.

We are now ready to show compact convergence of $P^{M}_{m,2-k,\ell}$. For each $\gamma\in\mathrm{SL}_2(Z)$, one uses (4.3) and $\mathrm{Im}(\gamma\tau)=\frac{y}{|j(\gamma,\tau)|^2}$ to obtain

$$\left|\left(g^{*}h_m\right)\Big|_{2-k+\ell}\gamma(\tau)\right|\ll\left|j(\gamma,\tau)\right|^{k-2-\ell}\left|g^{*}(\gamma\tau)\right|\cdot\left|h_m(\gamma\tau)\right|\ll\left|j(\gamma,\tau)\right|^{k-2-\ell}.$$

Since $k-2-\ell<-2$, one now concludes compact convergence of $P^{M}_{m,2-k,\ell}$ from compact convergence of the usual weight $2+\ell-k$ Eisenstein series. This completes the proof of the lemma. □

Remark. In Lemma 4.1 we restrict to $k\geq 12$ because otherwise no $g\in S_k$ exists. When generalizing to higher level with $k\geq 2$, the argument follows precisely as above, except that for $k=2$ the bound in (4.3) is replaced by $g^{*}(\tau)\ll 1+\log(\mathrm{Im}(\tau))$.

Having proven convergence, we now move on to the proof of Theorem 1.2.

Proof of Theorem 1.2. The automorphicity of $P^{M}_{m,2-k,\ell}(\tau)$ follows by the usual argument for Poincaré series. To be more precise, for $\delta\in\mathrm{SL}_2(\mathbb{Z})$, we have

$$\begin{aligned}P^{M}_{m,2-k,\ell}\Big|_{2-k+\ell}\delta(\tau)&=j(\delta,\tau)^{k-2-\ell}\sum_{\gamma\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})}\left(M^{-}h_m\right)\Big|_{2-k+\ell}\gamma(\delta\tau)\\&=\sum_{\gamma\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})}\left(M^{-}h_m\right)\Big|_{2-k+\ell}(\gamma\delta)(\tau)=P^{M}_{m,2-k,\ell}(\tau),\end{aligned}$$

where the absolute convergence proven in Lemma 4.1 is used in the last equality. We next note that, for each $\gamma\in\mathrm{SL}_2(\mathbb{Z})$

$$L_M(\gamma,\tau):=M^{-}\Big|_{2-k}(\gamma-1)(\tau)$$

is a polynomial in $j(\gamma,\tau)$ of degree at most $k-2$ (for example, see [30]) and is hence holomorphic for $\tau\in\mathbb{H}$. Moreover, for each $\gamma_0\in\Gamma_\infty$ one has

$$L_M(\gamma_0\gamma,\tau)=L_M(\gamma,\tau).$$

Since $P^M_{m,2-k,\ell}$ and $P_{m,\ell}$ both converge compactly for $\ell > k \geq 12$, the Poincaré series

$$Q^M_{m,2-k,\ell}(\tau) := \sum_{\gamma\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})} L_M(\gamma,\tau)\, j(\gamma,\tau)^{-\ell} e^{2\pi i m\gamma(\tau)} = P^M_{m,2-k,\ell}(\tau) - M^-(\tau)\, P_{m,\ell}(\tau) \quad (4.5)$$

converges compactly for $\ell > k \geq 12$ and is holomorphic for $\tau \in \mathbb{H}$. Since $L_M(\gamma,\tau) = 0$ for every $\gamma \in \Gamma_\infty$, $Q^M_{m,2-k,\ell}$ vanishes towards the cusp $i\infty$. From (4.5) one now deduces that $P^M_{m,2-k,\ell} \in \mathbb{M}_{2-k,\ell}$ is the completion (1.1) of $Q^M_{m,2-k,\ell} \in \mathscr{S}^+_{2-k,\ell}$.

The function $M^+P_{m,\ell}$ is also clearly holomorphic on $\mathbb{H}$. However, since both $P^M_{m,2-k,\ell}$ and $MP_{m,\ell}$ are automorphic of weight $2-k+\ell$, it follows that

$$H^M_{m,2-k,\ell} := P^M_{m,2-k,\ell} - MP_{m,\ell} = Q^M_{m,2-k,\ell} - M^+P_{m,\ell} \in M^!_{2-k+\ell},$$

because it is both automorphic and weakly holomorphic.

In order to finish the proof of Theorem 1.2, it remains to show that $Q^M_{m,2-k,\ell}$ ($m \in \mathbb{Z}$) and $P_{r,2-k+\ell}$ with $r \in \mathbb{N}$ (resp. $r \in \mathbb{Z}$) span $\mathscr{S}^+_{2-k,\ell}$ (resp. $\mathscr{M}^+_{2-k,\ell}$). Assume that $h \in \mathscr{M}^+_{2-k,\ell}$ and choose $g_1,\ldots,g_n \in S_k$ and $f_1,\ldots,f_n \in M^!_\ell$ so that

$$\mathcal{M}_h := h + \sum_{i=1}^n g_i^* f_i \in \mathbb{M}_{2-k,\ell}.$$

Since the Poincaré series $P_{m,\ell}$ with $m \in \mathbb{Z}$ span the space $M^!_\ell$, there exist $c_{i,j} \in \mathbb{C}$, $r_i \in \mathbb{N}$, and $m_{i,j} \in \mathbb{Z}$ $(1 \leq j \leq r_i)$ such that

$$f_i = \sum_{j=1}^{r_i} c_{i,j} P_{m_{i,j},\ell}.$$

Thus

$$\mathcal{M}_h = h + \sum_{i=1}^n \sum_{j=1}^{r_i} c_{i,j} g_i^* P_{m_{i,j},\ell}. \quad (4.6)$$

Using the surjectivity of ξ_{2-k} onto S_k, proven by Bruinier and Funke [15], for each g_i there exists $M_i \in H^+_{2-k}(N)$ satisfying $\xi_{2-k}(M_i) = g_i$. By (4.5) and (4.6),

$$\begin{aligned}\mathcal{H} := h - \sum_{i=1}^n \sum_{j=1}^{r_i} c_{i,j} Q^{M_i}_{m_{i,j},2-k,\ell} &= \mathcal{M}_h - \sum_{i=1}^n \sum_{j=1}^{r_i} c_{i,j} g_i^* P_{m_{i,j},\ell} - \sum_{i=1}^n \sum_{j=1}^{r_i} c_{i,j}\left(P^{M_i}_{m_{i,j},2-k,\ell} - g_i^* P_{m_{i,j},\ell}\right)\\ &= \mathcal{M}_h - \sum_{i=1}^n \sum_{j=1}^{r_i} c_{i,j} P^{M_i}_{m_{i,j},2-k,\ell} \in \mathbb{M}_{2-k,\ell}.\end{aligned}$$

Since the function $\mathcal{H}$ is clearly holomorphic on $\mathbb{H}$ by definition, $\mathcal{H} \in M^!_{2-k+\ell}$, which is spanned by the Poincaré series $P_{r,2-k+\ell}$ with $r \in \mathbb{Z}$.

Furthermore, if $h \in \mathscr{S}^+_{2-k,\ell}$, then $\mathcal{H}$ also decays exponentially at $i\infty$ and hence $\mathcal{H} \in S_{2-k+\ell}$. Since the space $S_{2-k+\ell}$ is spanned by the Poincaré series $P_{r,2-k+\ell}$ with $r \in \mathbb{N}$, the proof is now complete. □

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