ON JACOBI POINCARÉ SERIES OF SMALL WEIGHT

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1. Introduction and Statement of results

In this paper we study Poincaré series of small weight for the generalized Jacobi group

$$\Gamma_g^J := \mathrm{SL}_2(\mathbb{Z}) \ltimes (\mathbb{Z}^g \times \mathbb{Z}^g) \qquad (g \in \mathbb{N}).$$

We show that they form a generating system for the vector space of Jacobi cusp forms $J_{k,m}^{\mathrm{cusp}}$. As two applications, we estimate Fourier coefficients of Siegel modular forms and construct lifting maps from $J_{k,m}^{\mathrm{cusp}}$ to a subspace of elliptic modular forms. It is likely that one can generalize our results to certain congruence subgroups as done in [Br2].

In the following, let k and n be positive integers, $r \in \mathbb{Z}^g$, and m a positive definite symmetric half-integral (i.e., $2m$ has integral entries and even diagonal elements) $g \times g$ matrix such that $D := \det \left(\begin{smallmatrix} 2n & r^t \\ r & 2m \end{smallmatrix}\right)$ is positive. For $s \in \mathbb{C}$ and $(\tau, z) \in \mathbb{H} \times \mathbb{C}^g$, we define the Jacobi-Poincaré series of exponential type

$$P_{k,m;(n,r),s}(\tau, z) := \sum_{\gamma \in \left(\Gamma_g^J\right)_\infty \backslash \Gamma_g^J} \left(\frac{v}{|c\tau + d|^2}\right)^s \cdot e^{n,r}|_{k,m}\gamma(\tau, z). \tag{1.1}$$

Here $e^{n,r}(\tau, z) := e(n\tau + r^t z) := e^{2\pi i (n\tau + r^t z)}$, $\left(\Gamma_g^J\right)_\infty := \left\{ \left(\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right), (0, \mu)\right) \middle|\ n \in \mathbb{Z}, \mu \in \mathbb{Z}^g \right\}$, $v :=\mathrm{Im}(\tau)$, and $|_{k,m}$ is the usual slash operator for the Jacobi group defined in Section 2. One can show (see [Br1]) that this series is absolutely and locally uniformly convergent on $\mathbb{H} \times \mathbb{C}^g$ if $\sigma := \mathrm{Re}(s) > \frac{1}{2}(g - k + 2)$. If $s = 0$ and $k > g + 2$, then we obtain the usual Jacobi-Poincaré series as defined in [GKZ] and [BK]. By construction we have

$$P_{k,m;(n,r),s}|_{k,m}\gamma(\tau, z) = P_{k,m;(n,r),s}(\tau, z) \qquad \left(\text{for all } \gamma \in \Gamma_g^J, (\tau, z) \in \mathbb{H} \times \mathbb{C}^g\right). \tag{1.2}$$

In Section 3 we show that the Poincaré series $P_{k,m;(n,r),s}$ have an analytic continuation to $\sigma > \frac{1}{2}\left(\frac{g}{2} - k + 2\right)$ following [BK] and [Br2]. Then we observe that the corresponding Petersson coefficients formula holds in this range too via analytic continuation, a point missed in [BK] and [Br2]. This enables us to extend their results to a larger domain with a shorter proof.

Theorem 1.1. *Assume the notation above.*

(1) *The series $P_{k,m;(n,r),s}$ has an analytic continuation to $\sigma > \frac{1}{2}\left(\frac{g}{2} - k + 2\right)$ given by its Fourier expansion. Moreover, if $s + k - \frac{g}{2} + 1$ is not a non-positive integer, and $\phi(\tau, z) = \sum_{n', r'} c(n', r') e\left(n'\tau + r'^t z\right) \in J_{k,m}^{cusp}$, then one has*

$$\left\langle \phi, P_{k,m;(n,r),s} \right\rangle = \lambda_{k,m,D,s} \cdot c(n, r), \tag{1.3}$$

Date: February 20, 2007.

where

$$\lambda_{k,m,D,s} := 2^{-k-s+1} \cdot \Gamma\left(k - \frac{g}{2} - 1 + s\right) \cdot \pi^{-k+\frac{g}{2}+1-s} \cdot (\det(2m))^{k-\frac{g}{2}-\frac{3}{2}+s} \cdot D^{-k+\frac{g}{2}+1-s}.$$

(2) *If* $k > \frac{g}{2}+2$, *then the series* $P_{k,m;(n,r)} := P_{k,m;(n,r),0}$ *generate the space of Jacobi cusp forms* $J_{k,m}^{cusp}$ *and we have for* $\phi(\tau,z) = \sum_{n',r'} c(n',r')e\left(n'\tau + r'^t z\right) \in J_{k,m}^{cusp}$

$$\left\langle \phi, P_{k,m;(n,r)} \right\rangle = \lambda_{k,m,D} \cdot c(n,r), \tag{1.4}$$

where $\lambda_{k,m,D} := \lambda_{k,m,D,0}$.

In Section 4, we estimate the Fourier coefficients $b_{n,r}\left(P_{k,m;(n,r)}\right)$ of the Poincaré series $P_{k,m;(n,r)}$ for $k \geq g+1$ and $g > 2$. The case $k \geq g+2$ is contained in [BK] and [Br2]. These estimates require using Theorem 1.1 and then estimating sums involving Kloosterman sums and Bessel functions. In the case $k = g+1$ refined estimates for Kloosterman sums are required. We spit the sumation into more ranges than in [BK] and [Br2], use different estimates for Kloosterman sums in each range, and then optimize the cutoff points. This enables us to obtain estimates of the same quality as in [BK] and [Br2]. We show

Theorem 1.2. *If* $k \geq g+1$ *and* $g > 2$, *then*

$$\left|b_{n,r}\left(P_{k,m;(n,r)}\right)\right| \ll_{k,\epsilon} 1 + D^{\frac{g}{2}+\epsilon} \cdot \det(2m)^{-\frac{g+1}{2}+\epsilon}. \tag{1.5}$$

Section 5 contains the main result of this paper. We enlarge the range for estimates to $k > \frac{g}{2}+2$. The ideas used here are fundamentally different from those contained in [BK] and [Br2]. We employ the theta decomposition to reduce the estimation of the the Poincaré series $P_{k,m;(n,r),s}(\tau,z)$ to the estimation of the Fourier coefficients of certain one dimensional Poincaré series. The difficulty here is that those Poincaré series involve multiplier systems which are not characters. We show

Theorem 1.3. *If* $k > \frac{g}{2}+2$, *then*

$$\left|b_{n,r}\left(P_{k,m;(n,r)}\right)\right| \ll_{k} 1 + \frac{D}{\det(2m)}. \tag{1.6}$$

As an application of Theorems 1.2 and 1.3, we obtain estimates for Fourier coefficients of Siegel modular forms for a much wider range than known before. Let us first describe what is known. Let F be a cusp form of weight k with respect to the Siegel modular group $\Gamma_g := \mathrm{Sp}_g(\mathbb{Z}) \subset \mathrm{GL}_{2g}(\mathbb{Z})$ with Fourier coefficients $a(T)$, where T is a positive definite symmetric half-integral $g \times g$ matrix. It is well-known that

$$a(T) \ll_F (\det T)^{\frac{k}{2}}.$$

Resnikoff and Saldaña (cf. [RS]) conjectured that for every $\epsilon > 0$

$$a(T) \ll_{\epsilon,F} (\det T)^{\frac{k}{2}-\frac{g+1}{4}+\epsilon}. \tag{1.7}$$

For $g = 1$ this conjecture is true (cf.[De] and [DS]), but for arbitrary g there are known counter examples (cf. [K2]). For $k \geq g+1$, the best known estimate is

$$a(T) \ll_{\epsilon,F} (\det T)^{\frac{k}{2}-c_g+\epsilon}, \tag{1.8}$$

where

$$c_g := \begin{cases} \frac{1}{2} & \text{if } g = 2 \quad \text{[BK]}, \\ \frac{1}{4} & \text{if } g = 3 \quad \text{[Bre]}, \\ \frac{1}{2g} + \left(1 - \frac{1}{g}\right)\alpha_g & \text{if } g > 3 \quad \text{[BK], [Br2]}. \end{cases}$$

Here

$$\alpha_g^{-1} := 4(g-1) + 4\left[\frac{g-1}{2}\right] + \frac{2}{g+2}. \tag{1.9}$$

One directly sees that $c_g \to 0$ for $g \to \infty$ (i.e., far from (1.7)).

To see how we can use Theorems 1.2 and 1.3 to obtain estimates for $a(T)$, we write $Z \in \mathbb{H}_g$ as $Z = \left(\begin{smallmatrix} \tau & z^t \\ z & \tau' \end{smallmatrix}\right)$, where $\tau \in \mathbb{H}$, $z \in \mathbb{C}^{g-1}$, and $\tau' \in \mathbb{H}_{g-1}$. Then the function $F(Z)$ has a so-called Fourier-Jacobi expansion

$$F(Z) = \sum_{m>0} \phi_m(\tau, z)\, e^{2\pi i \mathrm{tr}\,(m\tau')},$$

where m runs through all positive definite symmetric half-integral $(g-1)\times(g-1)$ matrices, and where the coefficients $\phi_m(\tau, z)$ are Jacobi cusp forms. Since the estimates in Theorems 1.2 and 1.3 are uniform in m, we can use them to obtain estimates for $a(T)$. This was first observed by Kohnen [K1] for $g = 2$, and generalized to general g by Kohnen and Böcherer for $k > g + 1$. The case $k = g + 1$ was considered in [Br2] using the "Hecke trick". Here we enlarge the range of weights to $k > \frac{g+3}{2}$. For $k = g$, we obtain estimates of the same quality as in [BK]. For $\frac{g+3}{2} < k < g$, we have a slightly weaker bound. Using Theorem 1.2 and 1.3, we show

Theorem 1.4. *We have for $k \geq g$:*

$$a(T) \ll (\det T)^{\frac{k}{2} - \frac{1}{2g} - \left(1 - \frac{1}{g}\right)\alpha_g + \epsilon}. \tag{1.10}$$

Theorem 1.5. *We have for $k > \frac{g+3}{2}$:*

$$a(T) \ll (\det T)^{\frac{k}{2} - \left(1 - \frac{1}{g}\right)\alpha_g + \epsilon}.$$

As another application of Theorem 1.1, we generalize results of [GKZ] and [Br3] and construct lifting maps from the vector space of Jacobi cusp forms to a certain subspace of elliptic modular forms. In their paper "Heegner points and derivatives of L-series II " [GKZ], Gross, Kohnen, and Zagier constructed certain lifting maps in the dimension 1 case of Jacobi forms, to obtain deep formulas relating height pairings of Heegner points to coefficients of Jacobi forms. In [Br3] lifting maps for higher genus were constructed for $k \geq \frac{g+3}{2}$. Using Theorem 1.1, we can extend this to $k \geq 3$, independent of g. Armed with the result of [Br3] and the one obtained here, following the approach of [EZ], one should then be able to develop a theory of newforms and hopefully use the Eichler-Shimura trace formula for elliptic cusp forms to compare the Hecke actions on these spaces in a nice compatible way. One then expects explicit formulas that express the central critical values of Hecke L-functions of elliptic Hecke eigenforms as squares of Fourier coefficients of generalized Jacobi forms.

Here we consider the case of general genus and all weights $k \geq 3$. In the following, let $n_0, k, g \in \mathbb{N}$ with $g \equiv 1 \pmod 8$, and m a positive definite symmetric half-integral $g \times g$ matrix $r_0 \in \mathbb{Z}^g$, $D_0 := \det\begin{pmatrix} 2n_0 & r_0^t \\ r_0 & 2m \end{pmatrix} > 0$ (under certain additional restictions given in Section 7).

For an integer l, let $S_k(l)^-$ be the subspace of elliptic cusp forms with respect to $\Gamma_0(l)$ that have eigenvalue -1 under the Fricke involution. We define the following lifting maps.

Definition 1.6. *For $\phi \in J^{cusp}_{k+\frac{g+1}{2},m}$ and $w \in \mathbb{H}$, we define*

$$\mathcal{S}_{D_0,r_0}(\phi)(w) := 2^{1-g} \sum_{n=1}^{\infty} \left(\sum_{d|n} \left(\frac{-D_0}{d} \right) d^{k-1} c_\phi \left(\frac{n^2}{d^2} n_0, \frac{n}{d} r_0 \right) \right) e^{2\pi i n w},$$

where $c_\phi(n,r)$ is the $(n,r)-$th Fourier coefficient of ϕ, and where $\left(\frac{\cdot}{d}\right)$ denotes the usual Kronecker symbol. For $f \in S_{2k}\left(\frac{1}{2}\det(2m)\right)^-$ and $(\tau, z) \in \mathbb{H} \times \mathbb{C}^g$, we define

$$\mathcal{S}^*_{D_0,r_0}(f)(\tau,z) := \left(\frac{i}{\det(2m)} \right)^{k-1} \sum_{D>0} r_{k,\frac{1}{2}\det(2m),D_0 D, r_0(2m)^* r^t, -D_0}(f) \cdot e^{2\pi i (n\tau + r^t z)},$$

where $D := \det\left(\begin{smallmatrix} 2n & r^t \\ r & 2m \end{smallmatrix}\right)$, $n \in \mathbb{N}$, $r \in \mathbb{Z}^g$, and where $r_{k,\frac{1}{2}\det(2m),D_0 D, r_0(2m)^* r^t, -D_0}(f)$ is a certain cycle integral, defined in Section 7.

Using Theorem 1.1, we show as in [GKZ] and [Br3]

Theorem 1.7. *Assuming the hypotheses in Section 7, the following are true:*

1. *If ϕ is an element of $J^{cusp}_{k+\frac{g+1}{2},m}$, then the function $\mathcal{S}_{D_0,r_0}(\phi)(w)$ is an element of $S_{2k}(\frac{1}{2}\det(2m))^-$.*
2. *If $f \in S_{2k}\left(\frac{1}{2}\det(2m)\right)^-$, then the function $\mathcal{S}^*_{D_0,r_0}(f)(\tau,z)$ is an element of $J^{cusp}_{k+\frac{g+1}{2},m}$.*
3. *The maps $\mathcal{S}_{D_0,r_0}$ and $\mathcal{S}^*_{D_0,r_0}$ are adjoint with respect to the Petersson scalar products.*

The paper is organized as follows. In Section 2 we recall basic facts about Jacobi cusp forms. In Section 3 we show the analytic continuation of the series $P_{k,m;(n,r),s}(\tau,z)$ to $\sigma > \frac{1}{2}\left(\frac{g}{2} - k + 2\right)$ and prove Theorem 1.1. In Section 4 we consider the case $k = g+1$ and show Theorem 1.2, refining arguments used in [Br2]. In Section 5 we prove Theorem 1.3. Using the theta decomposition, we reduce the estimation of the Poincaré series $P_{k,m;(n,r),s}(\tau,z)$ to the estimation of the Fourier coefficients of certain one dimensional Poincaré series. This approach differs from the one used in [K1], [BK], and [Br2]. In Section 6 we combine the results of Sections 3-5 to obtain Theorems 1.4 and 1.5. Section 7 is devoted to the construction of the lifting maps and the proof of Theorem 1.7.

Acknowledgements

The authors thank W. Kohnen for a very fruitful conversion. Moreover they thank B. Kane and K. Ono for helpful comments on an earlier version of this paper. They also acknowledge the referee for carefully reading the paper and making useful suggestions which improved the exposition of the paper.

2. Basic facts about Jacobi cusp forms

Here we recall some basic facts about Jacobi cusp forms. For details we refer the reader to [EZ] and [Zi]. The Jacobi group Γ_g^J acts on $\mathbb{H} \times \mathbb{C}^g$ in the usual way by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu) \right) \circ (\tau, z) := \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d} \right).$$

Let $k \in \frac{1}{2}\mathbb{Z}$ be a half-integer, m be a positive definite symmetric half-integral $g \times g$ matrix, $\gamma = \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), (\lambda, \mu) \right) \in \Gamma_g^J$, and $\phi : \mathbb{H} \times \mathbb{C}^g \to \mathbb{C}$. Then we define the following action

$$\phi|_{k,m}\gamma(\tau, z) := (c\tau + d)^{-k} \cdot e(-c(c\tau+d)^{-1}m[z + \lambda\tau + \mu] + m[\lambda]\tau + 2\lambda^t m z)) \cdot \phi(\gamma \circ (\tau, z)),$$

where $e(w) := e^{2\pi i w} \quad (\forall\, w \in \mathbb{C})$, and where $A[B] := B^t A B$ for matrices A and B of compatible sizes. Moreover we write $w^{\frac{1}{2}} := \sqrt{r} \cdot e^{\frac{i\phi}{2}}$ if $w = r \cdot e^{i\phi}$ with $-\pi < \phi \leq \pi$.

A holomorphic function $\phi : \mathbb{H} \times \mathbb{C}^g \to \mathbb{C}$ is called a Jacobi cusp form of weight k and index m with respect to Γ_g^J, if for all $\gamma \in \Gamma_g^J$ we have $\phi|_{k,m}\gamma(\tau, z) = \phi(\tau, z)$, and ϕ has a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{D > 0} c(n, r) e\left(n\tau + r^t z\right),$$

where $D := \det \left(\begin{smallmatrix} 2n & r^t \\ r & 2m \end{smallmatrix}\right)$ with $n \in \mathbb{N}$ and $r \in \mathbb{Z}^g$. Let us denote by $J_{k,m}^{\text{cusp}}$ the vector space of these Jacobi cusp forms. It is a finite dimensional Hilbert space with the Petersson scalar product

$$\langle \phi, \psi \rangle := \int_{\Gamma_g^J \backslash \mathbb{H} \times \mathbb{C}^g} \phi(\tau, z) \cdot \overline{\psi(\tau, z)} \cdot v^k \cdot \exp\left(-4\pi m[y] \cdot v^{-1}\right) dV_g^J,$$

where $dV_g^J = v^{-g-2}\, du\, dv\, dx\, dy$, $\tau = u + iv$, and $z = x + iy$.

3. Analytic continuation of $P_{k,m;(n,r),s}$ and Petersson coefficient formula

In this section we show the analytic continuation of the Poincaré series $P_{k,m;(n,r),s}$ defined in (1.1) and prove Theorem 1.1. The proof is basically the same as in [Br2] except for the simple observation in Lemma 3.3. For the convenience of the reader, we outline the argument here and refer to [Br2] for more detail.

Lemma 3.1. *(*[Br2]*, Lemma 3.1 and Theorem 3.4) The Poincaré series has an analytic continuation to* $\sigma > \frac{1}{2}\left(\frac{g}{2} - k + 2\right)$ *given by the Fourier expansion*

$$P_{k,m;(n,r),s}(\tau, z) = \sum_{\substack{n' \in \mathbb{Z} \\ r' \in \mathbb{Z}^g}} g^{\pm}_{k,m;(n,r);s,v}(n', r') e(n'\tau + r'^t z),$$

where

$$g^{\pm}_{k,m;(n,r);s,v}(n', r') := g_{k,m;(n,r);s,v}(n', r') + (-1)^k g_{k,m;(n,r);s,v}(n', -r').$$

Here

$$g_{k,m;(n,r);s,v}(n', r') := v^s \cdot \delta_m(n, r, n', r') + \sum_{c \geq 1} H_{m,c}(n, r, n', r') \cdot \Phi_{k,m,c,v}(n', r', s) \cdot c^{-k-2s},$$

where

$$\delta_m(n, r, n', r') := \begin{cases} 1 & \textit{if } D' = D,\ r' \equiv r \pmod{2m\mathbb{Z}^g}, \\ 0 & \textit{otherwise} \end{cases}$$

with $D' := \det \begin{pmatrix} 2n' & r'^t \\ r' & 2m \end{pmatrix}$*. Finally*

$$H_{m,c}(n, r, n', r') := \sum_{\substack{x \pmod{c} \\ y \pmod{c}^*}} e_c((m[x] + r^t x + n)\bar{y} + n'y + r'^t x).$$

Here x and y run over a complete set of representatives for $\mathbb{Z}^g/c\mathbb{Z}^g$ and $(\mathbb{Z}/c\mathbb{Z})^*$, respectively, $\bar{y}$ denotes an inverse of $y \pmod c$, and $e_c(x) := e\left(\frac{x}{c}\right)$. Moreover

$$\Phi_{k,m,c,v}(n',r',s) := (\det(2m))^{-\frac{1}{2}} \cdot i^{-\frac{g}{2}} \cdot v^{\frac{g}{2}-k-s+1} \cdot e_{2c}(r^t m^{-1} r') \times \int_{-\infty}^{\infty} (u+i)^{\frac{g}{2}-k-s} \cdot (u-i)^{-s} \cdot e\left(-(2\det(2m))^{-1}\left(D'v(u+i) + \frac{D}{vc^2(u+i)}\right)\right) du.$$

We next show that in the case $s = 0$ the Poincaré series are Jacobi cusp form.

Lemma 3.2. *For $k > \frac{g}{2}+2$ the function $P_{k,m;(n,r)}(\tau,z) := P_{k,m;(n,r),0}(\tau,z)$ is an element of $J_{k,m}^{cusp}$. It has the Fourier expansion*

$$P_{k,m;(n,r)}(\tau,z) = \sum_{D'>0} g^{\pm}_{k,m;(n,r)}(n',r')\, e(n'\tau + r'^t z),$$

where

$$g^{\pm}_{k,m;(n,r)}(n',r') := g_{k,m;(n,r)}(n',r') + (-1)^k g_{k,m;(n,r)}(n',-r'),$$

with

$$g_{k,m;(n,r)}(n',r') := \delta_m(n,r,n',r') + 2\pi i^k \cdot (\det(2m))^{-\frac{1}{2}} \cdot (D'/D)^{\frac{k}{2}-\frac{g}{4}-\frac{1}{2}} \times \sum_{c\geq 1} e_{2c}(r^t m^{-1} r') \cdot H_{m,c}(n,r,n',r') \cdot J_{k-\frac{g}{2}-1}\left(\frac{2\pi\sqrt{D'D}}{\det(2m)\cdot c}\right) \cdot c^{-\frac{g}{2}-1}.$$

Proof. The fact that $P_{k,m;(n,r)}(\tau,z)$ satisfies the correct transformation law under Γ_g^J follows directly from (1.2). Plugging $s = 0$ into Lemma 3.1 gives that it has the correct Fourier expansion. One can show exactly as in [Br2] that the Fourier coefficients of $P_{k,m;(n,r)}(\tau,z)$ are independent of $v = \operatorname{Im}(\tau)$ and are 0 unless $D' > 0$. Now the proof follows as in [BK] page 504. □

We next show that the Petersson coefficient formula holds.

Lemma 3.3. *Formulas (1.3) and (1.4) are true.*

Proof. From the proof of Theorem 3.4 in [Br2] it follows that the Fourier coefficients of $P_{k,m;(n,r),s}$ have at most polynomial growth and the coefficients of $P_{k,m;(n,r)}$ have exponential decay. Together with (1.2) and Lemma 3.2, this implies that the left hand sides of (1.3) and (1.4) are well defined and absolutely convergent. For $\sigma > \frac{1}{2}(g-k+2)$ formula (1.3) follows by the usual unfolding argument as in [Br2]. The series $P_{k,m;(n,r),s}$ has an analytic continuation to $\sigma > \frac{1}{2}\left(\frac{g}{2}-k+2\right)$. On the other hand, the right hand side of (1.3) clearly has a meromorphic continuation to the whole complex plane, with at most simple poles at s satisfying $s+k-\frac{g}{2}+1 \leq 0$ is an integer. Now the lemma is clear. □

Lemma 3.3 implies that $P_{k,m;(n,r)}$ form a generating system for $J_{k,m}^{\mathrm{cusp}}$ whenever $k > \frac{g}{2}+2$. Moreover we obtain an estimate for the Fourier coefficients of a Jacobi cusp form.

Lemma 3.4. *Suppose that $k > \frac{g}{2}+2$, and $\phi \in J_{k,m}^{cusp}$ with Fourier coefficients $c(n,r)$. Then we have*

$$|c(n,r)| \ll_k \left|b_{n,r}\left(P_{k,m;(n,r)}\right)\right|^{\frac{1}{2}} \frac{D^{\frac{k}{2}-\frac{g}{4}-\frac{1}{2}}}{(\det 2m)^{\frac{k}{2}-\frac{1}{4}(g+3)}} \cdot \parallel \phi \parallel .$$

Proof. Using (1.4) and the Cauchy-Schwarz inequality we find

$$\begin{aligned}|c(n,r)|^2 &= \lambda_{k,m,D}^{-2} \cdot \left|\left\langle \phi, P_{k,m;(n,r)}\right\rangle\right|^2 \\ &\leq \lambda_{k,m,D}^{-2} \cdot \parallel \phi \parallel^2 \cdot \left\langle P_{k,m;(n,r)}, P_{k,m;(n,r)}\right\rangle = \lambda_{k,m,D}^{-1} \cdot b_{n,r}(P_{k,m;(n,r)}) \cdot \parallel \phi \parallel^2\end{aligned}$$

which immediately gives the lemma. □

4. The case $k = g + 1$ and proof of Theorem 1.2

In this section we estimate the Fourier coefficients of the Poincaré series $P_{k,m;(n,r)}$ for the special case $k = g + 1$ with $g > 2$. By Lemma 3.2 we have to estimate

$$S_{m,n,r} := \sum_{c \geq 1} c^{-\frac{g}{2}-1} \left|H_{m,c}(n,r,n,\pm r)\right| \left| J_{\frac{g}{2}}\left(\frac{2\pi D}{\det(2m)c}\right)\right|. \tag{4.1}$$

Estimating (4.1) requires more care than was necessary for the estimates in [Br2]. We use more refined estimates for Kloosterman sums, and split the summation into 3 ranges. In each range, we use different estimates for Kloosterman sums and Bessel functions and then optimize the cutoff points. This enables us to obtain estimates of the same quality as in [BK] and [Br2].

We need the following two estimates on Kloosterman sums which can be found in [BK] and [Br1] (implicitely in the proof of the analytic continuation of the Poincaré series), respectively:

$$|H_{m,c}(n,r,n,\pm r)| \ll c^{g+\epsilon} \cdot (D,c), \tag{4.2}$$

$$|H_{m,c}(n,r,n,\pm r)| \leq (2\det(2m)D))^{\frac{g}{2}} \cdot c^{\frac{g}{2}+1+\epsilon}. \tag{4.3}$$

Moreover, for $l, x > 0$, we have (see [Ba] pages 4 and 74)

$$J_l(x) \ll_l min\left\{x^{-\frac{1}{2}}, x^l\right\}. \tag{4.4}$$

Now write

$$S_{m,n,r} = \sum_{d|D} \sum_{\substack{c \geq 1 \\ (c,D/d)=1}} (cd)^{-\frac{g}{2}-1} \left|H_{m,dc}(n,r,n,\pm r)\right| \left|J_{\frac{g}{2}}\left(\frac{A}{c}\right)\right|,$$

with $A = \frac{2\pi D}{d\det(2m)}$. We first estimate

$$B_{m,n,r,d} := \sum_{\substack{c \geq 1 \\ (c,D/d)=1}} (cd)^{-\frac{g}{2}-1} \left|H_{m,dc}(n,r,n,\pm r)\right| \left|J_{\frac{g}{2}}\left(\frac{A}{c}\right)\right|. \tag{4.5}$$

For this we split the sum into three parts. A part with $c \leq A$, a part with $A \leq c \leq B$ and a part with $c \geq B$, where $B := (D \cdot \det(2m))^4$. To estimate the sum with $c \leq A$ we use the first estimate in (4.4) and (4.2). Thus we can estimate the contribution to $B_{m,n,r,d}$ against

$$A^{-\frac{1}{2}} \cdot d^{\frac{g}{2}+\epsilon} \sum_{c \leq A} c^{\frac{g}{2}-\frac{1}{2}+\epsilon} = d^{\frac{g}{2}+\epsilon} \cdot A^{\frac{g}{2}+\epsilon}. \tag{4.6}$$

Next we estimate the sum in the range $A \le c \le B$. For this we use the second estimate in (4.4) and (4.2). This shows that the contribution to $B_{m,n,r,d}$ can be estimated against

$$d^{\frac{g}{2}+\epsilon} \cdot A^{\frac{g}{2}} \sum_{A \le c \le B} c^{-1+\epsilon} \ll d^{\frac{g}{2}+\epsilon} \cdot A^{\frac{g}{2}} \cdot B^{\epsilon}. \tag{4.7}$$

To estimate the sum with $B \le c$ we use the second estimate in (4.4) and (4.3). This shows that the contribution to $B_{m,n,r,d}$ can be estimated against

$$(\det(2m)D)^{\frac{g}{2}} \cdot A^{\frac{g}{2}} \cdot d^{\epsilon} \sum_{c \ge B} c^{-\frac{g}{2}+\epsilon} \ll d^{\epsilon} \cdot (\det(2m)D)^{\frac{g}{2}} \cdot A^{\frac{g}{2}} \cdot B^{-\frac{g}{2}+1+\epsilon}. \tag{4.8}$$

Recall our choice $B = (D \cdot \det(2m))^4$. Combining (4.6), (4.7), and (4.8) gives

$$B_{m,n,r,d} \ll d^{\frac{g}{2}+\epsilon} \cdot A^{\frac{g}{2}+\epsilon} (\det(2m)D)^{\epsilon} \ll D^{\frac{g}{2}+\epsilon} \cdot \det(2m)^{-\frac{g}{2}+\epsilon}.$$

Thus

$$S_{m,n,r} \ll \det(2m)^{-\frac{g}{2}+\epsilon} \cdot D^{\frac{g}{2}+\epsilon} \sum_{d|D} 1 \ll \det(2m)^{-\frac{g}{2}+\epsilon} \cdot D^{\frac{g}{2}+\epsilon}.$$

This immediately gives (1.5) in Theorem 1.2.

5. Theta Decomposition of Poincare Series

In this section we study the Poincaré series defined in (1.1) by using its theta decomposition. Let $T = \left(\begin{smallmatrix} n & r^t/2 \\ r/2 & m \end{smallmatrix}\right)$ be a positive definite half-integral $(g+1) \times (g+1)$ matrix with $D = \det(2T) = t\left(n - \frac{1}{4}m^{-1}[r]\right) > 0$ and $t := 2\det 2m$. For $l \in \mathbb{Z}^g/2m\mathbb{Z}^g$, we define the theta series

$$\Theta_l(\tau, z) := \sum_{\lambda \in \mathbb{Z}^g} e\left(m\left[\lambda + \frac{1}{2}m^{-1}l\right]\tau + 2\left(\lambda + \frac{1}{2}m^{-1}l\right)^t mz \right). \tag{5.1}$$

This theta series is known to be a Jacobi form of weight $\frac{g}{2}$ for the principal congruence Jacobi group $\Gamma(t) \ltimes (\mathbb{Z}^g \times \mathbb{Z}^g)$. In fact, the vector-valued theta function

$$\Theta(\tau, z) := (\Theta_l(\tau, z))_{l \in \mathbb{Z}^g/2m\mathbb{Z}^g} \tag{5.2}$$

has the following transformation law for every $M \in \mathrm{SL}_2(\mathbb{Z})$

$$\Theta|_{\frac{g}{2},m} M(\tau, z) := \left(\Theta_l|_{\frac{g}{2},m} M(\tau, z)\right)_{l \in \mathbb{Z}^g/2m\mathbb{Z}^g} = U(M)\Theta(\tau, z) \tag{5.3}$$

for some unitary matrix $U(M)$. Let $\chi^i_j(M)$ be the (i,j)-th entry of $U(M)$.

Lemma 5.1. *Assuming the notation from above we have:*

(1) $|\chi^i_j(M)| \le 1$.
(2) $U(\gamma M) = U(M\gamma) = U(M)$ *for* $\gamma \in \Gamma(t)$.
(3) $\chi^r_l(\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) M) = e(-\frac{Db}{t})\, \chi^r_l(M)$.
(4) $\chi^i_j(-M) = i^{-g}\, \chi^i_{-j}(M)$.

Proof. (1) is clear from the fact that $U(M)$ is unitary.
(2) follows from the fact that Θ_l is a modular form for $\Gamma(t)$.
(3) follows from the fact that $\Theta|_{g/2}\left(\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}\right) = A\Theta$, where A is the diagonal matrix with the (n,l)th entry being $e\left(\frac{D_l b}{t}\right)$, with $D_l := t\left(n - \frac{1}{4}m^{-1}[l]\right)$.
(4) follows from $\Theta_l|_{g/2}(-I) = i^{-g}\Theta_{-l}$, where I is the identity matrix. □

By Lemma 5.1, it is easy to see that the Poincaré series

$$P_{s,l}(\tau) := v^s \sum_{M\in\Gamma_\infty\backslash\Gamma} \chi_l^r(M)\cdot(c\tau+d)^{-k+\frac{g}{2}}\cdot|c\tau+d|^{-2s}\cdot e\left(\frac{D\,(a\tau+b)}{t\,(c\tau+d)}\right) \tag{5.4}$$

is well-defined and is a (non-holomorphic) modular form of weight $k-\frac{g}{2}$ for $\Gamma(t)$. We remark that it is not a modular form for $\mathrm{SL}_2(\mathbb{Z})$ since χ_l^r is not a character. The following proposition describes how this Poincaré series is related to the Jacobi Poincaré series $P_{k,m;(n,r),s}(\tau,z)$.

Proposition 5.2. *One has*

$$P_{k,m;(n,r),s}(\tau,z) = \sum_{l\in\mathbb{Z}^g/2m\mathbb{Z}^g} \Theta_l(\tau,z)P_{s,l}(\tau).$$

In particular, $b_{n,r}(P_{k,m;(n,r),s}) = b_{\frac{D}{t}}(P_{s,r})$, *with* $b_{\frac{D}{t}}(P_{s,r})$ *the* $\frac{D}{t}$*-th Fourier coefficient of* $P_{s,r}$.

Proof. We choose the elements $\left(\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right),(a\lambda,b\lambda)\right)$ as a set of representatives of $(\Gamma_g^J)_\infty\backslash\Gamma_g^J$, where $c,d\in\mathbb{Z}$ with $(c,d)=1$, $\lambda\in\mathbb{Z}^g$, and where, for each pair (c,d), we have chosen $a,b\in\mathbb{Z}$ such that $ad-bc=1$. It is not hard to see from the definition that

$$P_{k,m;(nr),s}(\tau,z) = \frac{v^s}{t}\sum_{M\in\Gamma(t)_\infty\backslash\Gamma}(c\tau+d)^{-(k-\frac{g}{2})}\cdot|c\tau+d|^{-2s}\cdot e\left(\frac{n(a\tau+b)}{c\tau+d}+r^tz\frac{1}{c\tau+d}\right)$$
$$\sum_{\lambda\in\mathbb{Z}^g} e\left(\left(m[\lambda]+r^t\lambda\right)\tau+2\lambda^tmz\right)|_{\frac{g}{2},m}M.$$

Rewriting

$$\sum_{\lambda\in\mathbb{Z}^g} e\left(\left(m[\lambda]+r^t\lambda\right)\tau+2\lambda^tmz\right) = e\left(-\frac{1}{4}m^{-1}\left[r^t\right]\tau-r^tz\right)\Theta_r(\tau,z)$$

one sees from (5.3) that $P_{k,m;(r,n),s}(\tau,z)$ equals

$$\begin{aligned}&\frac{v^s}{t}\sum_{M\in\Gamma(t)_\infty\backslash\Gamma}(c\tau+d)^{-(k-\frac{g}{2})}\cdot|c\tau+d|^{-2s}\cdot e\left(\frac{D\,(a\tau+b)}{t\,(c\tau+d)}\right)\cdot\Theta_r|_{\frac{g}{2},m}M(\tau,z)\\ =&\sum_{l\in\mathbb{Z}^g/2m\mathbb{Z}^g}\Theta_l(\tau,z)\cdot P_{s,l}(\tau),\end{aligned}$$

as claimed. Since $P_{s,l}$ is independent of z, it is easy to check the identity between the Fourier coefficients in the proposition. □

Proposition 5.2 leads to the study of the one variable Poincaré series $P_{s,l}(\tau)$. As usual, we break the sum in $P_{s,l}$ into three parts: $c=0$, $c>0$, and $c<0$. For $c\neq 0$ we use the identity

$$\frac{a\tau+b}{c\tau+d} = \frac{a}{c}-\frac{1}{c^2\left(\tau+\frac{d}{c}\right)}$$

and write d as $d+\lambda tc$ with $\lambda \in \mathbb{Z}$ and d running modulo ct. Here we need $P_{s,l}(\tau+t) = P_{s,l}(\tau)$, which follows from Lemma 5.1. A simple calculation gives

$$
\begin{aligned}
(5.5)\quad P_{s,l}(\tau) = v^s \left(1+(-1)^k\right) e\left(\frac{D\tau}{t}\right) \\
+ v^s \sum_{c>0} c^{\frac{g}{2}-k-2s} \sum_{\substack{d \pmod{tc} \\ (d,c)=1}} \left(\chi_l^r(c,d) + (-1)^k \chi_{-l}^r(c,d)\right) F_s\left(\tau + \frac{d}{c}\right).
\end{aligned}
$$

Here for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$

$$\chi_l^r(c,d) := \chi_l^r(M) e\left(\frac{Da}{tc}\right),$$

is well-defined (independent of the choice of M) by Lemma 5.1, and

$$F_s(\tau) := \sum_{\lambda \in \mathbb{Z}} (\tau+\lambda t)^{-(k-\frac{g}{2})} \cdot |\tau + \lambda t|^{-2s} \cdot e\left(-\frac{D}{tc^2(\tau+\lambda t)}\right). \tag{5.6}$$

Lemma 5.3. *The function $F_s(\tau)$ has Fourier expansion*

$$F_s(\tau) = \sum_{\mu \in \mathbb{Z}} \Phi_{k,c,v}(\mu, s) \cdot e\left(\frac{\mu\tau}{t}\right),$$

with

$$
\begin{aligned}
(5.7)\quad \Phi_{k,c,v}(\mu, s) = v^{-k+\frac{g}{2}-2s+1} \cdot \frac{1}{t} \cdot e^{\frac{2\pi\mu v}{t}} \\
\int_{\mathbb{R}} (u+i)^{-(k-\frac{g}{2}+s)} \cdot (u-i)^{-s} \cdot e\left(-\frac{1}{t}\left(\mu v u + \frac{D}{c^2 v(u+i)}\right)\right) du.
\end{aligned}
$$

Moreover,

(1) *If $\sigma > \frac{1}{2}\left(1+\frac{g}{2}-k\right)$, then the coefficients $\Phi_{k,c,v}(\mu, s)$ are holomorphic functions in s. If K is any compact set in the right half plane $\sigma > \frac{1}{2}\left(1+\frac{g}{2}-k\right)$, then we have for $s \in K$*

$$\Phi_{k,c,v}(\mu, s) \ll_{K,v,D,t} e^{\frac{2\pi\mu v}{t}(1-sign(\mu)v_1)},$$

where v_1 is a positive constant.

(2) *The function $F_s(\tau)$ has an analytic continuation to $\sigma > \frac{1}{2}\left(1+\frac{g}{2}-k\right)$.*

(3) *If $k > \frac{g}{2}+1$, then $F_0(\tau)$ is a holomorphic function of τ with Fourier expansion*

$$F_0(\tau) = 2\pi i^{\frac{g}{2}-k+2} \cdot t^{-1} \sum_{\mu>0} \left(\frac{D}{\mu}\right)^{\frac{1}{4}(g+2-2k)} \cdot c^{k-\frac{g}{2}-1} \cdot J_{k-\frac{g}{2}-1}\left(\frac{4\pi\sqrt{D\mu}}{ct}\right) \cdot e(\mu\tau). \tag{5.8}$$

Proof. Clearly $F_s(\tau)$ has period t and thus a Fourier expansion

$$F_s(\tau) = \sum_{\mu\in\mathbb{Z}} a_\mu(v) \cdot e\left(\frac{\mu\tau}{t}\right),$$

where

$$a_\mu(v) = \frac{1}{t}\int_0^t F_s(\tau) \cdot e\left(-\frac{\mu\tau}{t}\right) du = \frac{1}{t}\int_{\mathbb{R}} \tau^{-\left(k-\frac{g}{2}+s\right)} \cdot \overline{\tau}^{-s} \cdot e\left(-\frac{D}{tc^2\tau} - \frac{\mu\tau}{t}\right) du.$$

Making the substitution $u \mapsto uv$ gives (5.7).

Claim (1) follows directly from Lemma 3.5 of [Br1] with $c_1 = k - \frac{g}{2}$, $c_2 = \frac{v\mu}{t}$, and $c_3 = \frac{D}{tc^2 v}$.
(2) follows from (1).
(3) If $k > \frac{g}{2} + 1$, then one has

$$\Phi_{k,c,v}(\mu, 0) = \frac{1}{t} \int_{iv-\infty}^{iv+\infty} \tau^{\frac{g}{2}-k} \cdot e\left(-\frac{1}{t}\left(\mu\tau + \frac{D}{c^2\tau}\right)\right) d\tau.$$

For $\mu > 0$, the substitution $\tau = \frac{i}{c} \cdot \left(\frac{D}{\mu}\right)^{1/2} \cdot w$ gives

$$\begin{aligned}
\Phi_{k,c,v}(\mu, 0) &= \frac{1}{t}\left[\frac{i}{c}\sqrt{\frac{D}{\mu}}\right]^{\frac{g}{2}+1-k} \int_{v'-i\infty}^{v'+i\infty} w^{\frac{g}{2}-k} \cdot \exp\left(\frac{2\pi\sqrt{D\mu}}{ct}\left(w - w^{-1}\right)\right) dw \\
&= \frac{1}{t}\left[\frac{i}{c}\sqrt{\frac{D}{\mu}}\right]^{\frac{g}{2}+1-k} 2\pi i \cdot J_{k-\frac{g}{2}-1}\left(\frac{4\pi\sqrt{D\mu}}{ct}\right)
\end{aligned}$$

as claimed. The vanishing of the Fourier coefficients for $\mu \le 0$ can be established if we deform the path of integration up to infinity. $\square$

From this we obtain the Fourier expansion of $P_{s,l}(\tau)$. Combining the following theorem with Proposition 5.2 gives another proof of the analytic continuation of the Poincaré series $P_{k,m;(n,r),s}(\tau, z)$.

Theorem 5.4. (1) *The function $P_{s,l}(\tau)$ has an analytic continuation to $\sigma > \frac{1}{2}\left(\frac{g}{2} + 2 - k\right)$ with the following Fourier expansion:*

$$P_{s,l}(\tau) = v^s \sum_{\mu\in\mathbb{Z}} b_{\frac{\mu}{t}}(P_{s,l}) \cdot e\left(\frac{\mu\tau}{t}\right)$$

with

$$b_{\frac{\mu}{t}}(P_{s,l}) = \left(1 + (-1)^k\right) \cdot \delta_{\mu,D} + \sum_{c>0} c^{\frac{g}{2}-k-2s} \cdot K_\mu(c, tc) \cdot \Phi_{k,c,v}(\mu, s).$$

Here $\delta_{\mu,D} := \begin{cases} 1 & \text{if } \mu = D, \\ 0 & \text{otherwise,} \end{cases}$ *and*

$$K_\mu(c, tc) := \sum_{\substack{d \pmod{tc} \\ (c,d)=1}} e\left(\frac{\mu d}{tc}\right)\left(\chi_l^r(c,d) + (-1)^k \chi_{-l}^r(c,d)\right).$$

(2) *If $k > \frac{g}{2} + 2$, then $P_r(\tau) := P_{0,r}(\tau)$ has the Fourier expansion:*

$$P_r(\tau) = \sum_{\mu>0} b_{\frac{\mu}{t}}(P_r) \cdot e\left(\frac{\mu\tau}{t}\right)$$

where

$$b_{\frac{\mu}{t}}(P_r) = \left(1 + (-1)^k\right) \cdot \delta_{\mu,D} - 2\pi i^{\frac{g}{2}-k}\left(\frac{D}{\mu}\right)^{\frac{1}{4}(g+2-2k)} \sum_{c>0} \frac{1}{tc} K_\mu(c, tc) \cdot J_{k-\frac{g}{2}-1}\left(\frac{4\pi\sqrt{D\mu}}{tc}\right).$$

Proof. (1) The Fourier expansion of $P_{s,l}(\tau)$ follows from (5.5) and Lemma 5.3 for $\sigma > \frac{1}{2}\left(-k+\frac{g}{2}+1\right)$. Notice that

$$|K_\mu(c, tc)| \leq 2tc. \tag{5.9}$$

Now, using the Fourier expansion and Lemma 5.3 (1), one sees that for $\sigma > \frac{1}{2}\left(\frac{g}{2}+2-k\right)$

$$\left|b_{\frac{\mu}{t}}(P_{s,l})\right| \ll_{K,v,D,t} \delta_{\mu,D} + e^{\frac{2\pi\mu v}{t}(1-\mathrm{sign}(\mu)v_1)} \sum_{c>0} c^{\frac{g}{2}+1-k-2\sigma} \ll_{K,v,D,t} \delta_{\mu,D} + e^{\frac{2\pi\mu v}{t}(1-\mathrm{sign}(\mu)v_1)}.$$

Thus

$$|P_{s,l}(\tau)| \ll_{K,\nu,D,t} 1 + \sum_{\mu\in\mathbb{Z}} e^{-\frac{2\pi|\mu|v_1}{t}},$$

which is absolutely convergent. This proves (1).
(2) By Lemma 5.3 and (1), one sees that $P_r(\tau)$ is a holomorphic function of τ in the upper half plane. The Fourier expansion formula follows from (1) and Lemma 5.3. □

Corollary 5.5. *(Theorem 1.3) Assuming the hypothesis above, we have*

$$\left|b_{n,r}\left(P_{k,m;(n,r),s}\right)\right| = \left|b_{\frac{D}{t}}(P_{0,r})\right| \ll_k 1 + \frac{D}{\det(2m)}.$$

Proof. The first identity is already contained in Proposition 5.2. Set $A := \frac{4\pi D}{t}$. Similarly to the proof of (1.5), Theorem 5.4, (5.9), and (4.4) give,

$$\left|b_{\frac{D}{t}}(P_{0,r})\right| \ll_k 1 + \sum_{c>0}\left|J_{k-\frac{g}{2}-1}\left(\frac{A}{c}\right)\right| \ll_k 1 + \frac{D}{\det(2m)}.$$

□

6. Proof of Theorems 1.4 and 1.5

Here we prove Theorems 1.4 and 1.5. For this we recall the following Lemma from [BK].

Lemma 6.1. *If ϕ_m is the mth Fourier Jacobi coefficient of a Siegel cusp form F, then we have*

$$\parallel \phi_m \parallel \ll_{\epsilon,F} (\det 2m)^{\frac{k}{2}-\alpha_g+\epsilon},$$

where α_g is defined in (1.9).

Define

$$m_{g-1}(T) := \min\left\{T[U]_{g-1} \,|U \in \mathrm{GL}_g(\mathbb{Z})\right\},$$

where $T[U]|_{g-1}$ denotes the determinant of the leading $(g-1)$ rowed submatrix of $T[U]$. To prove Theorems 1.4 and 1.5, we may assume that $T = \begin{pmatrix} n & \frac{r^t}{2} \\ \frac{r}{2} & m \end{pmatrix}$ with $\det m = m_{g-1}(T)$ since both sides of the estimates are invariant under replacing T with $T[U]$ with $U \in \mathrm{GL}_g(\mathbb{Z})$. Now by Lemma 3.4 (with $g-1$ instead of g) and Lemma 6.1

$$a(T) \ll \left|b_{n,r}\left(P_{k,m;(n,r)}\right)\right|^{\frac{1}{2}} \cdot D^{\frac{k}{2}-\frac{g}{4}-\frac{1}{4}} \cdot (\det(2m))^{\frac{g}{4}+\frac{1}{2}-\alpha_g+\epsilon}. \tag{6.1}$$

We now use this estimate to prove Theorems 1.4 and 1.5. Recall $D = \det(2T)$.

Proof of Theorem 1.4. From Theorem 1.3 we get

$$b_{n,r}\left(P_{k,m;(n,r)}\right) \ll \det(2m)^{-\frac{g}{2}}\left(\det(2m)^{\frac{g}{2}} + D^{\frac{g-1}{2}+\epsilon}\cdot\det(2m)^{\epsilon}\right).$$

Combining this with (6.1) gives

$$a(T) \ll \det(2m)^{\frac{1}{2}-\alpha_g+\epsilon}\cdot D^{\frac{k}{2}-\frac{g}{4}-\frac{1}{4}}\left(\det(2m)^{\frac{g}{2}} + D^{\frac{g-1}{2}+\epsilon}\cdot\det(2m)^{\epsilon}\right)^{\frac{1}{2}}.$$

Using reduction theory we can assume that

$$\det m = m_{g-1}(T) \ll (\det T)^{1-\frac{1}{g}}. \tag{6.2}$$

This directly gives Theorem 1.4. □

Proof of Theorem 1.5. By (6.1) and Corollary 5.5, we have

$$a(T) \ll \det(2m)^{\frac{g}{4}-\alpha_g+\epsilon} D^{\frac{k}{2}-\frac{g}{4}-\frac{1}{4}}\left(\det(2m)+D\right)^{\frac{1}{2}}.$$

Theorem 1.5 now follows directly using (6.2). □

7. Proof of Theorem 1.7

In this section we construct lifting maps from the vector space of Jacobi cusp forms to a subspace of elliptic modular forms. Since we have shown the properties of the Poincaré series given in Theorem 1.1, we can proceed as in [Br2]. For the reader convenience we recall the arguments here. First we recall some facts from [GKZ] about quadratic forms, the generalized genus character, and geodesic cycle integrals. For $a, b, c \in \mathbb{Z}$ let us define the integral binary quadratic form

$$[a,b,c](x,y) := ax^2 + bxy + cy^2.$$

The group $\mathrm{SL}_2(\mathbb{Z})$ acts on these forms in the usual way by

$$[a,b,c]\circ\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)(x,y) := [a,b,c](\alpha x+\beta y, \gamma x+\delta y) \quad (x,\, y\in\mathbb{Z}).$$

Let $\Delta > 0$ be a discriminant (of a binary quadratic form) and denote by $\mathcal{D}_\Delta$ the set of integral binary quadratic forms with discriminant $\Delta = 4ac - b^2 > 0$. Furthermore for a positive integer l, denote by $\mathcal{D}_{l,\Delta}$ the subset of $\mathcal{D}_\Delta$ of all quadratic forms with the additional condition that $a \equiv 0 \pmod{l}$. Moreover for integers $\rho \pmod{2l}$ with $\Delta \equiv \rho^2 \pmod{4l}$, let

$$\mathcal{D}_{l,\Delta,\rho} := \left\{[a,b,c]\in\mathcal{D}_\Delta \,|\, a\equiv 0 \pmod{l}, b\equiv\rho \pmod{2l}\right\}.$$

Both sets $\mathcal{D}_{l,\Delta}$ and $\mathcal{D}_{l,\Delta,\rho}$ are $\Gamma_0(l)$ invariant. For a fundamental discriminant $-D_0$ that divides Δ with $-D_0$ and $-\frac{\Delta}{D_0}$ are squares $\pmod{4l}$, define for $Q = [al, b, c] \in \mathcal{D}_{l,\Delta}$ the generalized genus character:

$$\chi_{D_0}(Q) := \begin{cases} \left(\frac{-D_0}{n}\right) & \text{if } (a,b,c,D_0) = 1,\\ 0 & \text{otherwise.}\end{cases}$$

Here n is an integer coprime to D_0 represented by the form $[al_1, b, cl_2]$ for some decomposition $l = l_1 l_2$, $l_i > 0$ $(i = 1, 2)$. It is easy to show that such an n always exists and that the value of $\left(\frac{-D_0}{n}\right)$ is independent of the choice of l_1, l_2, and n.

Define for $f \in S_{2k}(l)$ and $Q = [a, b, c] \in \mathcal{D}_{l,\Delta,\rho}$ the cycle integral

$$r_{k,l,Q}(f) := \int_{\gamma_Q} f(z) \cdot Q(z,1)^{k-1}\, dz,$$

where γ_Q is the image in $\Gamma_0(l)\backslash\mathbb{H}$ of the semicircle $a|z|^2 + bx + c = 0$ ($x = \mathrm{Re}(z)$), orientated from $\frac{-b-\sqrt{\Delta}}{2a}$ to $\frac{-b+\sqrt{\Delta}}{2a}$ if $a \neq 0$ or, if $a = 0$, the vertical line $bx + c = 0$, orientated from $-\frac{c}{b}$ to $i\infty$ if $b > 0$ and from $i\infty$ to $-\frac{c}{b}$ if $b < 0$. It is not hard to see that the above definition makes sense (i.e., the integral is invariant with respect to the subgroup of $\Gamma_0(l)$ preserving Q) and depends only on the $\Gamma_0(l)$ equivalence class of Q. Furthermore, we define

$$r_{k,l,\Delta,\rho,D_0}(f) := \sum_{Q \in \mathcal{D}_{l,\Delta,\rho}/\Gamma_0(l)} \chi_{D_0}(Q) \cdot r_{k,l,Q}(f).$$

Next we define a kernel functions for the cycle integrals

$$f_{k,l,\Delta,\rho,D_0}(z) := \sum_{Q \in \mathcal{D}_{l,\Delta,\rho}} \frac{\chi_{D_0}(Q)}{Q(z,1)^k} \qquad (z \in \mathbb{H}).$$

It is known from [GKZ] that the series $f_{k,l,\Delta,\rho,D_0}(z)$ is absolutely and locally uniformly convergent for $k > 1$ and is an element of $S_{2k}(l)^-$. Moreover for $k = 1$ the series is continued, using the "Hecke-trick", and again is an element of $S_{2k}(l)^-$.

Lemma 7.1. *The Fourier expansion of $f_{k,l,\Delta,\rho,D_0}(z)$ $(k \geq 1)$ is given by*

$$f_{k,l,\Delta,\rho,D_0}(z) = \sum_{m=1}^{\infty} c_{k,l}^{\pm}(m, \Delta, \rho, D_0) e^{2\pi i m z},$$

where

$$c_{k,l}^{\pm}(m, \Delta, \rho, D_0) := c_{k,l}(m, \Delta, \rho, D_0) + (-1)^{k+1} c_{k,l}(m, \Delta, -\rho, D_0),$$

with

$$c_{k,l}(m, \Delta, \rho, D_0) := i^k \cdot (-1)^{-\frac{1}{2}} \cdot \frac{(2\pi)^k}{(k-1)!} \cdot (m^2/\Delta)^{\frac{k-1}{2}} \cdot \Bigg[D_0^{-\frac{1}{2}} \cdot \epsilon_l(m, \Delta, \rho, D_0)$$
$$+ i^{k+1} \cdot \pi \cdot \sqrt{2} \cdot \left(m^2/\Delta\right)^{\frac{1}{4}} \cdot \sum_{a \geq 1} (la)^{-\frac{1}{2}} \cdot S_{la}(m, \Delta, \rho, D_0) \cdot J_{k-\frac{1}{2}}\left(\frac{\pi m \sqrt{\Delta}}{la}\right) \Bigg].$$

Here

$$\epsilon_l(m, \Delta, \rho, D_0) := \begin{cases} \left(\frac{-D_0}{m/f}\right) & \text{if } \Delta = D_0^2 \cdot f^2 \ (f > 0),\ f|m,\ \ -D_0 f \equiv \rho \pmod{2l} \\ 0 & \text{otherwise} \end{cases},$$

$$S_{la}(m, \Delta, \rho, D_0) = \sum_{\substack{b(2la) \\ b \equiv \rho (2l) \\ b^2 \equiv \Delta (4la)}} \chi_{D_0}\left(\left[al, b, \frac{b^2 - \Delta}{4la}\right]\right) \cdot e\left(\frac{mb}{2la}\right).$$

The following theorem is known from [GKZ].

Theorem 7.2. *For $f \in S_{2k}(l)^-$ we have*

$$\langle f, f_{k,l,\Delta,\rho,D_0}\rangle = \pi \cdot \binom{2k-2}{k-1} \cdot 2^{-2k+2} \cdot \Delta^{-k+1/2} \cdot r_{k,l,\Delta,\rho,D_0}(f),$$

where $< \cdot, \cdot >$ denotes the usual Petersson scalar product for elliptic cusp forms with respect to $\Gamma_0(l)$.

Proof of Theorem 1.7. Before we give the proof, we state the needed conditions precisely as follows:

(1) $-D_0$ is a a square $\pmod{\frac{1}{2}\det(2m)}$ and is a fundamental discriminant.
(2) If $p|\gcd\left(\frac{1}{2}\det(2m), D_0\right)$, then $\mathrm{ord}_p\left(\frac{1}{2}\det(2m)\right) \le \mathrm{ord}_p(D_0)$.
(3) If $2 \neq p|\gcd\left(\frac{1}{2}\det(2m), D_0\right)$, then there always exists a matrix $U \in \mathrm{GL}_2(\mathbb{Z})$, such as $(2m)[U] \equiv \mathrm{diag}(m_1, \cdots, m_{g-1}, 0) \pmod p$. We require $\prod_{1 \le i < g} m_i$ to be a square $\pmod p$.

Quadratic forms with the above conditions indeed exist (for an example see [Br1]). It can easily be shown that the last two conditions are satisfied if $\det(2m) \cdot D_0$ is square-free. Moreover it can be shown for $g = 1$ that the conditions are equivalent to the conditions given in [GKZ].

To prove Theorem 1.7, define

$$\Omega_{k,m,D_0,r_0}(w;\tau,z) := c_{k,m,D_0} \cdot \sum_{D>0} D^{k-\frac{1}{2}} \cdot f_{k,\frac{1}{2}\det(2m),D_0D,r(2m)^*r_0^t,D_0}(w) \cdot e(n\tau + r^t z), \tag{7.1}$$

where

$$c_{k,m,D_0} := \frac{(-2i)^{k-1} \cdot D_0^{k-\frac{1}{2}}}{\left(\frac{1}{2}\det(2m)\right)^{k-1} \cdot \pi \cdot \binom{2k-2}{k-1}}.$$

Here for a matrix A we denote by A^* the adjoint matrix of A. One can easily see, using the Fourier expansion of $f_{k,\frac{1}{2}\det(2m),D_0D,r(2m)^*r_0^t,D_0}(w)$, that the series $\Omega_{k,m,D_0,r_0}(w;\tau,z)$ is absolutely convergent. As a function of w it is clearly an element of $S_{2k}\left(\frac{1}{2}\det(2m)\right)^-$. On the other hand, the same argument as in [Br3], Section 3 gives

$$\begin{aligned}\Omega_{k,m,D_0,r_0}(w;\tau,z) &= c_{k,m,D_0} \cdot \frac{i^{k-1}\cdot(2\pi)^k}{(k-1)!} \\ &\times \sum_{l\ge 1} l^{k-1}\left(\sum_{dd'=l}\left(\frac{-D_0}{d}\right)\cdot d'^k \cdot P_{k+\frac{g+1}{2},m,(n_0d'^2,r_0d')}(\tau,z)\right)e^{2\pi i l w}.\end{aligned} \tag{7.2}$$

This shows that $\Omega_{k,m,D_0,r_0}(w;\tau,z)$ is a Jacobi form in the variables τ and z. Now for $f \in S_k\left(\frac{1}{2}\det(2m)\right)^-$ we have by Theorem 7.2 and (7.1) that

$$\mathcal{S}^*_{D_0,r_0}(f)(\tau,z) = \langle f, \Omega_{k,m,D_0,r_0}(\cdot;-\bar\tau,-\bar z)\rangle \tag{7.3}$$

is a Jacobi form, proving Theorem 1.7 (2).

Next, for $\phi \in J^{cusp}_{k+\frac{g+1}{2},m}$, we have by Theorem 1.1 and (7.2)

$$\mathcal{S}_{D_0,r_0}(\phi)(\omega) = \langle \phi, \Omega_{k,m,D_0,r_0}(-\bar\omega,\cdot,\cdot)\rangle, \tag{7.4}$$

proving Theorem 1.7 (1). Here we also used the fact $\Omega_{k,m,D_0,r_0}(w;-\bar\tau,-\bar z) = \overline{\Omega_{k,m,D_0,r_0}(-\bar w;\tau,z)}$. Theorem 1.7 (3) is now clear from (7.3) and (7.4). $\square$

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