Symmetric superspaces: slices, radial parts, and invariants

Alexander Alldridge (Cologne)

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Plan of the talk

- Recollections
- Even type and odd type
- Results in even type
- Radial parts
- Results in odd type

Non-graded recollections. I. The “group case”

$\mathfrak{g}$	reductive Lie algebra
$\mathfrak{h}$	Cartan subalgebra
$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{u}$	Borel subalgebra
$W = W(\mathfrak{g} : \mathfrak{h})$	Weyl group

Theorem (Chevalley, Harish-Chandra).

1. The map $\mathrm{res}_{\mathfrak{h}} : p \longmapsto p|_{\mathfrak{h}^*}$ is an algebra isomorphism $S(\mathfrak{g})^{\mathfrak{g}} \longrightarrow S(\mathfrak{h})^W$, and $S(\mathfrak{h})^W$ is a polynomial algebra in $r = \mathrm{rk}\,\mathfrak{g}$ indeterminates.
2. There is an algebra isomorphism

$$\Gamma : \mathcal{Z}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{g})^{\mathfrak{g}} \longrightarrow S(\mathfrak{h})^W,$$

given by the projection

$$\mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{h}) \oplus (\mathfrak{u}\mathfrak{U}(\mathfrak{g}) + \mathfrak{U}(\mathfrak{g})\mathfrak{u}^-) \longrightarrow \mathfrak{U}(\mathfrak{h}) = S(\mathfrak{h}),$$

followed by the shift $p \longmapsto p(\cdot - \varrho)$, $\varrho = \frac{1}{2}\,\mathrm{tr}_{\mathfrak{u}}\,\mathrm{ad}\,|_{\mathfrak{h}}$.

Non-graded recollections. I. The case of symmetric pairs

$(\mathfrak{g}, \theta)$	reductive symmetric Lie algebra
$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$	θ-eigenspace decomposition
$\mathfrak{a} \subseteq \mathfrak{p}$	Cartan subspace
$\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$	minimal θ-parabolic
$W = W(\mathfrak{g} : \mathfrak{a})$	little Weyl group

Theorem (Chevalley, Harish-Chandra).

1. The map $\mathrm{res}_{\mathfrak{a}} : p \longmapsto p|_{\mathfrak{a}^*}$ is an algebra isomorphism $S(\mathfrak{p})^{\mathfrak{k}} \longrightarrow S(\mathfrak{a})^W$, and $S(\mathfrak{a})^W$ is a polynomial algebra in $r = \mathrm{rk}\,\Sigma$ indeterminates, $\Sigma = \Delta(\mathfrak{g} : \mathfrak{a})$.
2. There is an exact sequence

$$0 \longrightarrow (\mathfrak{k}\mathfrak{U}(\mathfrak{g}))^{\mathfrak{k}} \longrightarrow \mathfrak{U}(\mathfrak{g})^{\mathfrak{k}} \xrightarrow{\Gamma} S(\mathfrak{a})^W \longrightarrow 0$$

where Γ is given by the projection

$$\mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{a}) \oplus (\mathfrak{n}\mathfrak{U}(\mathfrak{g}) + \mathfrak{U}(\mathfrak{g})\theta(\mathfrak{n})) \longrightarrow \mathfrak{U}(\mathfrak{a}) = S(\mathfrak{a}),$$

followed by the shift $p \longmapsto p(\cdot - \varrho)$, $\varrho = \frac{1}{2}\,\mathrm{tr}_{\mathfrak{n}}\,\mathrm{ad}\,|_{\mathfrak{a}}$.

Remark. The “group case” is recovered for $(\mathfrak{g} \times \mathfrak{g}, (1\,2))$.

Graded group cases

$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$	contragredient Lie superalgebra
$\mathfrak{h} \subseteq \mathfrak{g}_{\bar{0}}$	Cartan subalgebra
$\mathfrak{b}$	Borel subalgebra
$W = W(\mathfrak{g}_{\bar{0}} : \mathfrak{h})$	Weyl group

Theorem (Sergeev, Kac, Gorelik).

1. $\mathrm{res}_{\mathfrak{h}} : S(\mathfrak{g})^{\mathfrak{g}} \longrightarrow S(\mathfrak{h})^W$ is an injection whose image $I(\mathfrak{h})$ consists of all $p \in S(\mathfrak{h})^W$,

$$p(\lambda) = p(\lambda + \alpha), \quad \forall \lambda \in \mathfrak{h}^*, \alpha \in \overline{\Delta}_{\bar{1}}, \langle \lambda + \varrho, \alpha \rangle = 0.$$

   Here, $\overline{\Delta}_{\bar{1}} := \Delta_{\bar{1}} \setminus \mathbb{Q}\Delta_{\bar{0}}$ are the purely odd roots.
2. $\Gamma : \mathcal{Z}(\mathfrak{g}) \longrightarrow S(\mathfrak{h})^W$ is an injective algebra morphism whose image is $I(\mathfrak{h})$.

Remark (Sergeev, Stembridge). $I(\mathfrak{gl}(m|n))$ is not finitely generated for $m, n \geqslant 1$.

Symmetric superpairs

$(\mathfrak{g}, \theta)$ symmetric contragredient Lie superalgebra

Theorem (Serganova, Chuah). If $\mathfrak{k} = \ker(1 - \theta)$ is non-degenerate, then $(\mathfrak{g}, \mathfrak{k})$ is one of:

1. Lie algebra symmetric pairs,
2. parity involution pairs $(\mathfrak{g}, \mathfrak{g}_{\bar{0}})$,
3. group type pairs $(\mathfrak{g} \times \mathfrak{g}, \mathfrak{g})$,
4. entries of the following list and their simple subquotients.

$AI\|AII$	$\mathfrak{gl}(p\|2q)$	$\mathfrak{osp}(p\|2q)$
$AIII\|AIII$	$\mathfrak{gl}(p+q\|r+s)$	$\mathfrak{gl}(p\|r) \times \mathfrak{gl}(q\|s)$
$BDI\|CII$	$\mathfrak{osp}(p+q\|2r+2s)$	$\mathfrak{osp}(p\|2r) \times \mathfrak{osp}(q\|2s)$
$DIII\|CI$	$\mathfrak{osp}(2p\|2q)$	$\mathfrak{gl}(p\|q)$
$(BDI\|CII)_\alpha$	$D(2,1;\alpha)$	$\mathfrak{osp}(2\|2) \times \mathfrak{o}(2)$
BDI_I	$F(4)$	$\mathfrak{sl}(4\|1)$
BDI_{II}	$F(4)$	$\mathfrak{osp}(2\|4) \times \mathbb{C}$
BDI_{III}	$F(4)$	$\mathfrak{osp}(4\|2) \times \mathfrak{sl}(2)$
G_I	$G(3)$	$\mathfrak{osp}(3\|2) \times \mathfrak{sl}(2)$
G_{II}	$G(3)$	$\mathfrak{osp}(4\|2)$

Remark. $\mathfrak{g} = F(4), G(3)$ type II $\Longrightarrow$ trivial automorphism of $\mathfrak{g}_{\bar{0}} \longmapsto (\mathfrak{g}, \mathfrak{g}_{\bar{0}})$
$|\mathrm{Aut}_2(\mathfrak{g}_{\bar{0}}) \setminus \{\mathrm{id}\}| = 3, 1$, parity self-dual in $F(4)$ case, non self-dual in $G(3)$ case

Even type and odd type

$$\mathfrak{a}_{\bar{0}} \subseteq \mathfrak{p}_{\bar{0}} \qquad \text{even Cartan subspace}$$
$$\mathfrak{a} := \mathfrak{z}_{\mathfrak{p}}(\mathfrak{a}_{\bar{0}}) \qquad \text{Cartan subspace}$$

Definition. We say $(\mathfrak{g}, \mathfrak{k})$ is of *even type* if $\mathfrak{a} = \mathfrak{a}_{\bar{0}}$ and *odd type* otherwise.

Of the symmetric pairs listed above, the following are of even type:

1. Lie algebra symmetric pairs,
2. group type symmetric pairs,
3. the following types: $AI|AII$, $DIII|CI$, $(BDI|CII)_{\alpha}$, BDI_{I}, BDI_{II}, BDI_{III}, G_{I}, and
4. the types $AIII|AIII$, $BDI|CII$ for $(p-q)(r-s) \geqslant 0$.

Thus, the following are the odd type pairs:

1. parity involution pairs,
2. type G_{II}, and
3. the types $AIII|AIII$, $BDI|CII$ for $(p-q)(r-s) < 0$.

Invariants in even type. I. Functions

$\overline{\Sigma}_{\bar{1}} := \Sigma_{\bar{1}} \setminus \mathbb{Q}\Sigma_{\bar{0}}$ purely odd restricted roots

Theorem (A–Hilgert–Zirnbauer 2010). $\mathrm{res}_{\mathfrak{a}} : S(\mathfrak{p})^{\mathfrak{k}} \longrightarrow S(\mathfrak{a})^W$ is an injection whose image is

$$I(\mathfrak{a}) := \bigcap_{\alpha \in \overline{\Sigma}_{\bar{1}}} I_\alpha$$

where for $\langle \alpha, \alpha \rangle = 0$:

$$I_\alpha := \{p \in S(\mathfrak{a})^W \mid \partial_\alpha^k p \in (\check{\alpha}^k), k = 0, \ldots, \tfrac{1}{2} \dim \mathfrak{g}_{\bar{1}}^\alpha\}$$

and for $\langle \alpha, \alpha \rangle \neq 0$:

$$I_\alpha := \{p \in S(\mathfrak{a})^W \mid \partial_\alpha^k p \in (\check{\alpha}), k = 1, 3, 5, \ldots, \dim \mathfrak{g}_{\bar{1}}^\alpha - 1\}.$$

Example. $\mathfrak{g} = \mathfrak{osp}(2|2q)$, $\mathfrak{k} = \mathfrak{osp}(1|2q)$:

$$I(\mathfrak{a}) = \mathbb{C}[a^2, a^{2q+1}] \cong \mathbb{C}[X, Y]/(X^{2q+1} - Y^2)$$

Invariants in even type. II. Operators

Theorem (A 2012). $\Gamma : \mathfrak{U}(\mathfrak{g})^{\mathfrak{k}} \longrightarrow S(\mathfrak{a})^W$ has kernel $(\mathfrak{k}\mathfrak{U}(\mathfrak{g}))^{\mathfrak{k}}$ and image

$$J(\mathfrak{a}) := \bigcap_{\alpha \in \overline{\Sigma}_{\bar{1}}} J_\alpha$$

where for $\langle \alpha, \alpha \rangle = 0$:

$$J_\alpha := I_\alpha$$

and for $\langle \alpha, \alpha \rangle \neq 0$ and $2q = \dim \mathfrak{g}^\alpha_{\bar{1}}$:

$$J_\alpha := S(\mathfrak{a})^W \cap \mathbb{C}[(\check{\alpha}^2 - q^2), (\check{\alpha} - q)(\check{\alpha}^2 - q^2)^q].$$

We have $\operatorname{gr} J(\mathfrak{a}) = I(\mathfrak{a})$, but in general $J(\mathfrak{a}) \neq I(\mathfrak{a})$.

Example. $\mathfrak{g} = \mathfrak{osp}(2|2q)$, $\mathfrak{k} = \mathfrak{osp}(1|2q)$:

$$J(\mathfrak{a}) = \mathbb{C}[a^2 - q^2, (a - q)(a^2 - q^2)^q] \cong \mathbb{C}[X, Y]/(X^{2q+1} - Y^2 - 2qX^qY)$$

Idea of proof: Radial parts

Theorem. For any differential operator D on $\mathfrak{p}$, there is a unique differential operator $\overline{D}$ on $\mathfrak{a}'$ such that

$$\overline{D}(f|_{\mathfrak{a}'}) = D(f)|_{\mathfrak{a}'}$$

for any locally defined $\mathfrak{k}$-invariant analytic function.

Theorem (A–Hilgert–Zirnbauer 2010). We have

$$I_\alpha = \bigcap_{D \in S(\mathfrak{p}_{\bar{1}}^\alpha)} \operatorname{dom} \overline{D}$$

This follows from the fact that for a symplectic basis $(z_i, \tilde{z}_i)$ of $\mathfrak{p}_{\bar{1}}^\alpha = (1+\theta)(\mathfrak{g}_{\bar{1}}^\alpha)$, we have

$$\overline{z_I \tilde{z}_I} = (-1)^{\frac{k(k+1)}{2}} \sum_{j=0}^{k-1} \frac{(k-1+j)!}{2^j (k-1-j)!} \frac{(-\langle \alpha, \alpha \rangle)^j}{\breve{\alpha}^{k+j}} \partial_\alpha^{k-j}, \quad k = |I|.$$

One might hope for a characterisation in terms of operators with simpler radial parts. Moreover, one would like to give meaning to radial parts in general.

Radial parts and slices. I

G	real or complex Lie supergroup
$X, a : G \times X \longrightarrow X$	real or complex supermanifold with G-action
$Y \subseteq X$	locally closed subsupermanifold

Prime example:

$$G = \mathrm{Ad}_{\mathfrak{k}}, \quad X = \mathfrak{p}, \quad Y = \mathfrak{a}'.$$

Definition. We say that Y is a *weak slice* if

$$T_y X = T_y Y \oplus \mathrm{im}(\mathfrak{g} \longrightarrow T_y X), \quad \forall y \in Y_0$$

and a *strong slice* if in addition

$$(\forall y' \in U(y) : (v_X)(y') = 0) \quad \Longrightarrow \quad (v_X|_Y)_y = 0, \quad \forall v \in \mathfrak{g}, y \in Y_0.$$

The conditions are equivalent if X, Y, G are non-graded.

Proposition. In the above situation, $\mathfrak{a}$ is always a weak slice, and it is a strong slice if and only if $(\mathfrak{g}, \mathfrak{k})$ is of even type.

Radial parts and slices. II

Theorem (A–Coulembier 2015). Let Y be a strong slice. For any differential operator D on X, there is a unique differential operator $\overline{D}$ on Y such that

$$\overline{D}(f|_Y) = D(f)|_Y$$

for any locally defined (locally) G-invariant analytic (smooth) function f.

When Y is only a weak slice, consider the *Weyl groupoid*

$$\widetilde{W}_Y := \mathrm{Trans}_G(Y)/\mathrm{Fix}_G(Y), \quad \mathrm{Trans}_G(Y) := \{(y,g) \mid g \cdot y \in Y\},$$
$$\mathrm{Fix}_G(Y) := \{(y,g) \mid g \cdot y = y\}.$$

Theorem (A–Coulembier 2015). Let Y be a weak slice and assume the quotient $\pi_Y : Y \longrightarrow Y/\widetilde{W}_Y$ exists as a reasonable superspace. For any differential operator D on X, there is a unique differential operator $\overline{D}$ along π_Y such that

$$\overline{D}(\bar{f}) = D(f)|_Y, \quad f|_Y = \pi_Y^\sharp(\bar{f})$$

for any locally defined (locally) G-invariant analytic (smooth) function f. If D is G-invariant, then $\overline{D}$ descends to an operator on $Y/\widetilde{W}_Y$.

Proof no. 1

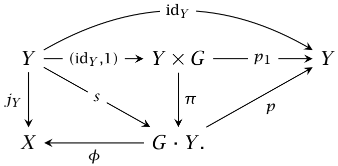
Proof of Theorem 1.

There is a local isomorphism onto an open subspace:

$$\phi : G \cdot Y := (Y \times G)/\mathrm{Fix}_G(Y \subseteq X) \longrightarrow X$$

Shrinking Y and X, we may assume it is an isomorphism.

There is a commutative diagram:



For any $D \in \Gamma(\mathcal{D}_X)$, we may define $\bar{D} \in \Gamma(\mathcal{D}_Y)$ by the prescription

$$\bar{D} := j_Y^{\sharp} \circ D \circ \phi^{-1\sharp} \circ p^{\sharp}.$$

This shows existence, and uniqueness follows similarly. □

Proof no. 2

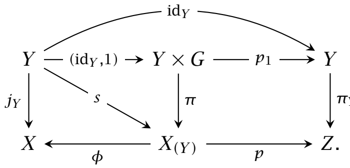
Proof of Theorem 2.

There is an open G-equivariant embedding

$$\phi : X_{(Y)} := (Y \times G)/\mathrm{Trans}_G(Y) \longrightarrow X$$

Shrinking X, we may assume it is an isomorphism.

Let $Z := Y/\tilde{W}_Y$. There is a commutative diagram:



For any $D \in \Gamma(\mathcal{D}_X)$, we may define $\bar{D} \in \Gamma(\mathcal{D}_{Y \to Z})$ by the prescription

$$\bar{D} := j_Y^{\sharp} \circ D \circ \phi^{-1\sharp} \circ p^{\sharp}.$$

This shows existence, and uniqueness follows similarly. □

Radial parts in odd type

Theorem (A–Coulembier 2015). The Weyl groupoid $\tilde{W}_{\mathfrak{a}'}$ is $\mathfrak{a}' \times \widetilde{W}$, where $\widetilde{W}$ is the Weyl group

$$\widetilde{W} = N_{K_0}(\mathfrak{a}) / Z_{K_0}(\mathfrak{a}) = W \times \overline{W}$$

where

$$W = W(\mathfrak{g}_{\bar{0}} : \mathfrak{a}_{\bar{0}}), \quad \overline{W} = N_{K_0}(\mathfrak{a}) / (Z_{K_0}(\mathfrak{a}_{\bar{0}}) \cap N_{K_0}(\mathfrak{a}_{\bar{1}})).$$

W acts only on $\mathfrak{a}_{\bar{0}}$, and $\overline{W}$ acts only on $\mathfrak{a}_{\bar{1}}$. The quotient $\mathfrak{a}' / \widetilde{W}$ is reasonable.

Example. For $(\mathfrak{g}, \mathfrak{k}) = G_{II}$:

$$\overline{W} \cong \mathrm{SL}(2, \mathbb{C}).$$

Proposition (A–Coulembier 2015). Let $\alpha \in \Sigma$. The radial part of the Laplacian L_α of the symmetric pair $(\mathfrak{g}_\alpha, \mathfrak{k}_\alpha)$ has the shape

$$L_{\mathfrak{a}} + \frac{\dim \mathfrak{g}^\alpha + \dim \mathfrak{g}^{2\alpha}}{\breve{\alpha}} \partial_\alpha, \quad \alpha \notin \Sigma_{\bar{0}} \cap \Sigma_{\bar{1}}$$

or

$$L_{\mathfrak{a}} + \Big(\dim \mathfrak{g}^\alpha + \dim \mathfrak{g}^{2\alpha} - \mathrm{tr}\Big(\frac{1}{1 + G_\alpha / \breve{\alpha}^2}\Big)\Big) \frac{1}{\breve{\alpha}} \partial_\alpha + \sum_{ab} \eta_{ab} \, \mathrm{tr}\Big(\frac{\partial_{\theta_a} G_\alpha / \breve{\alpha}^2}{1 + G_\alpha / \breve{\alpha}^2}\Big) \partial_{\theta_b}, \quad \alpha \in \Sigma_{\bar{0}} \cap \Sigma_{\bar{1}}.$$

Invariant functions in odd type

Theorem (A–Coulembier 2015).

1. $\mathrm{res}_{\mathfrak{a}} : S(\mathfrak{p})^{\mathfrak{k}} \longrightarrow S(\mathfrak{a})$ is injective, and its image is

$$S(\mathfrak{a})^{\widetilde{W}} \cap \bigcap_{\alpha\in\Sigma} K_\alpha, \qquad K_\alpha := \bigcap_{k=0}^{\infty} \mathrm{dom}(\overline{L}_\alpha^k).$$

2. When $\alpha \in \overline{\Sigma}_{\bar{1}}$, then finite intersections up to $k = \frac{1}{2}\dim\mathfrak{g}^\alpha$ are sufficient.
3. If $(\mathfrak{g},\mathfrak{k})$ is of even type, then one may omit K_α for $\alpha \in \mathbb{Q}\Sigma_{\bar{0}} \cap \Sigma_{\bar{1}}$.
4. For $\alpha \in \Sigma_{\bar{0}}$, s_α-invariance may be replaced by K_α.
5. Similar statements hold for analytic and smooth functions.

For parity involution pairs, this just states that

$$S(\mathfrak{p})^{\mathfrak{k}} = (\textstyle\bigwedge \mathfrak{g}_{\bar{1}})^{\mathfrak{g}_{\bar{0}}} = S(\mathfrak{a})^{\widetilde{W}}.$$

Remark. This explains the $(\mathfrak{osp}(2|2q), \mathfrak{osp}(1|2q))$ result, as

$$L_\alpha^m x^n = (\partial_x^2 - 2qx^{-1}\partial_x)^m x^n = 4^m (n/2)_m ((n-1)/2 - q)_m x^{n-2m}$$

so $p \in \bigcap_m \mathrm{dom}(L_\alpha^m)$ if and only if $p_{2j-1} = 0$ for $j = 1, \ldots, q$.

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Thank you for your attention.