Quiver Grassmannians, Schubert varieties and degenerate flag varieties

Type A and C

Giovanni Cerulli Irelli

Dipartimento di Matematica, Sapienza-Universitá di Roma (ITALY)

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Outline

1. Summary
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4. Degenerate Flag varieties
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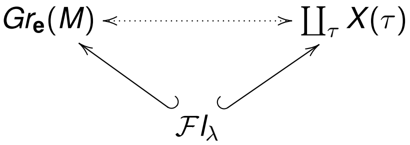
Compare the three projective varieties

The main subjects of this talk are three different projective varieties of different nature:

- Quiver Grassmannians of type A: $Gr_{\mathbf{e}}(M)$;
- Schubert varieties of type A: $X(\tau) \subseteq \mathcal{F}l(d_1, \cdots, d_s)$;
- Degenerate flag varieties of type A: $\mathcal{F}l_\lambda$.

The aim of the talk is to show that they are comparable:



Definition

Let $K = \mathbb{C}$ be the field of complex numbers and let $n \geq 1$ be a positive integer. We consider the algebra $A = T_n(K)$ of lower triangular $n \times n$ matrices: as a vector space this has basis the elementary matrices $\{e_{ji} | 1 \leq i \leq j \leq n\}$ and multiplication is given by the usual rule

$$e_{ji} \cdot e_{k\ell} = \delta_{ik} e_{j\ell}$$

So

$$A \simeq \begin{pmatrix} K & 0 & \cdots & 0 \\ K & K & & \\ \vdots & & \ddots & \\ K & \cdots & & K \end{pmatrix}$$

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A complete set of pairwise orthogonal idempotents of $A = T_n(K)$ is given by the diagonal elementary matrices $\{e_i := e_{ii} | 1 \leq i \leq n\}$ and as a left A–module we have

$$_AA = Ae_1 \oplus Ae_2 \oplus \cdots \oplus Ae_n$$

Notice that the projective indecomposable A–module Ae_i is nothing but the "i–th column of A":

$$P_i := Ae_i = \begin{pmatrix} & & i & & & \\ 0 & & & & & \\ \vdots & \ddots & & & & \\ 0 & \cdots & K & & & \\ 0 & \cdots & K & 0 & & \\ \vdots & & \vdots & \vdots & \ddots & \\ 0 & \cdots & K & 0 & \cdots & 0 \end{pmatrix}$$

Let M be a finite dimensional A–module. As a K–vector space, M decomposes as $M = e_1M \oplus e_2M \oplus \cdots \oplus e_nM$ and we define

$$M_i := e_iM.$$

The *dimension vector* of M is the integer vector

$$\mathbf{dim}(M) = (\mathrm{dim}_K M_1, \mathrm{dim}_K M_2, \cdots, \mathrm{dim}_K M_n).$$

The action of a basis element e_{ji} defines a linear map

$$f_{ji} : M_i \to M_j.$$

Since $e_{ji} = e_{j(j-1)} \cdot e_{(j-1)(j-2)} \cdot \cdots \cdot e_{(i+1)i}$ the linear map f_{ji} is the composite of linear maps

$$f_{ji} : \; M_i \xrightarrow{f_{(i+1)i}} M_{i+1} \xrightarrow{f_{(i+2)(i+1)}} \cdots \longrightarrow M_{j-1} \xrightarrow{f_{j(j-1)}} M_j$$

We hence see that an A–module M determines a diagram

$$M \mapsto \; M_1 \xrightarrow{f_{21}} M_2 \xrightarrow{f_{32}} \cdots \xrightarrow{f_{(n-1)n}} M_n$$

and viceversa.
Let $\varphi : M \to N$ an A-morphism, i.e. $\varphi(a \cdot m) = a \circ \varphi(m)$, then $\varphi(e_i M) = e_i \varphi(M) \subseteq e_i N$. We hence see that $\varphi = \sum_{i=1}^n \varphi_i$ where $\varphi : M_i \to N_i$ is a linear map such that the squares of the diagram

$$\begin{array}{ccc} M \\ \downarrow \varphi \\ N \end{array} \qquad \begin{array}{ccccccc} M_1 & \xrightarrow{f_{21}} & M_2 & \xrightarrow{f_{32}} & \cdots & \xrightarrow{f_{(n-1)n}} & M_n \\ \downarrow \varphi_1 & & \downarrow \varphi_2 & & & & \downarrow \varphi_n \\ N_1 & \xrightarrow{g_{21}} & N_2 & \xrightarrow{g_{32}} & \cdots & \xrightarrow{g_{(n-1)n}} & N_n \end{array}$$

commute.

Definition

An A-submodule $L \subseteq_A M$ of M, is a vector subspace of M invariant under A. In other words an A-submodule is a pair (L, ι) where L is an A-module and $\iota : L \to M$ is a monomorphism.

Definition

A quiver Grassmannian of type A is the closed subvariety

$$Gr_{\mathbf{e}}(M) := \{L \subseteq_A M | \, \mathbf{dim} L = \mathbf{e}\} \subseteq Gr_{e_1}(M_1) \times \cdots \times Gr_{e_n}(M_n)$$

where M is an A-module and $\mathbf{e} \in \mathbb{Z}^n_{\geq 0}$ is a dimension vector.

First examples

- If $\mathbf{e} = \mathbf{dim}\, M$ or $\mathbf{e} = \mathbf{0}$ then $Gr_{\mathbf{e}}(M) = \{point\}$.
- If $e_i > \dim M_i$ for some $i = 1, \cdots, n$, then $Gr_{\mathbf{e}}(M)$ is empty.
- If M is semisimple, then $Gr_{\mathbf{e}}(M) = \prod_{i=1}^{n} Gr_{e_i}(M_i)$
- For $n = 2$, let $M = K^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} K^2$ then

$$Gr_{(1,1)}(M) \simeq \{([X_0 : X_1], [Y_0 : Y_1]) \in \mathbb{P}^1 \times \mathbb{P}^1 | X_0 Y_0 = 0\}$$

is the cross of two $\mathbb{P}^1$'s in one point.

AR theory of A

The indecomposable A-modules are, up to isomorphisms,

$$M[i,j] := \; 0 \to \cdots \to 0 \to \underset{i}{K} \xrightarrow{1} \cdots \xrightarrow{1} \underset{j}{K} \to 0 \to \cdots \to 0$$

for all $1 \leq i \leq j \leq n$.
For example, the projective indecomposables are

$$P_i = M[i,n] \qquad (i = 1, \cdots, n)$$

and the injectives

$$I_k = M[1,k] \qquad (k = 1, \cdots, n)$$

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Irreducible maps

Recall that a map $f : M \to N$ between two indecomposable A–modules (for any algebra A) is called irreducible if any decomposition $f = g \circ h$ implies that either h is split-mono or g is split–epi. It is not difficult to show that the irreducible maps between indecomposable A–modules are of two "types"

$$M[i,j] \twoheadrightarrow M[i-1,j] \quad \text{and} \quad M[i,j] \hookrightarrow M[i,j-1]$$

for any $2 \leq i \leq j \leq n$.

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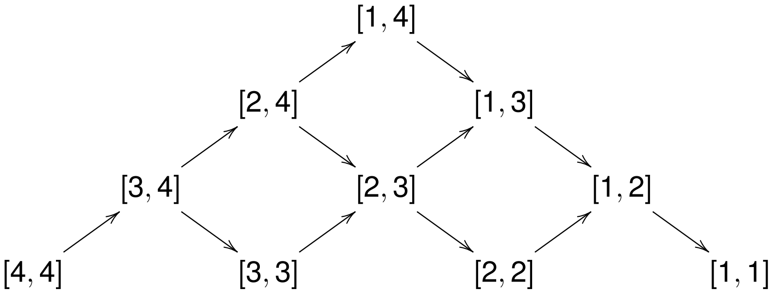
AR–quiver

The AR–quiver of an algebra B is the oriented graph with

- vertices: isoclasses of indecomposable B–modules;
- arrows: there is an arrow $[M] \to [N]$ if and only if there exists an irreducible map $M \to N$. Such arrow has multiplicity equal to the dimension of the "space of irreducible maps", which for our algebra A is at most 1.

The AR–quiver of the matrix algebra A, is (for n=4)



Projective modules as flags

The reason why we are working with the matrix algebra A, is that it enjoys the following wonderful property: an A-module P is projective if and only if the algebra A acts on P via injective linear maps. In other words P defines a flag

$$P = P_\bullet = (\ P_1 \hookrightarrow P_2 \hookrightarrow \cdots \hookrightarrow P_n\ )$$

of vector subspaces of P_n. Moreover, a sub representation $Q \subseteq_A P$ determines a sub-flag $Q_\bullet \subseteq_A P_\bullet$, in the sense that $Q_i \subseteq P_i$ for every $i = 1, \cdots, n$.

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Quiver Grassmannians inside flag varieties

- Let M be an A–module and let

$$0 \longrightarrow P^M \xrightarrow{\iota} Q^M \xrightarrow{\pi} M \longrightarrow 0$$

be a minimal projective resolution of M.

- The homomorphism $\iota : P^M \to Q^M$ provides a sub–flag $\iota(P^M_\bullet) \subseteq Q^M_\bullet$.
- Every sub representation $N \subseteq_A M$ lifts to a sub–flag $\pi^{-1}(N)_\bullet =: \widehat{N}_\bullet \subseteq Q^M_\bullet$ with the property that

$$\iota(P^M_\bullet) \subseteq \widehat{N}_\bullet \subseteq Q^M_\bullet$$

and $\mathbf{dim}\,\widehat{N} = \mathbf{dim}\,N + \mathbf{dim}\,P^M$.

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Quiver Grassmannians inside flag varieties

Definition

Given $\mathbf{e} = (e_1, \cdots, e_n) \in \mathbf{Z}^n_{\geq 0}$ and an A–module M, we denote by $Gr_{\mathbf{e}}(Q^M_\bullet \subseteq P^M_\bullet)$ the variety of flags $R_\bullet$ in P^M_n s.t.

1. $Q^M_\bullet \subseteq R_\bullet \subseteq P^M_\bullet$;
2. $\dim R_i = e_i + \dim Q^M_i$.

Quiver Grassmannians inside flag varieties

For a vector space V, and non–negative integers $0 \leq d_1 \leq d_2 \leq \cdots \leq d_r$, we denote by

$$\mathcal{F}l(d_1, \cdots, d_r; V) = \{R_\bullet = (R_1 \subseteq \cdots \subseteq R_r \subseteq V) | \dim R_k = d_k\}.$$

Then, by definition,

$$Gr_{\mathbf{e}}(Q_\bullet^M \subseteq P_\bullet^M) \subseteq \mathcal{F}l(e_1 + q_1, \cdots, e_n + q_n; P_n^M)$$

is a closed sub variety of a variety of (partial) flags of P_n^M where $q_i := \dim Q_i^M$.

Quiver Grassmannians inside flag varieties

Lemma

The map

$$Gr_{\mathbf{e}}(M) \longrightarrow Gr_{\mathbf{e}}(Q_\bullet^M \subseteq P_\bullet^M)$$

$$N \longmapsto \pi^{-1}(N)_\bullet$$

is an isomorphism of projective variety.

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Example

Let us consider the example above: $n = 2$, and

$M = K^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} K^2 = S_1 \oplus P_1 \oplus P_2$. The minimal projective resolution of M is

$$0 \longrightarrow Q^M = P_2 \xrightarrow{\iota} P^M = P_1 \oplus P_1 \oplus P_2 \xrightarrow{\pi} M \longrightarrow 0$$

Example

- The variety $Gr_{(1,1)}(\ Q_\bullet^M \hookrightarrow P_\bullet^M\ )$ is the variety of complete flags in $P_2^M = K^3 = \langle v_1, v_2, v_3 \rangle$

$$R_\bullet = (R_1 \subset R_2 \subset K^3) \qquad (\dim R_i = i)$$

such that

- $R_1 \subset \langle v_2, v_3 \rangle$ and
- $\langle v_3 \rangle \subset R_2$.
- In particular, notice that $Gr_{(1,1)}(Q_\bullet^M \subseteq P_\bullet^M)$ is stable under the action of the Borel $B \leq GL(P_2^M)$ of lower triangular matrices.

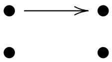
Coefficient quiver

Given an A-module M, by Krull-Remak-Schmidt theorem, we can write $M \simeq \bigoplus_{1 \leq i \leq j \leq n} M[i,j]^{m_{ij}}$. In particular, M has a normal form in which all its linear maps are permutation matrices (every column has at most one non–zero entry which is equal to 1).

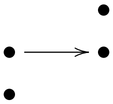
The coefficient quiver of M has vertices the basis vector of M in normal form, and there is an arrow $v \rightarrow f_{i+1,i}(v)$ if $f_{i+1,i}(v) \neq 0$. Notice that $f_{i+1,i}(v)$ is again a basis vector, by assumption.

Example

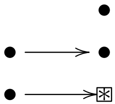
- The coefficient quiver of M is



- which can be rewritten as



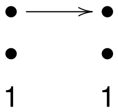
- and its minimal projective resolution looks like



where the $*$ represents the kernel P_2.

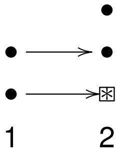
Example

We represent the quiver Grassmannian $Gr_{\mathbf{e}}(M)$ via its "torus fixed points": by putting the number e_i at the bottom of the i-th column of the coefficient quiver of M. For example,



represents the quiver Grassmannian of before.

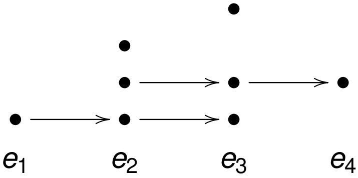
Similarly



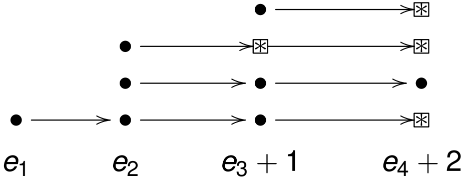
represents $Gr_{(1,1)}(Q_\bullet^M \subseteq P_\bullet^M)$ of before and the $*$ says that the corresponding line "must be taken".

Example

Let us consider the quiver Grassmannian (n=4)



and its corresponding subvariety of flags in K^4



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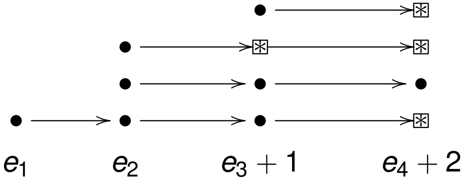
Example

From the picture



one immediately sees that $Gr_{\mathbf{e}}(Q_\bullet^M \subseteq P_\bullet^M)$ is not a B-stable subvariety of $\mathcal{F}l(e_1, e_2, e_3 + 1, e_4 + 2; K^4)$. In particular, it is not clear if its irreducible components share the geometric properties of Schubert varieties (projectively normal, rational singularities, Bott–Samelson resolution..).

Definition

Definition

We say that a quiver Grassmannian $Gr_{\mathbf{e}}(M)$ is Schubert if the corresponding variety $Gr_{\mathbf{e}}(Q_\bullet^M \subseteq P_\bullet^M)$ is stable under a Borel subgroup of $GL(P_n^M)$.

Example

- $Gr_{\mathbf{e}}(Q)$ is Schubert, for any projective A-module Q. Indeed, this is just a variety of partial flags in Q_n. The variety of full flags is $Gr_{(1,2,\cdots,n-1)}(P_1^n)$.
- $Gr_{\mathbf{e}}(E)$ is Schubert, for any injective A-module E.

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Linearly ordered modules

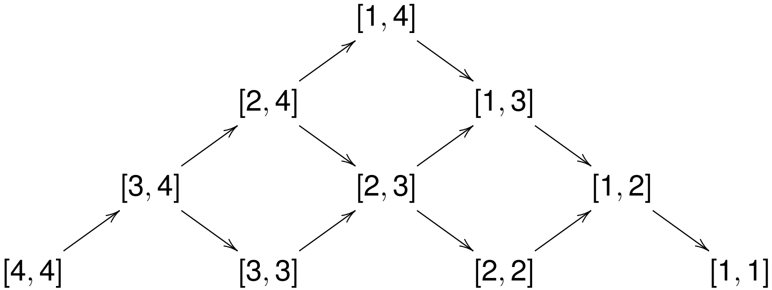
Definition

We say that an A-module whose indecomposable decomposition is of the form

$$M = M(1)^{a_1} \oplus \cdots \oplus M(s)^{a_s}$$

is linearly ordered if $[M(1)], \cdots, [M(s)]$ lie in a same path of the AR-quiver of A.

example



Theorem

Theorem (C. I.- Reineke, 2014)

$Gr_{\mathbf{e}}(M)$ is Schubert if and only if M is linearly ordered.

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Theorem (Evgeny Feigin)

Let $\mathfrak{g} = sl_{n+1}$. Then the complete degenerate flag variety associated with $\mathfrak{g}$ is the projective variety

$$\mathcal{F}l_{n+1} \simeq \{(V_1, \cdots, V_n) \in \prod_{i=1}^{n} Gr_i(K^{n+1}) | \, pr_{i+1} V_i \subseteq V_{i+1}\}$$

where $pr_i : K^{n+1} \to K^{n+1} : \sum a_k v_k \mapsto \sum_{i \neq k} a_k v_k$ is the projection along the i-th basis vector v_i.

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Degenerate Flag variety and quiver Grassmannians

The module over $A = T_n(K)$:

$$M: \; K^{n+1} \xrightarrow{pr_1} K^{n+1} \xrightarrow{pr_2} \cdots \qquad K^{n+1} \xrightarrow{pr_{n-1}} K^{n+1}$$

is nothing but

$$M \simeq P_1 \oplus \cdots P_n \oplus I_1 \oplus \cdots \oplus I_n = {}_AA \oplus D(A_A)$$

Corollary (C.I.- M. Reineke- E. Feigin, 2011)

$\mathcal{F}l_{n+1} \simeq Gr_{\mathbf{dim}_A A}({}_AA \oplus D(A_A))$

Degenerate Flag variety and Schubert quiver Grassmannians

Since $M = {}_AA \oplus D(A_A)$ is clearly linearly ordered, we get another proof of

Theorem (C.I.- M. Lanini, 2014)

Degenerate flag varieties are Schubert varieties.

Self-duality

There is an isomorphism of algebras

$$A \to A^{op} : \; e_{i,j} \mapsto e_{n-j,n-i}$$

which induces an equivalence of categories

$$S : A - mod \to A^{op} - mod$$

By composing with the standard duality
$D : A^{op} - mod \to A - mod$ we get a self-duality

$$\nabla : A - mod \to A - mod$$

$(A - mod, \nabla)$ is hence an Hermitian category in the terminology of W. Scharlau.

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Symplectic Modules

- A symplectic module is a pair (M, Ψ) where
- M is a finite-dimensional A-module
- $\Psi : M \to \nabla M$ is an isomorphism such that $\Psi + \nabla\Psi = 0$

Isotropic quiver Grassmannians

The underlying vector space of a symplectic module (M, Ψ) inherits a symplectic non-degenerate bilinear form $\langle -, - \rangle_\Psi$, "compatible with the action of A".

Definition

Given a dimension vector $\mathbf{e}$ one defines the isotropic quiver Grassmannian $Gr_{\mathbf{e}}^{Iso}(M, \Psi)$, as the subvariety of $Gr_{\mathbf{e}}(M)$ consisting of isotropic sub representations with respect to $\langle -, - \rangle_\Psi$.

Degenerate flag varieties of type C are isotropic quiver Grassmannians.

Why representation theory of quivers helps?

Let us show how having the representation theory of A at disposal can help. For example, let us prove that degenerate flag varieties are flat degeneration of flag varieties: This result was first proved by E. Feigin using explicit computations. Together with M. Reineke and E. Feigin we gave another proof using the interpretation in terms of quiver Grassmannians:

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Let $M = X \oplus Y$ be an A-module, such that $Ext^1(X, Y) = 0 = Ext^1(X, X) = Ext^1(Y, Y)$. Let $\mathbf{d} := \mathbf{dim}\, M$ and $\mathbf{e} := \mathbf{dim}\, X$. Let

$$Rep(\mathbf{d}, A) = \prod_{i=1}^{n-1} Hom_K(K^{d_i}, K^{d_{i+1}})$$

be the affine variety parameterizing A–modules of dimension vector $\mathbf{d}$.

Theorem (C.I.- M. Reineke- E. Feigin, 2011)

Let $\widetilde{M}$ be the generic representation in $Rep(\mathbf{d}, A)$. Then $Gr_{\mathbf{dim}\, X}(M)$ is a flat degeneration of $Gr_{\mathbf{dim}\, X}(\widetilde{M})$

Proof

- Let $Y \subset Rep(\mathbf{d}, A)$ be the open subset consisting of all representations Z whose orbit closure $\bar{\mathcal{O}}_Z$ contains the orbit $\mathcal{O}_M$;
- Let

$$Gr_{\mathbf{e}}^{Q}(\mathbf{d}) = \{((U_i)_{i=1}^{n} \times (f_{(i+1),i}) \in \left(\prod_{i=1}^{n} Gr_{e_i}(K^{d_i})\right) \times Rep(\mathbf{d}, A) | \\ f_{(i+1),i}(U_i) \subseteq U_{i+1}\}$$

- We have two projections:

$$\begin{array}{ccc} & Gr_{\mathbf{e}}^{Q}(\mathbf{d}) & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ \prod_{i=1}^{n} Gr_{e_i}(K^{d_i}) & & Rep(\mathbf{d}, A) \end{array}$$

Proof

- We have two projections:

$$\begin{array}{ccccc} & & Gr_{\mathbf{e}}^{Q}(\mathbf{d}) & & \\ & \swarrow^{p_1} & & \searrow^{p_2} & \\ \prod_{i=1}^{n} Gr_{e_i}(K^{d_i}) & & & & Rep(\mathbf{d}, A) \end{array}$$

- $Gr_{\mathbf{e}}(M) = p_2^{-1}(M)$.
- Let $q : \widetilde{Y} \to Y$ be the restriction of p_2 to $widetildeY := p_2^{-1}(Y)$. This is a proper morphism between two smooth and irreducible varieties.
- The general fiber of q is $Gr_{\mathbf{e}}(\widetilde{M})$, since the orbit of $\widetilde{M}$ is open in Y and the special fiber of q is $Gr_{\mathbf{e}}(M)$, since the orbit of M is closed in Y by definition.

Proof

- By semicontinuity, and an easy dimension estimate, all fibres of q have the same dimension.
- By Matsumura's theorem, a proper morphism between smooth and irreducible varieties with constant fibre dimension is already flat.

Corollary

Degenerate flag varieties are flat degeneration of flag varieties.

Thank you!