Spherical varieties: interactions with representation theory and generalizations Part I

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Outline

What is a spherical G-variety?

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  [Knop]: $\mathcal{V}(X)$ is a finitely generated convex cone in $\Xi(X)^*_{\mathbb{Q}}$:

$$v \longmapsto \varphi_v \quad \text{with} \quad \varphi_v(\lambda) := v(\lambda_f)$$

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- *$\Sigma(X)$ is a subset of $\Sigma(G)$.*
- *$\Sigma(G)$ is finite (resp. infinite) in char 0 (resp. char $p > 0$).*

Classification (char 0: 1983– 2011 // char $p > 0$ ...)

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- char 0: Existence of a classification based on the aforementioned combinatorial invariants (Luna & Vust// Luna, Losev, Bravi, Pezzini, C.-F.).
- char $p \neq 2$: classification of reductive spherical subgroups of a given simple G (Knop & Röhrle, 2014)

Real structures on spherical varieties

$\mathbb{k} = \mathbb{C}$

A *real structure* μ on X is ,

Given $\sigma : G \to G$ an anti-holomorphic involution, a real structure μ is called σ-equivariant if

$$\mu(gx) = \sigma(g)\mu(x) \quad \text{for all } (g, x) \in G \times X.$$

Theorem (Akhiezer & C.-F., 2014)

Let σ define the split real form of G. If $H \subset G$ is spherical and self-normalizing then there exists a unique σ-equivariant real structure on G/H.

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$$\overline{f \circ \mu} \in \mathcal{O}(X) \quad \text{for all } f \in \mathcal{O}(X).$$

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Real parts

Let μ be a σ-equivariant real structure on X. The corresponding real part is the set

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Theorem (Akhiezer & C.-F., 2014)

Let σ define the split real form of G, $H \subset G$ be spherical and self-normalizing and $X = G/H$.

- *The real part of X^{μ} is not empty.*
- *The group G_0^{σ} acts on X^{μ} with finitely many orbits.*
- *There are at most $2^{\mathrm{rank}(X)}$ G_0^{σ}-orbits in X^{μ}.*

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Note: The G_0^{σ}-orbits of X^{μ} are real spherical varieties.

Further results

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- Generalizations of the above theorems to any involution σ (Akhiezer; C.F.).
- In case X is a symmetric space, the G^σ-orbits of X^μ are in $1:1$-correspondence with T^σ/W_X (Borel & Ji).
- Case of rank one real reductive groups $G_\mathbb{R}$: classification of spherical subgroups of $G_\mathbb{R}$ by Knauss & Miebach (2014).

Quasi-projective spherical varieties

Theorem (Vinberg, Kimelfeld & al)

A quasi-projective G-variety X is spherical if and only if the G-module $H^0(X, \mathscr{L})$ is multiplicity-free, $\forall$ G-line bundles $\mathscr{L}$.

Affine spherical varieties

$\mathrm{Char}\mathbb{k} = 0$
Let X be affine and spherical. Then

$$\mathbb{k}[X] = \oplus_{\Gamma(X)} V(\lambda)$$

$\Gamma(X)$ is the *weight monoid of X*.
The *root monoid of X* is the submonoid $R(X)$ of Λ generated by

$$\{\lambda + \mu - \nu : \mathbb{k}[X]_\lambda \cdot \mathbb{k}[X]_\mu \supset \mathbb{k}[X]_\nu \quad (\lambda, \mu, \nu) \in (\Lambda^+)^3\}.$$

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Theorem (Losev)

$$X_1 \simeq_G X_2 \quad \Longleftrightarrow \quad \Gamma(X_1) = \Gamma(X_2) \quad \text{and} \quad \Sigma(X_1) = \Sigma(X_2).$$

Moduli schemes for affine spherical G-varieties with prescribed weight monoid

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Theorem (Alexeev-Brion, 2005)

Equivalence classes of pairs (X, φ), where X is an affine spherical G-variety with weight monoid Γ and φ is a fixed morphism $X//U \to \operatorname{Spec} \Bbbk[\Gamma]$ are parametrized by an affine connected scheme M_Γ.

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- *The G-isomorphism classes of ASV with weight monoid Γ are in bijection with T_{ad}-orbits in M_Γ.*
- *$X_0(\Gamma)$ is the unique fixed T_{ad}-fixed point of M_Γ.*

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Given X with $\Gamma(X) = \Gamma$, the multiplication $m = m(X)$ of X can be written as

$$m = \sum_{\lambda,\mu,\nu} m_{\lambda,\mu}^{\nu} \quad \text{with } (\lambda,\mu,\nu) \in \Gamma^3 \text{ and}$$

$m_{\lambda,\mu}^{\nu} : V(\lambda) \otimes V(\mu) \longrightarrow V(\nu)$ homomorphism of G-modules.

Case of $X_0 = X_0(\Gamma)$: $m(X_0)_{\lambda,\mu}^{\nu} \neq 0 \iff \nu = \lambda + \mu$.

The T_{ad}-action on M_Γ reads as a T-action on the set of multiplication laws m with

$$t.m_{\lambda,\mu}^{\nu} = t^{\lambda+\mu-\nu} m_{\lambda,\mu}^{\nu} \quad t \in T_{\mathrm{ad}}.$$

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Partial results:

- (Jansou, 2007) $\Gamma = \mathbb{N}\lambda \subset \Lambda^+$.
- (Bravi & C.-F., 2008) for Γ free and G-saturated; M_Γ is irreducible.
- (C.-F., 2011) for *canonical* Γ.
- (Bravi & van Steirteghem, 2014) for Γ free.

The tangent space of $T_{X_0} M_\Gamma$

Let $T^1(X_0)$ be the cokernel of

$$H^0(X_0, \mathcal{O}_{X_0} \otimes_{\mathbb{k}} V) \longrightarrow H^0(X_0, \mathcal{N}_{X_0})$$

where $\mathcal{N}_{X_0}$ denotes the normal sheaf of X_0 in V.

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Theorem (Avdeev & C.-F.)

Let Γ be saturated (i.e. normal). The tangent space of $T_{X_0} M_\Gamma$ is a multiplicity-free T_{ad}-module whose weights belong to $-\Sigma(G)$.

Further results

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- Combinatorial description of the irreducible components of M_Γ.
- There exist non-reduced moduli schemes M_Γ.
- New proof of the Uniqueness Theorem for affine spherical varieties.

Moduli scheme of stable projective spherical varieties

$\mathrm{Char}\,\mathbb{k} = 0$

- $(X, \mathscr{L})$ – polarized projective G-variety
- $R(X, \mathscr{L})$ the section ring of X.
- $\hat{X} = \mathrm{Spec}\, R(X, \mathscr{L})$ affine cone over $(X, \mathscr{L})$.

The moment polytope of X is the set $\Gamma(\hat{X}) \cap \left(\{1\} \times \Lambda^+_{\mathbb{R}}\right)$.

Existence of a moduli theory for projective *stable* varieties (Alexeev & Brion).