# Simple Specht Modules and Signed Young Modules

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# Overview

1. Combinatorics
2. (Signed) Young Permutation Modules
3. (Signed) Young Modules
4. Specht Modules
5. The Labelling Formula
6. Ideas of Proof
7. Some Consequences

# Combinatorics

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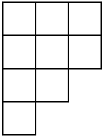
$[(3, 3, 2, 1)] =$  .

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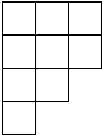
$[(3, 3, 2, 1)] =$  .

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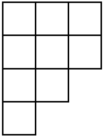

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- $\lambda \in \mathcal{P}(n) \rightsquigarrow \lambda' \in \mathcal{P}(n)$ is such that $[\lambda'] = [\lambda]^T$
- $\lambda \in \mathcal{P}(n) \rightsquigarrow$
  - repeatedly remove rim $p$-**hooks** from $[\lambda]$ to get the $p$-**core** $\tilde{\lambda}$
  - $\tilde{\lambda}$ is $p$-regular and $p$-restricted
  - number of rim $p$-hooks removed to obtain $\tilde{\lambda}$ is called the $p$-**weight** of $\lambda$

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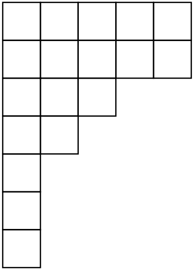

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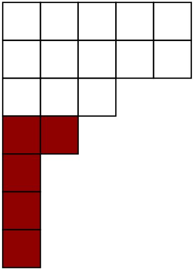

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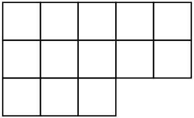

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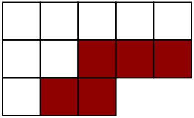

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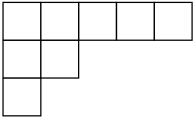

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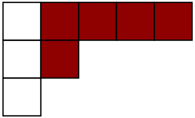

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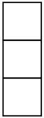

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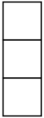

Thus $\lambda$ has 5-core $\tilde{\lambda} = (1^3)$ and 5-weight 3.

# Combinatorics

- $\lambda \in \mathcal{P}(n) \rightsquigarrow |\lambda| := n$
- The **dominance order** $\trianglerighteq$ on $\mathcal{P}(n)$ is the partial order defined by

$$\lambda \trianglerighteq \mu :\Leftrightarrow \sum_{i=1}^{l} \lambda_i \geqslant \sum_{i=1}^{l} \mu_i \,, \text{ for all } l \geqslant 0 \,.$$

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$$(\lambda|\mu) \trianglerighteq (\alpha|\beta) :\Leftrightarrow \sum_{i=1}^{l} \lambda_i \geqslant \sum_{i=1}^{l} \alpha_i \quad \text{and}$$

$$|\lambda| + \sum_{i=1}^{l} \mu_i \geqslant |\alpha| + \sum_{i=1}^{l} \beta_i \,,$$

$$\text{for all } l \geqslant 0 \,.$$

# (Signed) Young Permutation Modules

- $F$ a field of characteristic $p \geqslant 0$
- For $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathcal{P}(n)$, one has the **(standard) Young subgroup**

$$\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_k}$$

of $\mathfrak{S}_n$, and the **Young permutation $F\mathfrak{S}_n$-module**

$$M^\lambda := \mathrm{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}(F)\,.$$

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- If $\zeta = \emptyset$ then $M(\lambda|\zeta) = M^\lambda$.
- The indecomposable direct summands of (signed) Young permutation modules are called **indecomposable (signed) Young modules**. How can these be characterized?

# (Signed) Young Modules

**Theorem (James 1983)**

*The isoclasses of indecomposable Young $F\mathfrak{S}_n$-modules are labelled by $\mathcal{P}(n)$. For $\lambda \in \mathcal{P}(n)$, let $Y^\lambda$ be the corresponding indec. Young module. Then*

$$M^\lambda \cong Y^\lambda \oplus \bigoplus_{\mu \rhd \lambda} m_{\lambda,\mu} Y^\mu ,$$

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**Theorem (Donkin 2001)**

*Let $p \geqslant 3$. The isoclasses of indec. signed Young $F\mathfrak{S}_n$-modules are labelled by the pairs $(\lambda|p\mu) \in \mathcal{P}^2(n)$. If $(\lambda|p\mu) \in \mathcal{P}^2(n)$ then*

$$M(\lambda|p\mu) \cong Y(\lambda|p\mu) \oplus \bigoplus_{(\alpha|p\beta) \rhd (\lambda|p\mu)} m_{(\lambda|p\mu),(\alpha|p\beta)} Y(\alpha|p\beta) ,$$

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# (Signed) Young Modules

- Every Young module is a signed Young module. If $\lambda \in \mathcal{P}(n)$ then $Y^\lambda \cong Y(\lambda|\emptyset)$.
- If $(\lambda|p\mu) \in \mathcal{P}^2(n)$ then $Y(\lambda|p\mu) \otimes \mathrm{sgn}$ is an indecomposable signed Young module, since

$$Y(\lambda|p\mu) \otimes \mathrm{sgn} \mid M(\lambda|p\mu) \otimes \mathrm{sgn} \cong M(p\mu|\lambda)\,.$$

**Problem**

Find $(\alpha|p\beta) \in \mathcal{P}^2(n)$ such that $Y(\lambda|p\mu) \otimes \mathrm{sgn} \cong Y(\alpha|p\beta)$.

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- Given $\lambda \in \mathcal{P}(n)$, when is $S^\lambda$ simple? $\rightsquigarrow$ A combinatorial answer has been
  - given by James–Mathas (1999), for $p = 2$,
  - conjectured by James–Mathas, for $p \geqslant 3$. The proof is due to work of Lyle and Fayers (2003–2005).

# JM-partitions

$F$ a field of characteristic $p \geqslant 3$

- $S^\lambda$ simple Specht $F\mathfrak{S}_n$-module $\rightsquigarrow$ $\lambda$ is called a **JM-partition**
- JM-partitions are best characterized using abacus combinatorics, which we omit here.
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**Lemma (D.–Lim 2015)**

*If $\lambda \in \mathcal{P}(n)$ is a JM-partition then the p-core $\tilde{\lambda}$ can be obtained by removing only vertical and horizontal p-hooks from $[\lambda]$. The procedures of removing horizontal and vertical p-hooks, respectively, are independent of each other.*

# Simple Specht Modules

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**Remark**

Hemmer's result fails for $p = 2$. Suppose that $p = 2$, and let $S^\lambda$ be a simple Specht $F\mathfrak{S}_n$-module. Then, by James–Mathas, one of the following cases occurs:

- $\lambda$ is 2-regular and $S^\lambda \cong Y^\lambda$, or
- $\lambda$ is 2-restricted and $S^\lambda \cong Y^{\lambda'}$, or
- $\lambda = (2, 2)$, but $S^\lambda$ is not isomorphic to an indecomposable Young module.

# Simple Specht Modules

**Problem**

Let $p \geqslant 3$, and let $S^\lambda$ be a simple Specht $F\mathfrak{S}_n$-module. Determine $(\alpha|p\beta) \in \mathcal{P}^2(n)$ such that $S^\lambda \cong Y(\alpha|p\beta)$.

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In the following we shall establish a solution to the above problem. This is joint work with Kay Jin Lim (Singapore).

The combinatorial formula we are going to present has been conjectured, independently, by D. (2007), Lim (2009), Orlob (2009).

# $p$-adic expansion

Let $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathcal{P}(n)$, and let $p$ be any prime. Then $\lambda$ admits a **$p$-adic expansion**

$$\lambda = \sum_{i=0}^{r_\lambda} p^i \cdot \lambda(i)\,,$$

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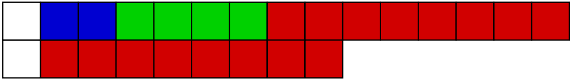
**Example**

$p := 2$, $\lambda := (15, 9) \in \mathcal{P}(24) \rightsquigarrow$



Thus $\lambda = 8 \cdot (1,1) + 4 \cdot (1) + 2 \cdot (1) + (1,1)$, and $\lambda(0) = (1,1)$.

# The Labelling

Let $p$ be a prime, and let $\Phi : \mathcal{P}(n) \to \mathcal{P}^2(n)$ be defined by

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**Theorem (D.–Lim 2015)**

*Let $p \geqslant 3$, let $F$ be a field of characteristic $p$, and let $S^\lambda$ be a simple Specht $F\mathfrak{S}_n$-module. Then $S^\lambda$ is isomorphic to the signed Young module $Y(\Phi(\lambda))$.*

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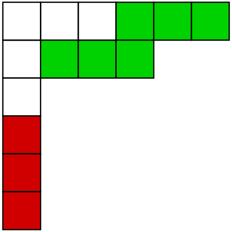
**Example**

$p := 3$, $\lambda := (6, 4, 1^4) \rightsquigarrow \Phi(\lambda) = ((6, 4, 1)|(3)) = ((3, 1^2) + (3^2)|(3))$.



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3. A reduction of the theorem to the case of simple Specht modules belonging to **Rouquier blocks**.
4. A generalization of **Young's Rule**, due to recent work of Lim and Tan. The latter determines the multiplicities of any given Specht $F\mathfrak{S}_n$-module as a factor of some Specht filtration of a signed Young permutation module.

# Relative Projectivity

$F$ a field of characteristic $p > 0$, $G$ any finite group, $FG$ the group algebra of $G$ over $F$

- If $p \nmid |G|$ then every $FG$-module is projective, i.e., a direct summand of a free $FG$-module.

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Let $H \leqslant G$. An $FG$-module $M$ is called **relatively** $H$**-projective** if $M \mid \mathrm{Ind}_H^G(\mathrm{Res}_H^G(M))$.

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**Theorem (J.A. Green 1959)**

*Let $M$ be an indecomposable $FG$-module, and let $P \leqslant G$ be minimal such that $M$ is relatively $P$-projective. Then $P$ is a $p$-group, unique up to $G$-conjugation.*

- One calls $P$ a **Green vertex** of $M$.

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- $N_{\mathfrak{S}_n}(\mathfrak{S}_\rho) \cong (\mathfrak{S}_{p^r} \wr \mathfrak{S}_{n_r}) \times \cdots \times (\mathfrak{S}_p \wr \mathfrak{S}_{n_1}) \times \mathfrak{S}_{n_0}$, and Donkin has given an explicit description of the Young–Green correspondent of $Y(\lambda|p\mu)$ w.r.t. $N_{\mathfrak{S}_n}(\mathfrak{S}_\rho)$.

# Twisting Formula

$F$ a field of characteristic $p > 0$

- $\alpha \in \mathcal{P}(n)$ a $p$-restricted partition, $D_\alpha$ the corresponding simple $F\mathfrak{S}_n$-module $\rightsquigarrow$ $D_\alpha \otimes \text{sgn}$ is also simple $\rightsquigarrow$ exists a $p$-restricted partition $\mathbf{m}(\alpha) \in \mathcal{P}(n)$ with $D_\alpha \otimes \text{sgn} \cong D_{\mathbf{m}(\alpha)}$

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*Let $(\lambda|p\mu) \in \mathcal{P}^2(n)$. Then one has an isomorphism of $F\mathfrak{S}_n$-modules*

$$Y(\lambda|p\mu) \otimes \mathrm{sgn} \cong Y(\mathbf{m}(\lambda(0)) + p\mu|\lambda - \lambda(0)) .$$

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Main idea of the proof: consider $p$-adic expansions of the partitions involved. Then use Donkin's result to show that $Y(\lambda|p\mu) \otimes \mathrm{sgn}$ and $Y(\mathbf{m}(\lambda(0)) + p\mu|\lambda - \lambda(0))$ have a common Young vertex $\mathfrak{S}_\rho$ and isomorphic Young–Green correspondents w.r.t. $N_{\mathfrak{S}_n}(\mathfrak{S}_\rho)$.

# Rouquier Blocks

$F$ a field of characteristic $p > 0$

- Nakayama Conjecture (proved 1947 by Brauer and Robinson): The blocks of $F\mathfrak{S}_n$ are labelled by pairs $(\kappa, w)$, where $\kappa$ is the $p$-core of a partition of $n$ and $w$ is the corresponding $p$-weight.

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- Rouquier blocks are usually better understood than arbitrary blocks of $F\mathfrak{S}_n$.
- Strategy used by Fayers and Hemmer: reduce statements about simple Specht modules to simple Specht modules belonging to Rouquier blocks.

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**Proposition**

*Let $p \geqslant 3$, and let $S^\lambda$ be a simple Specht $F\mathfrak{S}_n$-module. Then there is an $m \geqslant n$ and a simple Specht $F\mathfrak{S}_m$-module $S^\mu$ belonging to a Rouquier block such that*

$$S^\mu \mid \mathrm{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_m}(S^\lambda) \quad \textit{and} \quad S^\lambda \mid \mathrm{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_m}(S^\mu)\,.$$

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### Proposition (D.–Lim 2015)

*The theorem on signed Young module labels of simple Specht modules holds if it holds for simple Specht modules belonging to Rouquier blocks.*

# Specht Filtrations

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- An $F\mathfrak{S}_n$-module $M$ is said to admit a **Specht filtration** if there is a sequence of submodules

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**Lemma (D.–Lim 2015)**

*Let $p > 2$, and let $M$ be an $F\mathfrak{S}_n$-module admitting a Specht filtration. If $S^\lambda$ is a simple Specht $F\mathfrak{S}_n$-module with $S^\lambda \mid M$ then every Specht filtration of $M$ has a factor isomorphic to $S^\lambda$.*

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- Now use $S^\lambda \mid M(\alpha | p\beta)$ and Lim–Tan's twisted Young's Rule to obtain equality.

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# Some Consequences

Let $F$ be a field of characteristic $p \geqslant 3$. We summarize some $p$-local invariants of indecomposable signed Young modules. Our Labelling Formula then allows us to determine these invariants for every simple Specht $F\mathfrak{S}_n$-module as well.

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**(c)** *if $\mathfrak{S}_\rho$ is a Young vertex of $Y$ and $P_\rho \in \mathrm{Syl}_p(\mathfrak{S}_\rho)$ then $P_\rho$ is a Green vertex of $Y$. Moreover, $Y$ has trivial Green sources.*

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Let $F$ be a field of characteristic $p \geqslant 3$. We summarize some $p$-local invariants of indecomposable signed Young modules. Our Labelling Formula then allows us to determine these invariants for every simple Specht $F\mathfrak{S}_n$-module as well.

**Theorem (D.–Lim 2015)**

*Let $(\lambda|p\mu) \in \mathcal{P}^2(n)$, and let $Y := Y(\lambda|p\mu)$. Then*

**(a)** *$Y$ has complexity $|\mu| + (|\lambda| - |\lambda(0)|)/p$;*

**(b)** *$Y$ is*

- **(i)** *projective iff $\mu = \emptyset$ and $\lambda$ is p-restricted;*
- **(ii)** *non-projective periodic iff either $\mu = (1)$ and $\lambda$ is p-restricted, or $\mu = \emptyset$ and $\lambda - \lambda(0) = (p)$; in this case $Y$ has period $2p - 2$;*

**(c)** *if $\mathfrak{S}_\rho$ is a Young vertex of $Y$ and $P_\rho \in \mathrm{Syl}_p(\mathfrak{S}_\rho)$ then $P_\rho$ is a Green vertex of $Y$. Moreover, $Y$ has trivial Green sources. If $L$ is the Young–Green correspondent of $Y$ w.r.t. $N_{\mathfrak{S}_n}(\mathfrak{S}_\rho)$ then $\mathrm{Res}^{N_{\mathfrak{S}_n}(\mathfrak{S}_\rho)}_{N_{\mathfrak{S}_n}(P_\rho)}(L)$ is the Green correspondent of $Y$ w.r.t. $N_{\mathfrak{S}_n}(P_\rho)$.*