Categorical actions on unipotent representations of finite classical groups

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Motivation

Finite classical groups

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- Decomposition numbers

Unipotent characters of $\mathrm{GL}_n(q)$

There is a distinguished set of complex irreducible characters of $\mathrm{GL}_n(q)$, called the *unipotent characters*, labelled by partitions

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In fact there is a Hecke algebra $\mathcal{H}_q(A_{n-1})$ hiding there...

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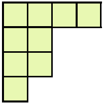
f  $=$

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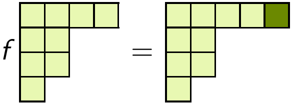

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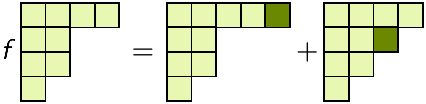

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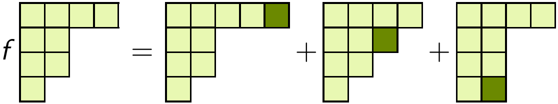

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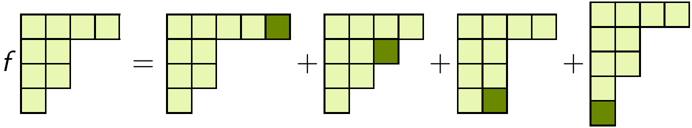

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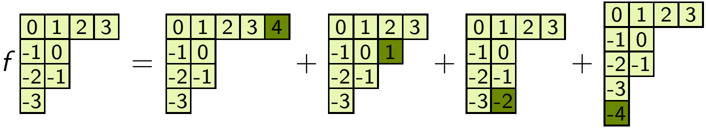

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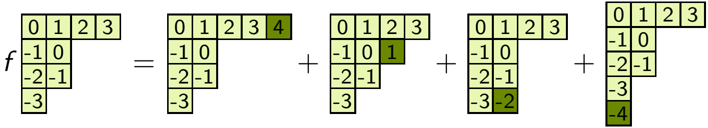

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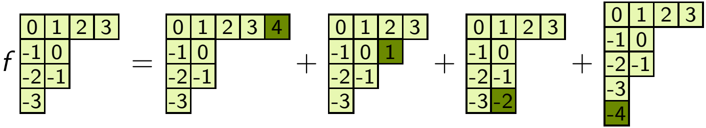

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$\langle e_a, f_a \rangle_{a\in\mathbb{Z}}$ induce an action of $\mathfrak{sl}_\infty$ on $\bigoplus \mathbb{C}\mathrm{Uch}(\mathrm{GL}_n(q))$ isomorphic to a Fock space representation of level 1.

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Through the decomposition map $\mathrm{Uch}(\mathrm{GL}_n(q)) \longrightarrow K_0(k\mathrm{GL}_n(q)\text{-mod}^{\mathrm{u}})$, the action of $\langle e_{\bar{a}}, f_{\bar{a}} \rangle_{\bar{a} \in \mathbb{Z}/d\mathbb{Z}}$ induces an action of $\widehat{\mathfrak{sl}}_d$ on

$$K_0(\mathcal{C}) = \bigoplus_{n \geq 0} K_0(k\mathrm{GL}_n(q)\text{-mod}^{\mathrm{u}})$$

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Chuang-Rouquier

Elements of $\widehat{\mathfrak{S}}_d$ lift to derived equivalences of $\mathcal{C}$. In particular, two blocks with same defect are derived equivalent.

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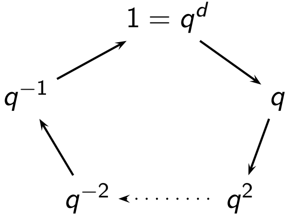


Decategorifying

Draw the quiver with

- vertices: eigenvalues of X
- edges: $a \longrightarrow qa$

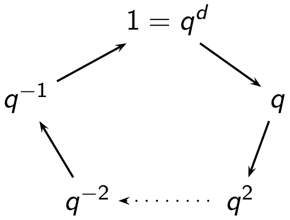
$\rightsquigarrow$ the Kac-Moody algebra associated to this quiver acts on $K_0(\mathcal{C})$

For parabolic induction and restriction we get

in char 0 $\quad \cdots q^{-2} \longrightarrow q^{-1} \longrightarrow 1 \longrightarrow q \longrightarrow q^2 \cdots \quad \mathfrak{sl}_\infty$

$$\Downarrow$$

in char ℓ $\quad \widehat{\mathfrak{sl}}_d$



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But branching rules are different!

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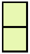
$e = [E]$ and $f = [F]$ act by removing/adding 2-hooks  or  on unipotent characters

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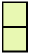
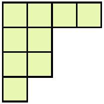
$e = [E]$ and $f = [F]$ act by removing/adding 2-hooks  or  on unipotent characters

f  $=$

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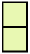
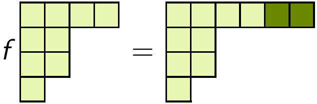
$e = [E]$ and $f = [F]$ act by removing/adding 2-hooks  or  on unipotent characters



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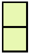
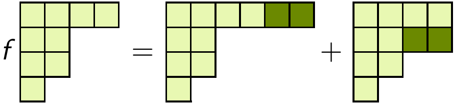
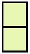
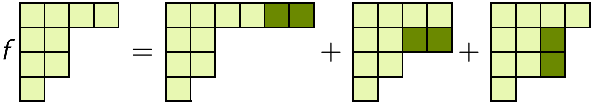
$e = [E]$ and $f = [F]$ act by removing/adding 2-hooks  or  on unipotent characters



$e = [E]$ and $f = [F]$ act by removing/adding 2-hooks  or  on unipotent characters



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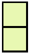
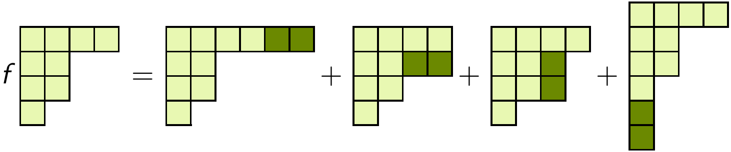
$e = [E]$ and $f = [F]$ act by removing/adding 2-hooks  or  on unipotent characters



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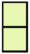
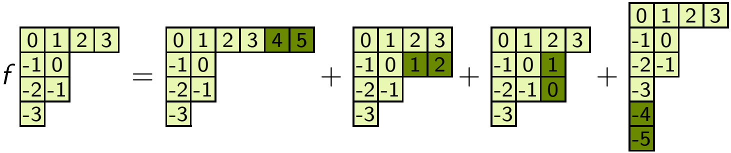
$e = [E]$ and $f = [F]$ act by removing/adding 2-hooks  or  on unipotent characters



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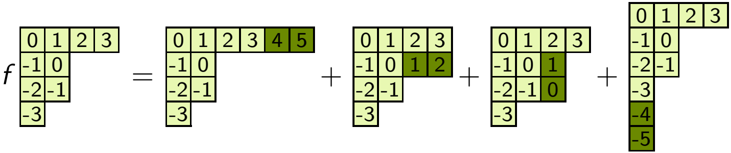

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Up to a sign normalization,

$$f_a = [f_a^{(\mathrm{GL})}, f_{-qa}^{(\mathrm{GL})}]$$

and $\langle e_a, f_a\rangle_{a\in\mathbb{Z}}$ induce an action of $\mathfrak{sl}_\infty^{\oplus 2}$ on $\bigoplus \mathbb{C}\mathrm{Uch}(\mathrm{GU}_n(q))$ isomorphic to a Fock space representation of level 2.

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f_a is constructed using the $(-q)^a$-eigenspace of a Jucys-Murphy element

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Moreover, weight spaces coincide with blocks.

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\mathcal{H}_{q^{2t+1},q^2}(B_m) = \langle X_1, T_1, \ldots, T_{m-1}\rangle & \xrightarrow{\sim} & \mathrm{End}_{\mathrm{GU}_{n+m}(q)}(F^m\chi_\mu)
\end{array}$$

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+ work in progress for type B and C (d odd)

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For $\mathrm{GL}_n(q)$, and for $\mathrm{GU}_n(q)$ with d even, the basis of simple modules in $K_0(\mathcal{C})$ corresponds to the (dual) canonical basis of the Fock spaces.

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Question

Is there any Lie-theoretic interpretation of the basis of simple modules when d is odd?