Categorification and Quantum Symmetric Pairs

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Schur-Weyl duality

Schur-Weyl duality

$$V_m$$

- V_m a complex vector space of dimension m

Schur-Weyl duality

$$\mathfrak{gl}_m \curvearrowright V_m$$

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- $\mathfrak{gl}_m$ the Lie algebra of endomorphisms of V_m

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- V_m a complex vector space of dimension m
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- S_n the symmetric group on n symbols
- $\mathcal{U}(\mathfrak{gl}_m)$ and S_n enjoy a double centralizing property

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$\rightsquigarrow$ categorify this set-up

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The category

$\mathcal{O}(n)$ the BGG category $\mathcal{O}$ of $\mathfrak{gl}_n$ with modules having integral weights

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- M admits a weight space decomposition for a fixed Cartan subalgebra with integral weights
- M is locally finite with respect to a fixed Borel subalgebra

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where the blocks are indexed by S_n orbits in $\mathbb{Z}^n$.

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Define $X(n,m) = \{\lambda \in \mathbb{Z}^n | 1 \leq \lambda_i \leq m\}$

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Let $\{v_1, \ldots, v_m\} \subset V_m$ be a basis with $\mathrm{wt}(v_i) = \varepsilon_i$.

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Let $\{v_1, \ldots, v_m\} \subset V_m$ be a basis with $\mathrm{wt}(v_i) = \varepsilon_i$. Then

$$\begin{array}{rcl} K_0(\mathcal{O}_{\leq m}) \otimes_{\mathbb{Z}} \mathbb{Q}(q) & \xrightarrow{\cong} & V_m^{\otimes n} \\ [\Delta(\lambda)] & \longmapsto & v_{\lambda_1} \otimes \ldots \otimes v_{\lambda_n} \end{array}$$

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Why is this a good candidate for a categorification?

$$\mathcal{U}_q(\mathfrak{gl}_m) \curvearrowright \underset{\text{(a)}}{V_m^{\otimes n}} \curvearrowleft \mathcal{H}_q(S_n)$$

$$\left\{ \begin{array}{c} S_n\text{-orbits in} \\ X(n,m) \end{array} \right\} \stackrel{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{compositions of } n \text{ into} \\ m \text{ pieces (with 0 allowed)} \end{array} \right\} =: C(n,m)$$

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Using this it follows that for $\underline{\mu} \in C(n,m)$

$$K_0(\mathcal{O}_{\underline{\mu}}) \otimes_{\mathbb{Z}} \mathbb{Q}(q) \quad \overset{\cong}{\longrightarrow} \quad \left(V_m^{\otimes n}\right)_{\mu_1\varepsilon_1+\ldots+\mu_m\varepsilon_m}$$

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Problem:

The q has no categorical meaning!

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Solution:

Replace $\mathcal{O}(n)$ by its graded analogue.

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Replace $\mathcal{O}(n)$ by its graded analogue. This is done block-wise:

$$\mathcal{O}_{\underline{\mu}} \cong A_{\underline{\mu}} - \mathrm{mod}$$

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Letting q act by the grading shift turns $K_0(\mathcal{O}_{\underline{\mu}}^{\mathrm{gr}})$ into a $\mathbb{Z}[q, q^{-1}]$-module.

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To make everything precise, all categories and functors need to be replaced by their graded analogues.

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$$\begin{aligned} \mathcal{F} &:= \mathrm{pr}_{\leq m} \circ (? \otimes L(\varepsilon_1)) \\ \mathcal{E} &:= \mathrm{pr}_{\leq m} \circ (? \otimes L(\varepsilon_1)^*) \end{aligned}$$

where $L(\varepsilon_1)$ is the vector representation of $\mathfrak{gl}_n$ and $\mathrm{pr}_{\leq m}$ is the projection onto $\mathcal{O}_{\leq m}$.

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These are exact functors, whose graded lifts are biadjoint up to a grading shift.

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Furthermore they can be decomposed

$$\begin{aligned} \mathcal{F} &= \bigoplus_{i=1}^{m-1} \mathcal{F}_i, \quad \text{with } \mathcal{F}_i \circ \mathrm{pr}_{\underline{\mu}} = \mathrm{pr}_{\underline{\mu}+\varepsilon_{i+1}-\varepsilon_i} \circ \mathcal{F} \circ \mathrm{pr}_{\underline{\mu}} \\ \mathcal{E} &= \bigoplus_{i=1}^{m-1} \mathcal{E}_i, \quad \text{with } \mathcal{F}_i \circ \mathrm{pr}_{\underline{\mu}} = \mathrm{pr}_{\underline{\mu}-\varepsilon_{i+1}+\varepsilon_i} \circ \mathcal{E} \circ \mathrm{pr}_{\underline{\mu}} \end{aligned}$$

$$\mathcal{U}_q(\mathfrak{gl}_m) \underset{(b)}{\curvearrowright} \mathcal{O}_{\leq m} \curvearrowleft \mathcal{H}_q(S_n)$$

Functors $\{\mathcal{D}_j^{\pm 1}\}_{1 \leq j \leq m}$ can be defined by fixed grading shifts on each block.

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$$\{\mathcal{F}_i\}_{1 \leq i \leq m-1}, \ \{\mathcal{E}_i\}_{1 \leq i \leq m-1}, \ \{\mathcal{D}_j^{\pm 1}\}_{1 \leq j \leq m}$$

satisfy the defining relations of $\mathcal{U}_q(\mathfrak{gl}_m)$.

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Theorem

The functors $\mathcal{F}_i$, $\mathcal{E}_i$, $\mathcal{D}_j$ give a (weak) categorification of part (b).

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Theorem

The functors $\mathcal{F}_i$, $\mathcal{E}_i$, $\mathcal{D}_j$ give a (weak) categorification of part (b).

Remark

For a strong categorification in the sense of Khovanov-Lauda or Rouquier one has to show that this really provides an action of the full 2-category $\dot{\mathcal{U}}$ corresponding to $\mathcal{U}_q(\mathfrak{gl}_m)$.

$$\mathcal{E}_i, \mathcal{F}_i \curvearrowright \mathcal{O}_{\leq m} \underset{(c)}{\curvearrowleft} \mathcal{H}_q(S_n)$$

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$\{H_i\}_{1 \leq j < n}$ Coxeter generators

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$\{H_i\}_{1 \leq j < n}$ Coxeter generators $\quad \rightsquigarrow \quad \{\mathcal{T}_i\}_{1 \leq j < n}$ twisting functors

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$\{\underline{H}_i\}_{1 \leq j < n}$ the KL generators

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$\{\underline{H}_i\}_{1 \leq j < n}$ the KL generators $\rightsquigarrow$ $\{\mathcal{Z}_i\}_{1 \leq j < n}$ Zuckerman functors

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Both sets of functors are not exact, so we have to pass to the derived category.

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Remark

As before one can also pass to a 2-category to get a stronger type of categorification.

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The definition of $\mathcal{O}^{\underline{k}}$ is exactly the same as $\mathcal{O}(n)$, except that we impose that $\mathfrak{p}_{\underline{k}}$ acts locally finite.

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$$K_0\left(\bigoplus_{\underline{\mu}\in C(n,m)} \mathcal{O}^{\underline{k}}_{\underline{\mu}}\right) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q) \cong \bigwedge^{\underline{k}} V_m$$

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$$\mathcal{E}_i, \mathcal{F}_i \curvearrowright \bigoplus_{\substack{\underline{k}\in C(n,r)\\ \underline{\mu}\in C(n,m)}} \mathcal{O}^{\underline{k}}_{\underline{\mu}} \stackrel{?}{\longleftrightarrow} \bigoplus_{\substack{\underline{k}\in C(n,r)\\ \underline{\mu}\in C(n,m)}} \mathcal{O}^{\underline{\mu}}_{\underline{k}} \curvearrowleft \mathcal{E}_i^\vee, \mathcal{F}_i^\vee$$

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Derive both sides and use Koszul duality.

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Beilinson-Ginzburg-Soergel, Backelin: This is an equivalence.

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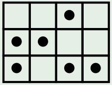
$[\Delta^{\underline{k}}(\lambda)] \rightsquigarrow$ 

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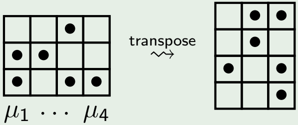

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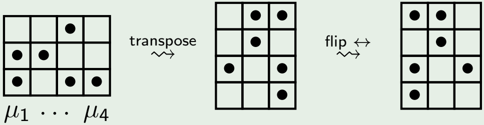

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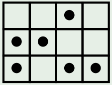
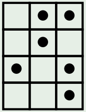
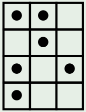
$[\Delta^{\underline{k}}(\lambda)] \rightsquigarrow$  $\mu_1 \cdots \mu_4$ $\overset{\text{transpose}}{\rightsquigarrow}$  $\overset{\text{flip} \leftrightarrow}{\rightsquigarrow}$  $\rightsquigarrow [K\Delta^{\underline{k}}(\lambda)]$

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$$X(n, m) = \{\lambda \in (\mathbb{Z} + \tfrac{1}{2})^n | \tfrac{1}{2} \leq |\lambda_i| \leq m - \tfrac{1}{2}\}$$

is stable under the Weyl group.

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The Weyl group $W(D_n)$ acts by permutations and even number of sign changes, hence

$$X(n, m) = \{\lambda \in (\mathbb{Z} + \tfrac{1}{2})^n | \tfrac{1}{2} \leq |\lambda_i| \leq m - \tfrac{1}{2}\}$$

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Other types:

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Define

$$\mathcal{O}_{\leq m} = \bigoplus_{\substack{\underline{\mu} \in C(n,m) \\ \varepsilon \in \mathbb{Z}/2\mathbb{Z}}} \mathcal{O}_{\underline{\mu},\varepsilon}$$

Twisting functors exists as before, now for the generators of the Weyl group of type D_n, thus $D^b(\mathcal{O}_{\leq m})$ gives a categorification of

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$B(m, \theta)$ is the quantum group analogue of the fixed points Lie subalgebra $\mathfrak{gl}_m \times \mathfrak{gl}_m \subset \mathfrak{gl}_{2m}$. It is a (right) coideal subalgebra of $\mathcal{U}_q(\mathfrak{gl}_{2m})$.

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Can we also get a skew Howe duality?

$$\mathcal{U}_q(\mathfrak{gl}_{2m}) \quad \curvearrowright \quad \bigwedge^n(V_{2m} \otimes V_r) \quad \curvearrowleft \quad \mathcal{U}_q(\mathfrak{gl}_r)$$

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To categorify $\bigwedge^n(V_{2m}\otimes V_r)$ we use the same approach with parabolic category $\mathcal{O}$ as before and decompose this with respect to the action of $\mathcal{U}_q(\mathfrak{gl}_{2m})$.

$$
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Theorem [E.-Stroppel]

The $\mathcal{B}_i$'s and $\bigoplus\mathcal{O}^{\underline{k}}_{\underline{\mu},\varepsilon}$ categorify the action of $B(m,\theta)$ on $\mathbb{V}$.

To obtain both actions at the same time we again use Koszul duality.

$$\bigoplus_{\substack{\underline{k}\in C(n,r)\\ \underline{\mu}\in C(n,m),\varepsilon\in\mathbb{Z}/2\mathbb{Z}}} D^b(\mathcal{O}^{\underline{k}}_{\underline{\mu},\varepsilon}) \xrightarrow{K} \bigoplus_{\substack{\underline{k}\in C(n,r)\\ \underline{\mu}\in C(n,m),\varepsilon\in\mathbb{Z}/2\mathbb{Z}}} D^b(\mathcal{O}^{\mu}_{\underline{k},\varepsilon})$$

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- Schur-Weyl duality between Brauer algebras and $\mathfrak{osp}(m|2k)$ Lie superalgebras in characteristic $\neq 2$ [E.-Stroppel, Lehrer-Zhang].

Thank You!