Representation type, boxes, and Schur algebras

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Notation

- k algebraically closed field
- $\operatorname{char} k = p \geq 0$
- A finite dimensional k-algebra
- $\operatorname{mod} A$ category of finite dimensional (left) A-modules
- $M \in \operatorname{mod} A \rightsquigarrow [M]$, the isomorphism class of M
- $\operatorname{ind} A = \{[M] \mid M \in \operatorname{mod} A \text{ indecomposable}\}$
- $\operatorname{ind}_d A = \{[M] \mid M \in \operatorname{ind} A, \dim M = d\}$

Finite representation type

Definition

An algebra A is called **representation-finite** if $|\operatorname{ind} A| < \infty$. Otherwise it is called **representation-infinite**.

Examples

- semisimple algebras are representation-finite
- kC_n is representation-finite
- $k(C_2 \times C_2)$ is representation-infinite for char $k = 2$

Schur algebras

Let $V = k^n$.

$$\mathrm{GL}_n \curvearrowright V^{\otimes d} \curvearrowleft \Sigma_d$$

The Schur algebra occurs in two ways:
$S(n,d) = \mathrm{End}_{k\Sigma_d}(V^{\otimes d}) = \mathrm{Im}(k\,\mathrm{GL}_n \to \mathrm{End}(V^{\otimes d}))$

Theorem (Schur 1901)

There is an idempotent $e \in S(n,d)$, such that

$$\begin{aligned} \operatorname{mod} S(n,d) &\to \operatorname{mod} eS(n,d)e \cong \operatorname{mod} k\Sigma_d \\ M &\mapsto eM \end{aligned}$$

is an equivalence for $\operatorname{char} k = 0$ *and* $n \geq d$.

For $n \geq d$: $\operatorname{mod} S(n,d) \cong$ category of strict polynomial functors

Representation-finite blocks of Schur algebras

Theorem (Xi '92–'93, Erdmann '93, Donkin, Reiten '94)

The Schur algebra $S(n, d)$ has finite representation type exactly in the following cases:

1. $n \geq 3, d < 2p$
2. $n = 2, d < p^2$
3. $n = 2, d = 5, 7$

The representation-finite blocks of $S(n, d)$ are Morita equivalent to the path algebra of

$$1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \cdots \underset{\beta_{m-1}}{\overset{\alpha_{m-1}}{\rightleftarrows}} m$$

with relations $\alpha_{i-1}\beta_{i-1} = \beta_i\alpha_i, \alpha_{m-1}\beta_{m-1}, \alpha_i\alpha_{i-1}, \beta_{i-1}\beta_i$

Tame and wild for A representation-infinite

Definition

An algebra A is called **tame** if for each dimension d there exist finitely many A-$k[x]$-bimodules $N_1^{(d)}, \ldots, N_{m(d)}^{(d)}$, finitely generated as $k[x]$-modules, such that

$$\operatorname{ind}_d A \subseteq \left\{ [N_i^{(d)} \otimes X] \,|\, X \in \operatorname{mod} k[x], i = 1, \ldots, m(d) \right\}.$$

Definition

An algebra A is called **wild** if there is an A-$k\langle x, y\rangle$-bimodule N such that N is finitely generated projective as a $k\langle x, y\rangle$-module and

$$N \otimes_{k\langle x,y\rangle} - \colon \operatorname{mod} k\langle x, y\rangle \to \operatorname{mod} A$$

preserves indecomposability and isomorphism classes.

Tame-wild dichotomy

Theorem (Drozd '80)

Any finite dimensional algebra is either representation-finite, tame or wild.

Tame Schur algebras

Theorem (Doty, Erdmann, Martin, Nakano '99)

The Schur algebra $S(n, d)$ is tame exactly in the following cases:

1. $p = 2$, $n = 2$, $d = 4, 9$
2. $p = 3$, $n = 3$, $d = 7$
3. $p = 3$, $n = 3$, $d = 8$
4. $p = 3$, $n = 2$, $d = 9, 10, 11$

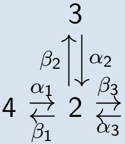
Blocks of tame Schur algebras

Theorem (Continued)

Their blocks are Morita equivalent to the following quivers with relations:

1. $3 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 1 \underset{\alpha_2}{\overset{\beta_2}{\rightleftarrows}} 2$, $\quad \beta_1\alpha_1 = \beta_2\alpha_2 = \alpha_2\beta_2\alpha_1 = \beta_1\alpha_2\beta_2 = 0$

2. $4 \underset{\beta_3}{\overset{\alpha_3}{\rightleftarrows}} 3 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} 2 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 1$, $\quad \beta_3\alpha_3 = \alpha_2\alpha_3 = \beta_3\beta_2 = 0,$
   $\alpha_3\beta_3 = \beta_2\alpha_2,\ \alpha_2\beta_2 = \beta_1\alpha_1$



3. $\beta_1\alpha_1 = \beta_1\alpha_2 = \beta_1\alpha_3 = 0,$
   $\beta_2\alpha_1 = \beta_2\alpha_2 = \beta_3\alpha_1 = 0,$
   $\alpha_3\beta_3 = \alpha_1\beta_1 + \alpha_2\beta_2$

4. $\beta_1\alpha_1 = \beta_2\alpha_2 = \beta_1\alpha_3 = 0$
   $\beta_2\alpha_3 = \beta_3\alpha_1 = \beta_3\alpha_2 = 0,$
   $\alpha_2\beta_2 = \alpha_3\beta_3$

Representation type of subcategories

Let A be a finite dimensional algebra.

Question

1. Under which conditions does there exist a tame-wild dichotomy theorem for $\mathcal{C} \subset \operatorname{mod} A$?
2. Is a particular $\mathcal{C}$ representation-finite, tame or wild?

Quasi-hereditary algebras

Definition

An algebra is called **quasi-hereditary** if there exist modules $\Delta(i)$ with $\mathrm{End}(\Delta(i)) = k$ and $\mathrm{Ext}^s(\Delta(i), \Delta(j)) \neq 0 \Rightarrow i < j$ and $A \in \mathcal{F}(\Delta)$, where
$\mathcal{F}(\Delta) := \{N \mid 0 = N_0 \subset N_1 \subset \cdots \subset N_t = N, \quad N_i/N_{i-1} \cong \Delta(j_i)\}$.

Example

The Schur algebra $S(n, d)$ is quasi-hereditary.
The $\Delta(i)$ are the **Weyl modules**.

Theorem (Hemmer, Nakano '04)

For char $k = p > 3$ and $n \geq d$ the Schur functor restricts to an equivalence $\mathcal{F}(\Delta) \to \mathcal{F}(S)$, where S are the Specht modules.

Filtered representation type

Question

1. Does there exist a tame-wild dichotomy theorem for $\mathcal{F}(\Delta) \subseteq \operatorname{mod} A$?
2. For a particular class of quasi-hereditary algebras, is $\mathcal{F}(\Delta)$ representation-finite, tame, or wild?

Differential biquivers - Motivation

Let $A = \begin{pmatrix} k & 0 \\ k & k \end{pmatrix} \cong k(\ 1 \xrightarrow{\ a\ } 2\ )$. Representations of A:

$$\begin{array}{ccc} V_1 & \xrightarrow{V_a} & V_2 \\ \downarrow{\scriptstyle \omega_1} & & \downarrow{\scriptstyle \omega_2} \\ W_1 & \xrightarrow{W_a} & W_2 \end{array} \qquad\qquad \begin{array}{ccc} 0 & \longrightarrow & k \\ \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ k & \xrightarrow{\mathrm{id}} & k \end{array}$$

Indecomposable representations $0 \to k$, $k \overset{\mathrm{id}}{\to} k$, $k \to 0$

The following also describes mod A:

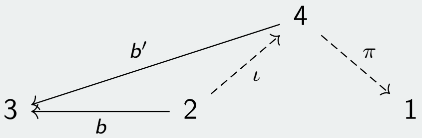
$$\begin{array}{ccccc} V_2' & & V_3' & & V_1' \\ \downarrow{\scriptstyle \omega_2} & \overset{\iota}{\searrow} & \downarrow{\scriptstyle \omega_3} & \overset{\pi}{\searrow} & \downarrow{\scriptstyle \omega_1} \\ W_2' & & W_3' & & W_1' \end{array} \qquad \begin{array}{ccccc} k & & 0 & & 0 \\ \downarrow & \overset{\mathrm{id}}{\searrow} & \downarrow & \searrow & \downarrow \\ 0 & & k & & 0 \end{array}$$

Differential biquivers

Definition

1. A **biquiver** is a quiver with two types of arrows, **solid** (deg $= 0$) and **dashed** (deg $= 1$).
2. A **differential biquiver** is a biquiver together with a linear map $\partial\colon kQ \to kQ$ of degree 1 such that $\partial(e_i) = 0$ for all i and ∂ is a differential, i.e. $\partial^2 = 0$ and $\partial(ab) = \partial(a)b + (-1)^{\deg a} a\partial(b)$.

Example



$$\partial(b) = b'\iota$$

Representations of differential biquivers

Definition

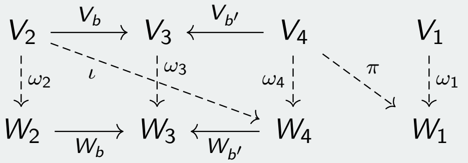
The category of **representations of a differential biquiver** is given as follows:

Objects: Representations of the solid part.

Morphisms: Pairs $((\alpha_i)_{i\in Q_0}, (\alpha_\varphi)_{\varphi \text{ dashed}})$ such that commutativity deformed by ∂(solid) holds.

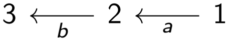
Composition: Given by ∂(dashed arrows)

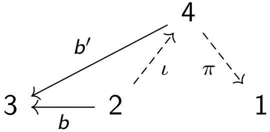
Example



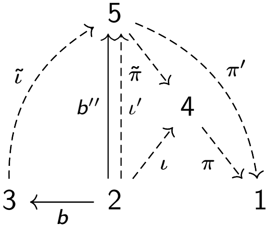
$$\omega_3 V_{b'} = W_{b'}\omega_4$$
$$\omega_3 V_b = W_b\omega_2 + W_{b'}\iota$$

Example of the reduction algorithm

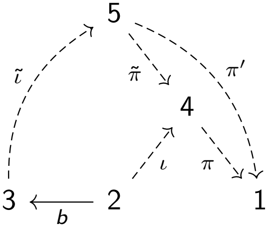




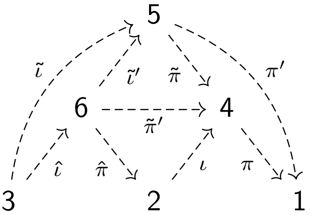
$\partial(b) = b'\iota$



$\partial(b'') = \iota', \partial(\pi') = \pi\tilde{\pi}$



$\partial(\pi') = \pi\tilde{\pi}$



$\partial(\pi') = \pi\tilde{\pi}, \partial(\tilde{\iota}) = \tilde{\iota}'\hat{\iota}$

$\partial(\tilde{\pi}') = \tilde{\pi}\tilde{\iota}' + \tilde{\iota}\hat{\pi}$

Passing to a hereditary situation

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Definition

Let $\mathcal{P}(A)$ be the category with

objects $P \xrightarrow{f} Q$, where $P, Q \in \operatorname{proj} A$

morphisms
$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ P' & \xrightarrow{f'} & Q' \end{array}$$
with $\beta f = f'\alpha$

Let $\mathcal{P}^1(A)$ be the full subcategory with $\operatorname{Im} f \subseteq \operatorname{rad} Q$.

Proposition

The functor $\operatorname{Coker}\colon \mathcal{P}^1(A) \setminus \langle (P \to 0) \rangle \to \operatorname{mod} A$, $f \mapsto \operatorname{Coker} f$ *is full, dense, preserves indecomposability and isomorphism classes.*

Transforming $\mathcal{P}(A)$ to a differential biquiver

Define the following biquiver:

- $2n$ vertices i and $\bar{i}$
- Solid arrow $i \to \bar{j}$ for each basis vector of $\operatorname{rad}_A(P_i, P_j)$
- Dashed arrow $i \dashrightarrow j$ for each basis vector of $\operatorname{rad}_A(P_i, P_j)$

In $\mathcal{P}^1(A)$:

objects: $P \to Q = \bigoplus P_i^{n_i} \xrightarrow{A_{ij}} \bigoplus P_j^{m_j}$

morphisms:
$$\begin{array}{ccc} P & \xrightarrow{\ f\ } & Q \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ P' & \xrightarrow{\ f'\ } & Q' \end{array}$$

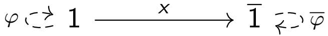
In $\operatorname{rep}(Q, \partial)$:

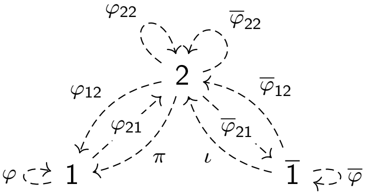
objects: $(k^{n_i}$ on $i) \xrightarrow{A_{ij}} (k^{m_j}$ on $\bar{j})$,

morphisms:
$$\begin{array}{ccc} k^{n_i} & \xrightarrow{\ A_{ij}\ } & k^{m_j} \\ \downarrow{\scriptstyle \alpha_{i\tilde{i}}} & & \downarrow{\scriptstyle \beta_{j\tilde{j}}} \\ k^{n_{\tilde{i}}} & \xrightarrow{\ A'_{\tilde{i}\tilde{j}}\ } & k^{m_{\tilde{j}}} \end{array}$$

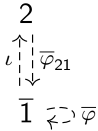
Example of the reduction algorithm

$k[x]/(x^2) \cong kC_2$





$$\begin{aligned}
\partial(\varphi) &= \iota\varphi_{21}\\
\partial(\overline{\varphi}) &= -\overline{\varphi}_{21}\iota\\
\partial(\varphi_{22}) &= -\varphi_{21}\pi\\
\partial(\varphi_{12}) &= -\varphi\pi + \pi\varphi_{22}\\
\partial(\overline{\varphi}_{22}) &= \iota\overline{\varphi}_{21}\\
\partial(\overline{\varphi}_{12}) &= \iota\overline{\varphi} - \overline{\varphi}_{22}\iota
\end{aligned}$$



Directed biquivers

Theorem (Koenig-K-Ovsienko '14)

For every quasi-hereditary algebra A, there exists a directed biquiver (with relations) (Q, I, ∂) with

$$\mathcal{F}(\Delta) \cong \mathrm{rep}(Q, I, \partial).$$

No relations if and only if A is **strongly quasi-hereditary**, *i.e.* $\operatorname{projdim} \Delta(i) \leq 1$ *for all i.*

Corollary

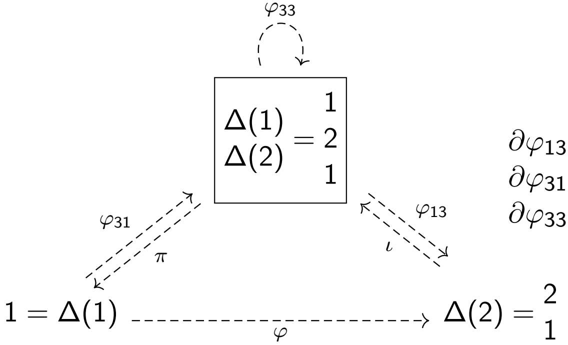
For every strongly quasi-hereditary algebra A, $\mathcal{F}(\Delta)$ is either representation-finite, tame or wild.

A strongly quasi-hereditary example

$$A = k(\ 1 \underset{b}{\overset{a}{\rightleftarrows}} 2\ )/(ab).\ \ \Delta(1) = S(1) = 1, \Delta(2) = P(2) = \begin{matrix} 2 \\ 1 \end{matrix}$$

Differential biquiver: $\Delta(1) \underset{\varphi}{\overset{a}{\rightrightarrows}} \Delta(2)$

Reduction:

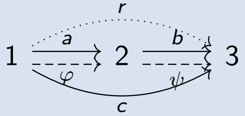


$$\partial\varphi_{13} = \varphi\pi$$
$$\partial\varphi_{31} = \iota\varphi$$
$$\partial\varphi_{33} = \iota\varphi_{13} + \varphi_{31}\pi$$

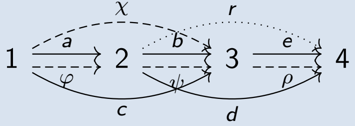
Boxes of tame Schur algebras

Proposition (K '14)

The following are the differential biquivers with relations for the tame Schur algebras:

1. 

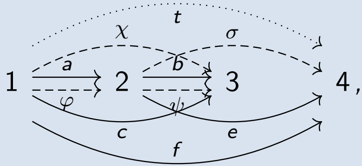
$, \partial(c) = b\varphi, \partial(r) = ba$

2. 

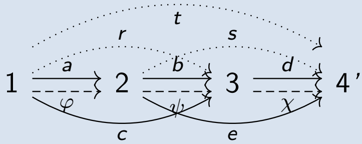
$$\partial(\chi) = \psi\varphi, \partial(r) = eb$$
$$\partial(c) = b\varphi + \psi a,$$
$$\partial(d) = e\psi + \rho b$$

Boxes of tame Schur algebras

Proposition (Continued)

3. 

$\partial(\chi) = \psi\varphi$, $\partial(t) = ea$,
$\partial(c) = b\varphi + \psi a$
$\partial(f) = e\varphi + \sigma a$

4. 

$\partial(r) = ba$, $\partial(s) = db$
$\partial(t) = ea + dc$,
$\partial(c) = b\varphi + \psi a$
$\partial(e) = d\psi$

Tame Schur algebras are filtered-finite

Theorem (K-Thiel '14)

The category $\mathcal{F}(\Delta)$ is representation-finite for the tame Schur algebras.

Proposition (Burt-Butler '91)

If the underlying solid quiver with relations is of finite representation type, then $\mathcal{F}(\Delta)$ is of finite representation type.

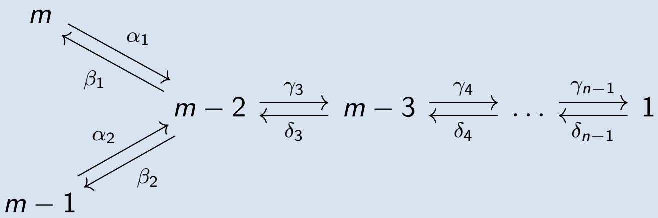
Proof.

- Compute their differential biquivers with relations,
- for 1 & 4 use the Proposition, the underlying algebras are special biserial
- for 2 & 3 use a computer to reduce. □

Filtered-finite wild Schur algebras

Theorem (Erdmann, de la Peña, Sáenz '02)

There are wild blocks of the Schur algebras $S(2,p^2), S(2,p^2+1)$ *with* $\mathcal{F}(\Delta)$ *representation-finite, e.g. the following for* $n \geq 5$*:*



with relations: $\beta_1\alpha_1 = \beta_2\alpha_2 = \beta_2\delta_3 = \beta_1\delta_3 = \gamma_3\alpha_2 = \gamma_3\alpha_1 = 0$
$\alpha_2\beta_2\alpha_1 = \beta_1\alpha_2\beta_2 = \gamma_{i+1}\gamma_i = \delta_i\delta_{i+1} = 0$, $\delta_3\gamma_3 = \alpha_2\beta_2$,
$\gamma_i\delta_i = \delta_{i+1}\gamma_{i+1}$.

General strategy for filtered representation type

- Take your favourite quasi-hereditary algebra,
- determine the standard modules,
- compute the (A_∞-structure on the) Ext-algebra of the standard modules,
- dualise, to get a differential biquiver (with relations),
- reduce this biquiver.