From groups to clusters

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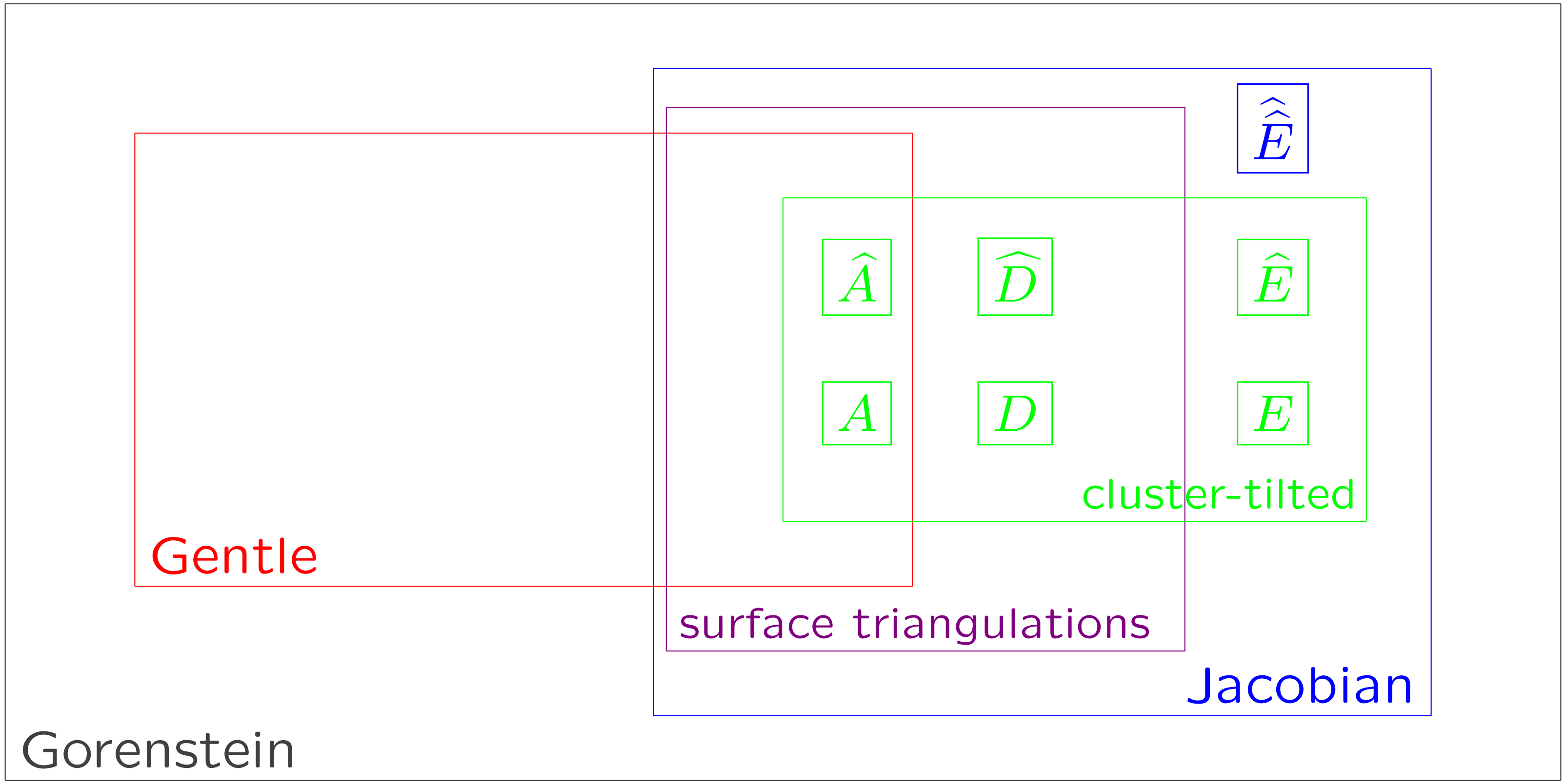
A map of finite-dimensional algebras



Symmetric algebras

A finite-dimensional algebra A over a field K is *symmetric* if $A \simeq \mathrm{Hom}_K(A, K)$ as A-A-bimodules.

Equivalently, there exists a linear form $\lambda\colon A \to K$ such that $\lambda(xy) = \lambda(yx)$ for all $x, y \in A$ and $\ker \lambda$ does not contain any non-zero left ideal of A.

The triangulated category $\mathrm{per}\, A$ of *perfect complexes* over a symmetric algebra A is 0-*Calabi-Yau*, i.e.

$$\mathrm{Hom}_{\mathrm{per}\, A}(X, Y) \simeq D\mathrm{Hom}_{\mathrm{per}\, A}(Y, X) \qquad X, Y \in \mathrm{per}\, A$$

Periodicity of modules

If A is self-injective, the *stable module category*

$$\underline{\text{mod}}\, A \simeq \mathcal{D}^b(\text{mod}\, A)/\text{per}\, A \qquad \text{[Rickard 1989]}$$

is triangulated with suspension Ω_A^{-1}.

syzygy $\Omega_A M \hookrightarrow P_M \twoheadrightarrow M$, *cosyzygy* $M \hookrightarrow I_M \twoheadrightarrow \Omega_A^{-1} M$.

$M \in \underline{\text{mod}}\, A$ is Ω-*periodic* if $\Omega_A^r M \simeq M$ for some $r > 0$.

When A is symmetric,

- $\Omega_A = \tau_A^2$, hence Ω-periodic implies τ-periodic.
- $\underline{\text{mod}}\, A$ is (-1)-Calabi-Yau, i.e.

$$\underline{\text{Hom}}_A(M, N) \simeq D\underline{\text{Hom}}_A(N, \Omega_A M).$$

Example. $A = K[x]/x^p$, $M = x^i A$, $\Omega_A^2 M \simeq M$ $(0 < i < p)$.

Algebras of quaternion type [Erdmann]

A finite-dimensional algebra A is of *quaternion type* if:

- A is symmetric, indecomposable;
- A is of tame representation type;
- $\Omega_A^4 M \simeq M$ for every non-projective A-module M;
- $\det C_A \neq 0$.

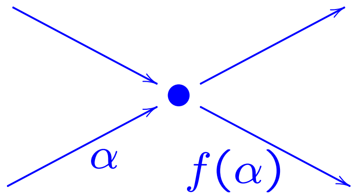
Erdmann produced a list of the possible quivers and relations of such algebras. In particular,

- The number of simple modules is at most 3.
- Blocks whose defect group is generalized quaternion are of quaternion type.

Triangulation quivers

A *triangulation quiver* is a pair (Q, f), where

- Q is a finite quiver (directed graph, loops and multiple edges allowed) such that the in-degree and out-degree of each vertex are 2;



- f is a permutation on the arrows of Q such that for each arrow α,
  - $f(\alpha)$ starts where α ends;
  - $f^3(\alpha) = \alpha$.

Triangulation quivers (continued)

- $\bar{\alpha}$ is the other arrow starting at the same vertex as α.
- We have another permutation g defined by $g(\alpha) = \overline{f(\alpha)}$.
- $\mathrm{PSL}_2(\mathbb{Z})$ acts on the set of arrows via $\alpha \mapsto f(\alpha)$ and $\alpha \mapsto \bar{\alpha}$.

Example. $\alpha \circlearrowleft \bullet \circlearrowright \beta$, $f(\alpha) = \alpha$, $f(\beta) = \beta$.

Surface triangulations

A *marked bordered surface* is a pair (S, M) consisting of:

- a compact, connected, oriented surface S (possibly with boundary),
- a finite set $M \subset S$ of *marked points*, containing at least one point on each boundary component of S.

(S, M) is *unpunctured* if $M \subset \partial S$.

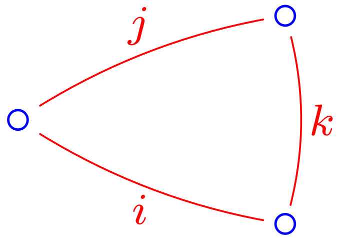
An *arc* is a path in S whose ends are marked points, considered up to isotopy.

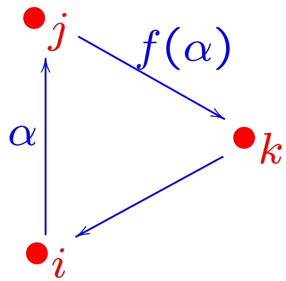
A *triangulation* of (S, M) is a maximal collection of compatible arcs.

Quivers from surface triangulations

A surface triangulation gives rise to a triangulation quiver

- whose *vertices* are the *sides* of the triangles,
- inside each triangle, a 3-cycle of arrows oriented clockwise according to the surface orientation





and $f(\alpha)$ follows α in the cycle,

- at each side on the boundary, a loop β with $f(\beta) = \beta$.

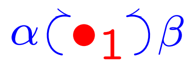
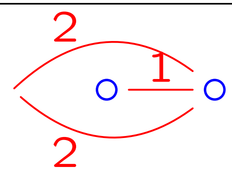
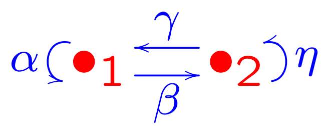
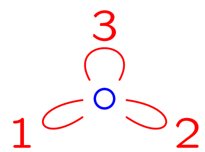
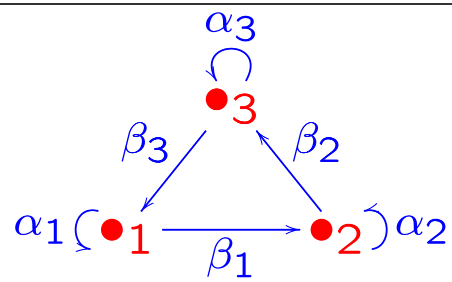
Triangulation vs. adjacency quivers

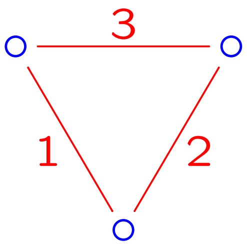
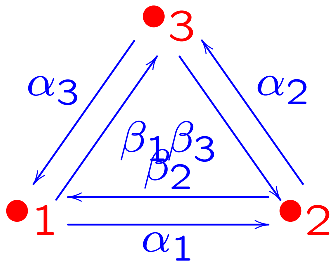
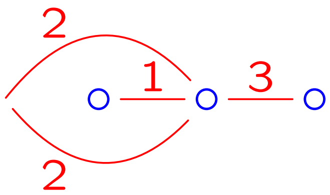
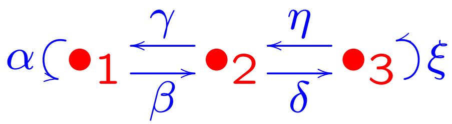
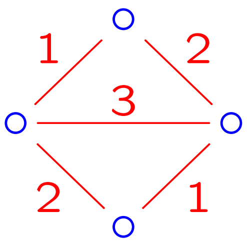
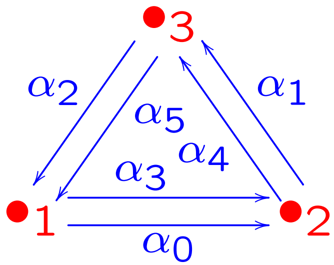
The construction of a triangulation quiver is inspired from that of the *adjacency quiver* [Fomin-Shapiro-Thurston 2008] of a triangulation, but there are some differences:

- Sides on the boundary are also considered;
- Self-folded triangles are treated differently;
- No removal of 2-cycles.

However, for any "nice" triangulation of a *closed* surface, the triangulation quiver and the adjacency quiver coincide.

Some triangulation quivers

Surface	**Triangulation**	**Quiver**
Monogon, unpunctured		 $(\alpha)(\beta)$
Monogon, one puncture		 $(\alpha\beta\gamma)(\eta)$
Triangle, unpunctured		 $(\alpha_1)(\alpha_2)(\alpha_3)(\beta_1\beta_2\beta_3)$

Surface	**Triangulation**	**Quiver**
Sphere, three punctures		 $(\alpha_1\alpha_2\alpha_3)(\beta_3\beta_2\beta_1)$
		 $(\alpha\beta\gamma)(\delta\xi\eta)$
Torus, one puncture		 $(\alpha_4\alpha_2\alpha_0)(\alpha_5\alpha_3\alpha_1)$

Path algebras of quivers

Q – quiver, K – field.

The *path algebra* KQ is the K-algebra

- spanned by all paths in Q,
- with multiplication given by composition of paths.

The *complete path algebra* $\widehat{KQ}$ is the completion of KQ with respect to the ideal generated by all arrows. It is a topological algebra.

Example. $\bullet\circlearrowright x$. The path algebra is $K[x]$, its completion is $K[[x]]$.

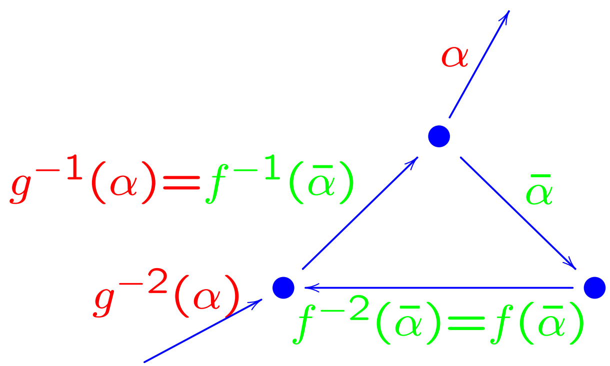
Algebras from triangulation quivers

Let (Q, f) be a triangulation quiver.

Any arrow α gives rise to a *cycle* ω_α and *almost cycle* ω'_α by

$$\omega_\alpha = \alpha \cdot g(\alpha) \cdot \ldots \cdot g^{n_\alpha - 1}(\alpha),$$
$$\omega'_\alpha = \alpha \cdot g(\alpha) \cdot \ldots \cdot g^{n_\alpha - 2}(\alpha),$$

where $n_\alpha \geq 1$ is the minimal $n \geq 1$ such that $g^n(\alpha) = \alpha$.



Note: ω'_α is parallel to $\bar{\alpha} \cdot f(\bar{\alpha})$.

Brauer graph algebra

Given a triangulation quiver (Q, f)
and g-invariant *multiplicities* $m_\alpha \in \mathbb{Z}_{>0}$, $m_{g(\alpha)} = m_\alpha$, define the *Brauer graph algebra*

$$\Gamma = KQ/\langle \bar{\alpha} \cdot f(\bar{\alpha}), \omega_\alpha^{m_\alpha} - \omega_{\bar{\alpha}}^{m_{\bar{\alpha}}} \rangle_{\text{all arrows } \alpha}.$$

Γ is:

- finite-dimensional,
- symmetric,
- special biserial, hence of tame representation type.

Example. For a sphere with three punctures, $\Gamma = KA_4$ if K is algebraically closed with $\text{char}\, K = 2$.

Triangulation algebra

Given a triangulation quiver (Q, f) and g-invariant *multiplicities* $m_\alpha \in \mathbb{Z}_{>0}$, $m_{g(\alpha)} = m_\alpha$, define the *triangulation algebra*

$$\Lambda = \widehat{KQ}/\overline{\langle \bar{\alpha} \cdot f(\bar{\alpha}) - \omega_\alpha^{m_\alpha - 1} \cdot \omega'_\alpha \rangle}_{\text{all arrows } \alpha}$$

(under the admissibility conditions $n_\alpha m_\alpha \geq 3$ for all α).

This concept unifies two classes of algebras:

- Jacobian algebras of quivers with potentials associated by [Labardini 2009] to triangulations of closed surfaces;
- Erdmann's algebras of quaternion type.

Theorem [L.] on triangulation algebras

Let Λ be a triangulation algebra. Then:

(a) Λ is finite-dimensional.

(b) Λ is symmetric.

(c) Λ is of tame representation type.

(d) Λ is *2-CY-tilted*, i.e. there is a 2-Calabi-Yau triangulated category $\mathcal{C}$ and a cluster-tilting object T in $\mathcal{C}$ such that $\Lambda \simeq \mathrm{End}_{\mathcal{C}}(T)$.

(e) $\Omega^4_\Lambda M \simeq M$ for all $M \in \underline{\mathrm{mod}}\,\Lambda$.

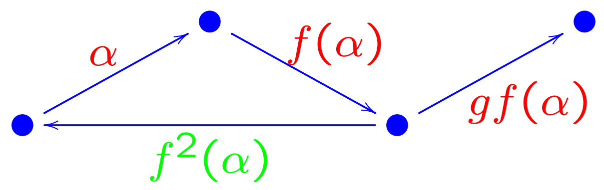
(f) Any algebra $\mathrm{End}_{\mathcal{C}}(T')$ for a cluster-tilting object T' in $\mathcal{C}$ reachable from T by a sequence of mutations is *derived equivalent* to Λ and has the same properties (a)-(e) above.

Significance

- New tame symmetric algebras with periodic modules which seem not to appear in [Erdmann-Skowronski 2006].
- New symmetric 2-CY-tilted algebras, in addition to the ones arising from odd-dimensional isolated hypersurface singularities [Burban-Iyama-Keller-Reiten 2008].
- New proof that the algebras in Erdmann's lists are of quaternion type.
- The algebras of quaternion type are 2-CY-tilted.
- The Jacobian algebras arising from triangulations of closed surfaces [Labardini 2009] are finite-dimensional.

Remarks on finite-dimensionality

Consider a *zig-zag* path $\alpha \cdot f(\alpha) \cdot g(f(\alpha))$



Repeatedly invoking the defining commutativity relations of the algebra Λ, one gets arbitrarily long paths

$$
\begin{aligned}
\alpha \cdot f(\alpha) \cdot gf(\alpha) &= \ldots \cdot \beta \cdot g(\beta) \cdot fg(\beta) \cdot \ldots \\
&= \ldots \cdot \gamma \cdot f(\gamma) \cdot gf(\gamma) \cdot \ldots \\
&= 0
\end{aligned}
$$

whose image vanishes since Λ is a quotient of a closure of an ideal.

Remarks on being 2-CY-tilted

2-Calabi-Yau triangulated categories with cluster-tilting object are generalizations of *cluster categories* [BMRRT 2006].

They play significant role in the additive categorification of skew-symmetric cluster algebras.

The 2-CY-tilted algebras are generalizations of *cluster-tilted algebras* [Buan-Marsh-Reiten 2006, Keller-Reiten 2007].

Quivers with potentials [Derksen-Weyman-Zelevinsky 2008] can be used to construct such categories and algebras [Amiot 2009, Keller 2011].

Potentials

A *potential* W on a quiver Q is a linear combination of cycles in $\widehat{KQ}$.

Its *Jacobian algebra* is the quotient of $\widehat{KQ}$ by the closure of the ideal generated by all the cyclic derivatives of W.

Finite-dimensional Jacobian algebras are 2-CY-tilted.

Example. The triangulation algebra

$$\Lambda = \widehat{KQ}/\overline{\langle \bar{\alpha} \cdot f(\bar{\alpha}) - \omega_\alpha^{m_\alpha - 1} \cdot \omega'_\alpha \rangle}_{\text{all arrows } \alpha}.$$

is a Jacobian algebra of a potential of the form

$$W = \sum_\beta \beta f(\beta) f^2(\beta) - \sum_\alpha \frac{1}{m_\alpha} \omega_\alpha^{m_\alpha}$$

under some restrictions on the characteristic of K.

Hyperpotentials

Sometimes the characteristic of K does not allow us to integrate the defining relations of an algebra to a potential.

Example. $K[x]/(x^{p-1})$ is a Jacobian algebra of a potential if and only if $\operatorname{char} K \neq p$.

To overcome this problem, we define a *hyperpotential* as a collection of elements $(\rho_\alpha)_{\text{arrows } \alpha} \subset \widehat{KQ}$ such that:

- If $i \xrightarrow{\alpha} j$, then ρ_α is a linear combination of paths from j to i,
- $\sum_\alpha [\alpha, \rho_\alpha] = 0$ in $\widehat{KQ}$.

(i.e. we consider $\mathsf{HH}_1(\widehat{KQ})$ instead of $\mathsf{HH}_0(\widehat{KQ})$).

Observation. All categorical constructions for potentials carry over to hyperpotentials.

Remarks on the remaining properties

The property $\Omega_\Lambda^4 M \simeq M$ as well as the derived equivalences in the theorem are true for any algebra Λ which is both *symmetric* and *2-CY-tilted*.

The derived equivalences between neighboring symmetric 2-CY-tilted algebras are afforded by tilting complexes of two-term projectives which have many incarnations:

- Okuyama-Rickard complexes;
- Silting mutation [Aihara-Iyama 2012];
- Perverse equivalence [Chuang-Rouquier].