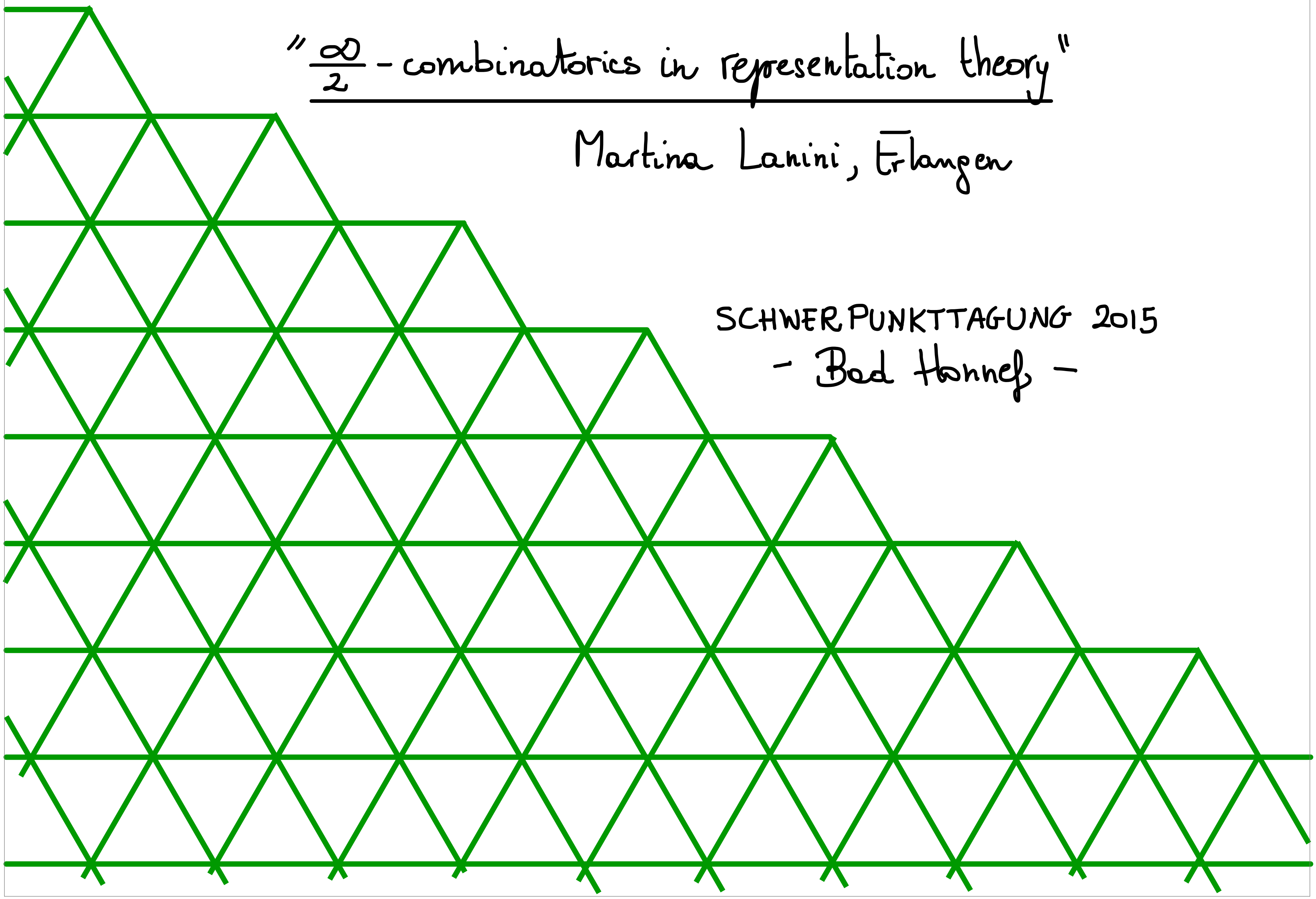
"$\frac{\infty}{2}$ - combinatorics in representation theory"

Martina Lanini, Erlangen

SCHWERPUNKTTAGUNG 2015

- Bad Honnef -



⚠ This talk is not meant to furnish an exhaustive list of occurrences of "$\frac{\infty}{2}$-structures" in representation theory

Too much for a 50-min talk!

Indeed, we find instances of $\frac{\infty}{2}$-structures in

- Lusztig's periodic module (while looking for evidences for his modular conjecture)
- Lusztig's "periodic Schubert varieties"
- Feigin-Frenkel construction of Wakimoto modules for affine KM-algebras $\rightsquigarrow$ $\frac{\infty}{2}$-flag variety $\mathcal{F}l^{\frac{\infty}{2}}$
- G,T-version of modular Lusztig conjecture
- Lusztig's conjecture for small quantum groups
- critical level analogue of Kazhdan-Lusztig conjecture (Feigin-Frenkel-Lusztig conj)
- Series of papers of (several subsets of) Arkhipov-Bezrukavnikov-Braverman-Feigin-Finkelberg-Gaitsgory-Kuznetsov-Mirković: $\mathcal{D}$-mods & perverse sheaves on "$\mathcal{F}l^{\frac{\infty}{2}}$" $\rightsquigarrow$ representation categories for
  - small quantum gps,
  - Lie alg's in char $p>0$
- ...

BRUHAT ORDER

Coxeter group

simple reflns

$(W, \mathcal{S})$ Coxeter system

reduced expression

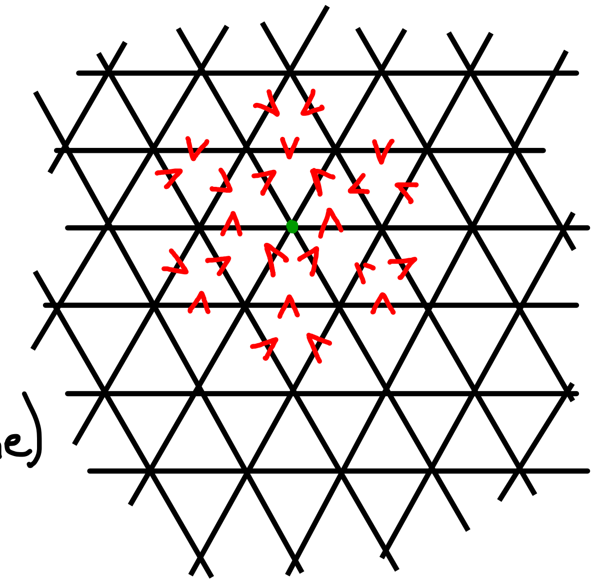
$w \in W \quad w = s_{i_1} \cdots s_{i_r} \quad s_{i_j} \in \mathcal{S}$

$v \overset{\text{def}}{\leq} w \iff v$ is a subword of w

In particular, want to consider $\leq$ on the (affine) Weyl grp $\widehat{W}$ of $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}K \oplus \mathbb{C}D$

⚠ In the picture only the weak Bruhat order is represented (we only want to compare with alcoves)

$\mathfrak{sl}_3$



BRUHAT ORDER

VS

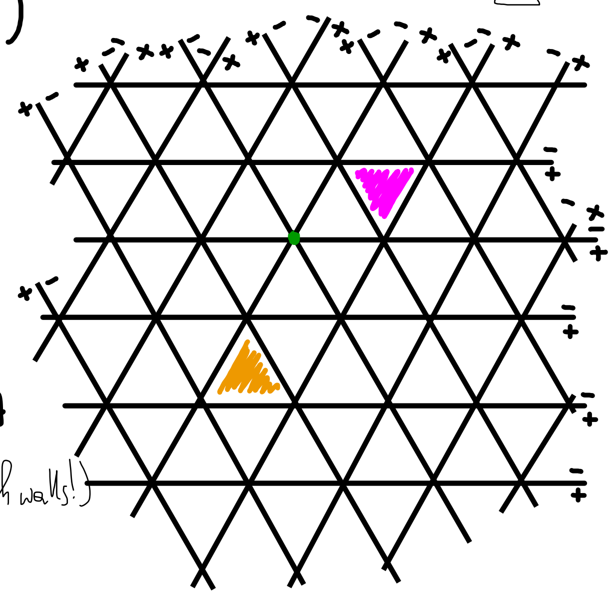
$\frac{\infty}{2}$-BRUHAT ORDER

To define the $\frac{\infty}{2}$-Bruhat order (Lusztig's generic order)

- first we consider the following orientation of the hyperplanes through the origin,
- then we impose that all parallel hyperplanes have same orientation
- finally declare $A \prec B$ if there is a path from A to B consisting only of positive crossings (not only through walls!)

$A \prec B$

⚠ as above



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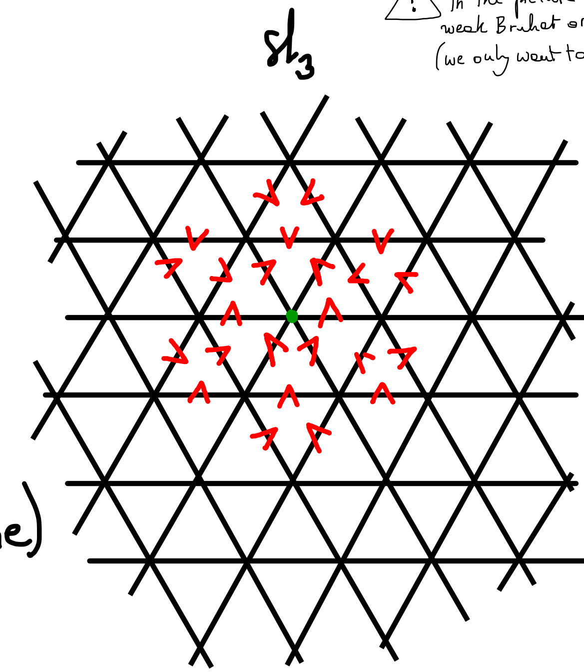

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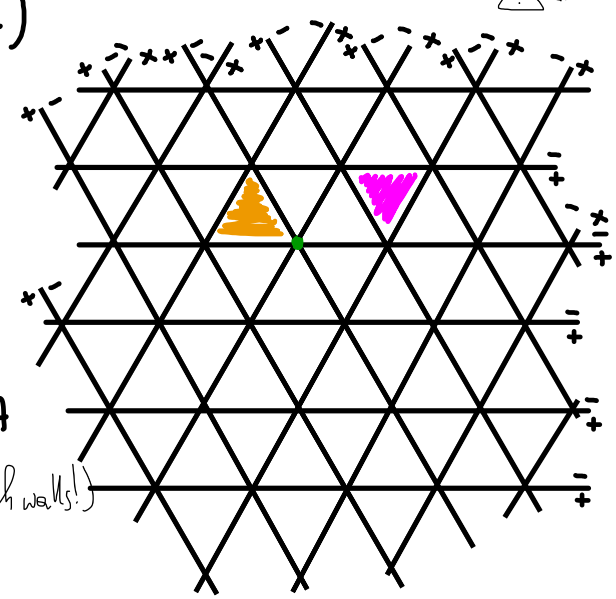
$A \not< B$

⚠ as above ↑



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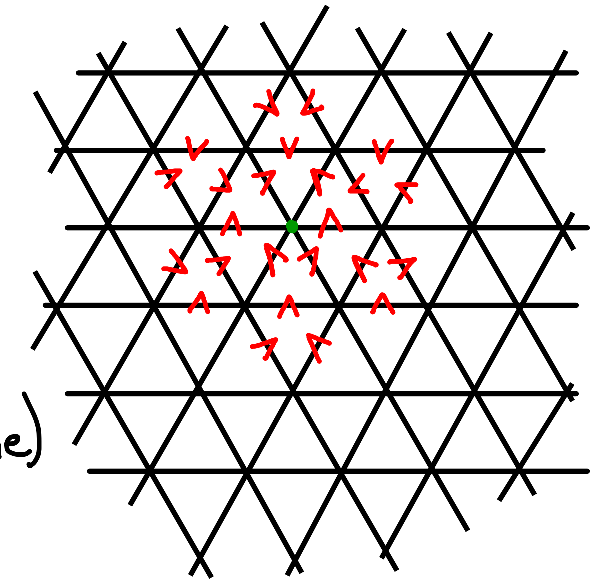

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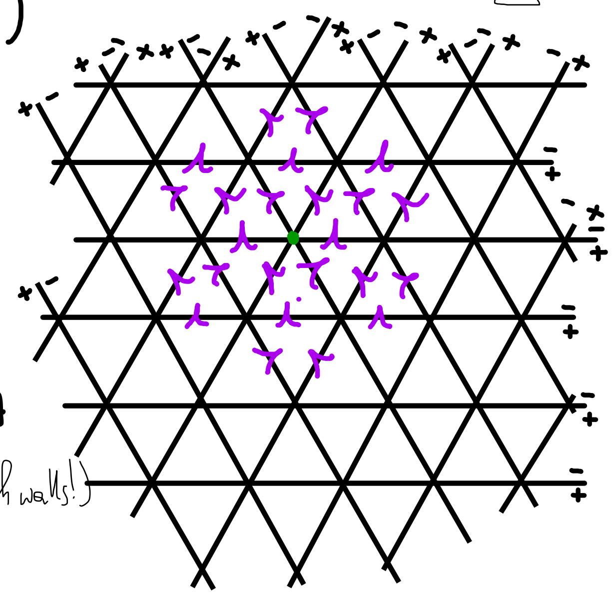

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$v \overset{b}{\leq} w \iff v$ is a subword of w

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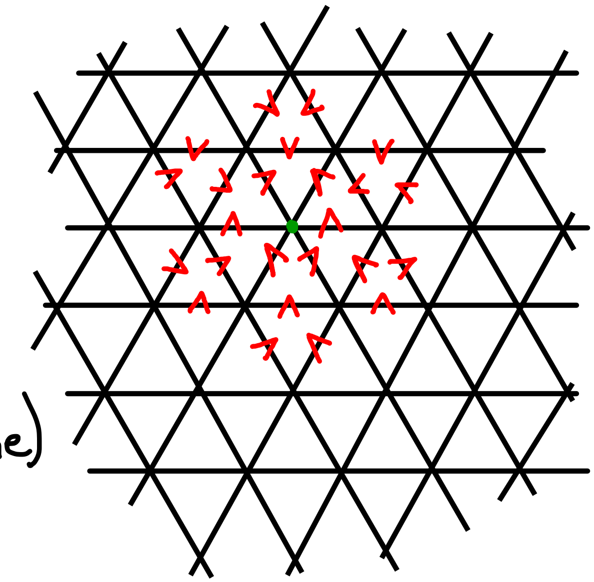

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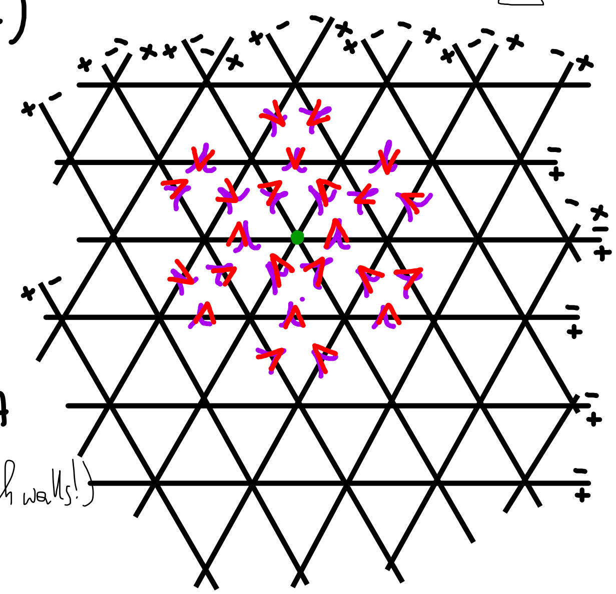

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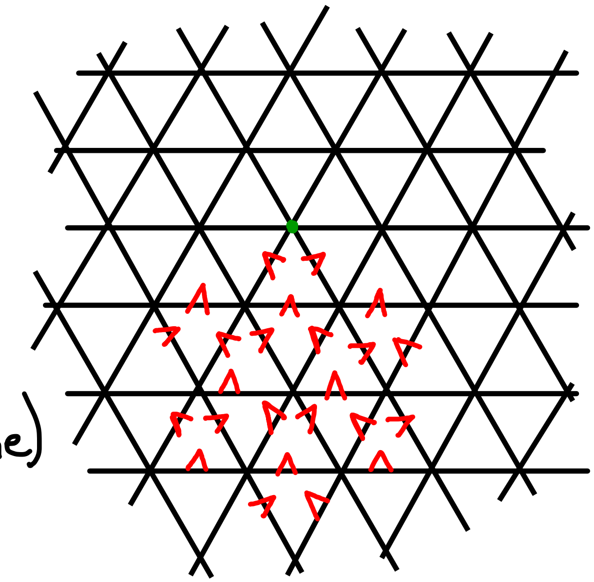

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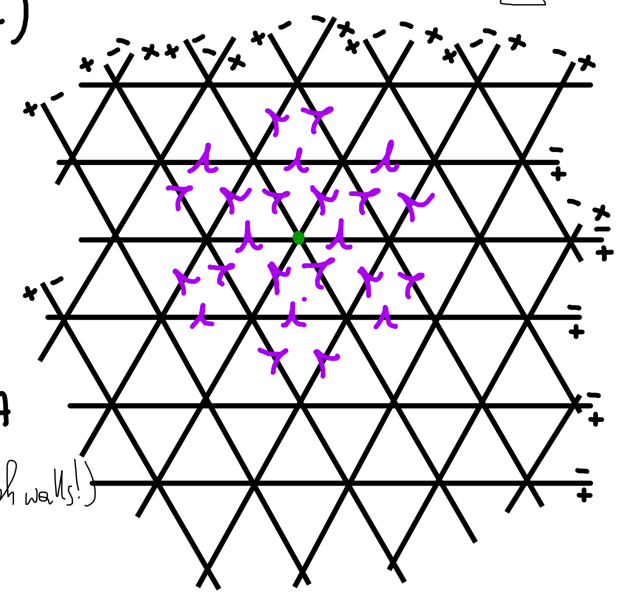

$\frac{\infty}{2}$-BRUHAT ORDER

BRUHAT ORDER **VS** $\frac{\infty}{2}$-BRUHAT ORDER

there exists $n_0 \in \mathbb{Z}_{\geq 0}$ such that

$$A - m\rho \leq B - m\rho \quad \forall m \geq n_0$$

$$\Big(\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha \Big)$$

positive roots of $\mathfrak{g}^\vee$

$$\Updownarrow$$

$$A \preccurlyeq B$$

$\mathfrak{sl}_3$

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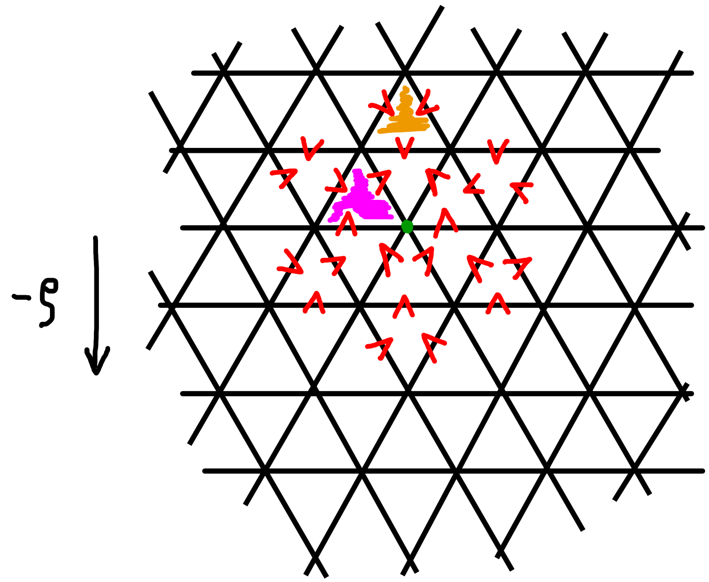

BRUHAT ORDER

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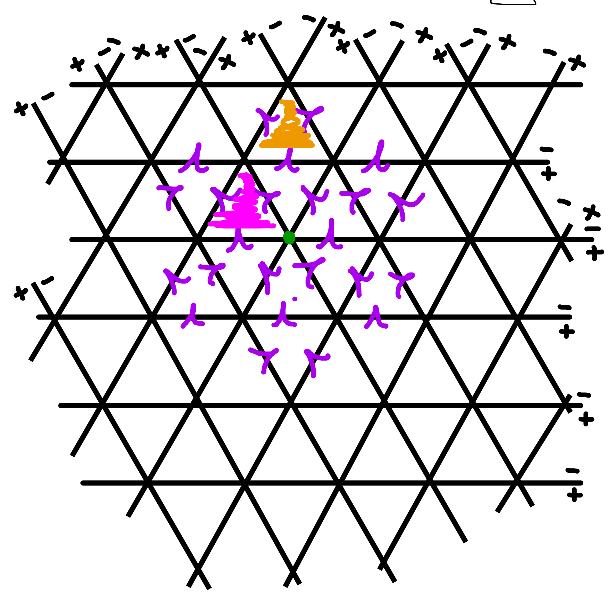

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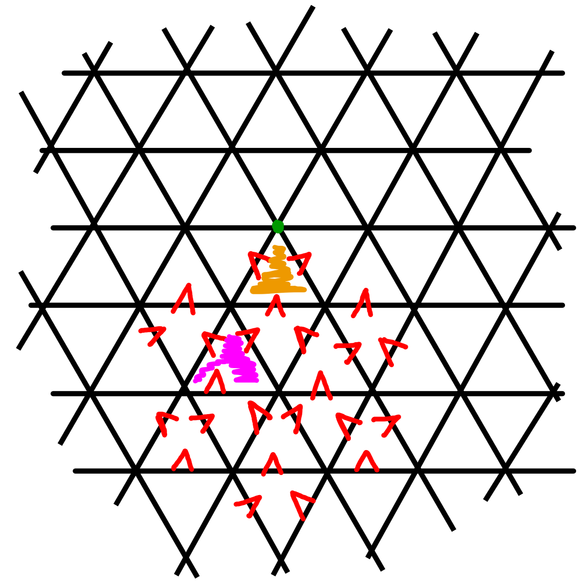
$-\rho$



BRUHAT ORDER

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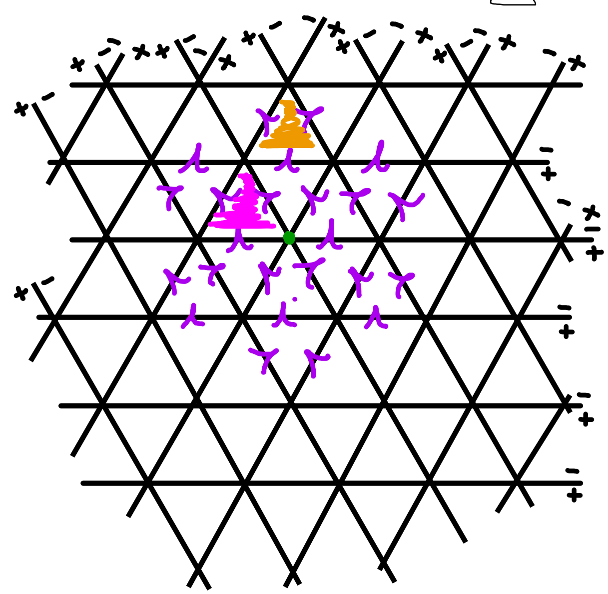

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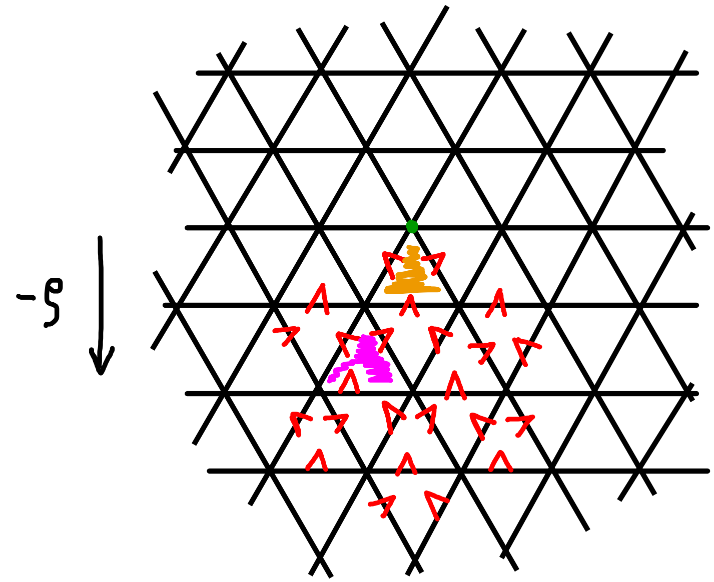

BRUHAT ORDER

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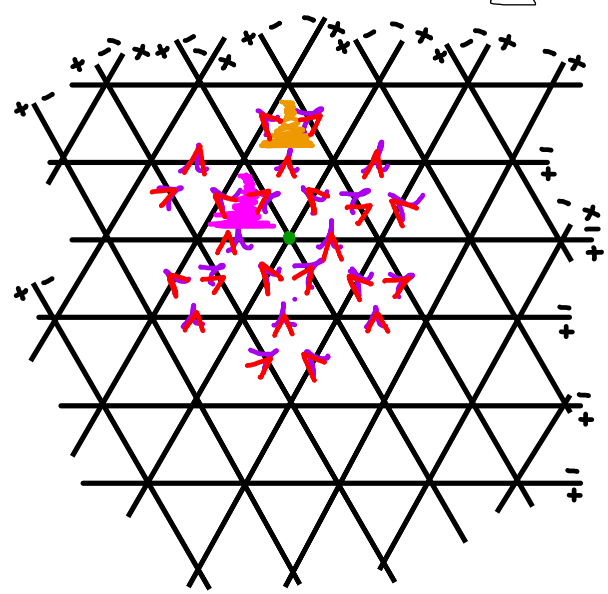

$\frac{\infty}{2}$-BRUHAT ORDER

Inclusion of orbit closures

$\mathfrak{g} \rightsquigarrow G^\vee : \mathrm{Lie}\, G = \mathfrak{g}$

$B^\vee \subset G^\vee$

$$B^\vee \circlearrowright G^\vee / B^\vee = \bigsqcup_{w \in W} B^\vee w B^\vee / B^\vee = \bigsqcup_{w \in W} X_w$$

$$\overline{X}_w = \bigsqcup_{x \leq w} X_x \qquad \dim \overline{X}_w = \ell(w)$$

BRUHAT ORDER

vs

$\frac{\infty}{2}$-BRUHAT ORDER

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Inclusion of orbit closures

BRUHAT ORDER

$\mathfrak{g} \rightsquigarrow G^\vee : \mathrm{Lie}\, G = \mathfrak{g}$

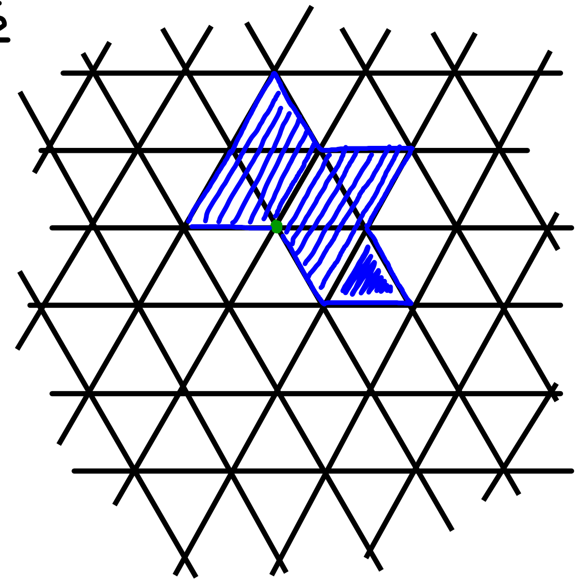
$$\begin{array}{ccc} G^\vee[[t]] & \xrightarrow{t \mapsto 0} & G^\vee \\ \cup & & \cup \\ I^\vee & \longrightarrow & B^\vee \end{array}$$

$I^\vee \circlearrowright$

$$G^\vee((t))/I^\vee = \bigsqcup_{w \in \widehat{W}} I^\vee w I^\vee / I^\vee = \bigsqcup_{w \in \widehat{W}} X_w$$

$$\overline{X_w} = \bigsqcup_{x \leq w} X_x \qquad \dim \overline{X_w} = \ell(w), \quad \operatorname{codim} \overline{X_w} = \infty$$

$\mathfrak{sl}_3$



BRUHAT ORDER

VS

$\frac{\infty}{2}$-BRUHAT ORDER

$G^\vee \supset B^\vee = N^\vee T^\vee$

unipotent rad of B

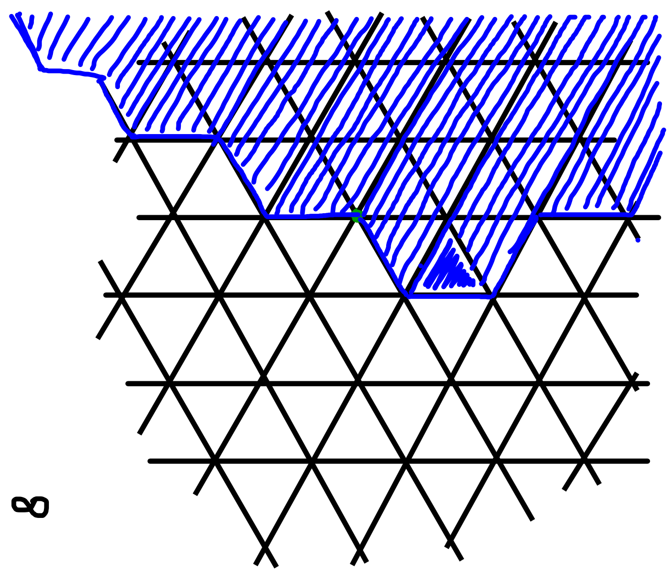
mxl torus

(⚠ Not rigorous!)

$I^\vee \circlearrowright$

$$G^\vee((t)) / N^\vee((t)) T^\vee[[t]] = \bigsqcup_{w \in \widehat{W}} X_w^{\frac{\infty}{2}}$$

$$\overline{X_w^{\frac{\infty}{2}}} = \bigsqcup_{x \preceq w} X_x^{\frac{\infty}{2}} \qquad \dim \overline{X_w^{\frac{\infty}{2}}} = \infty, \quad \operatorname{codim} \overline{X_w^{\frac{\infty}{2}}} = \infty$$



$\frac{\infty}{2}$-BRUHAT ORDER

VS

Moment graphs

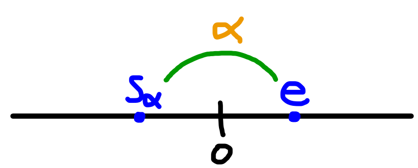
Fix a lattice of finite rank: $X \simeq \mathbb{Z}^r$, $r < \infty$.

A moment graph $\mathcal{G}$ on X is a graph whose edges are labelled by non-zero elements of X.

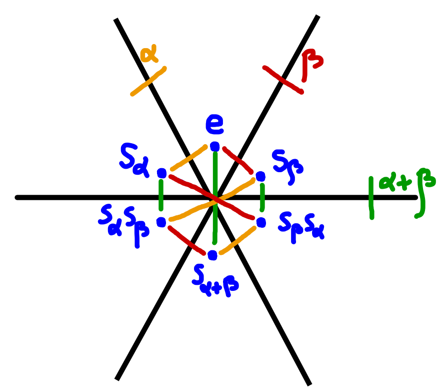
Example ① $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ simple fin. diml Lie alg. $/\mathbb{C}$, Borel, Cartan

Bruhat graph $\mathcal{G}$ on $\mathbb{Z}R$:   root lattice of $\mathfrak{g}^\vee$

$\mathfrak{sl}_2$



$\mathfrak{sl}_3$



vertices $\leftrightarrow$ Weyl gp W $\leftrightarrow$ Weyl chambers

labelled edges $\leftrightarrow$ $x \overset{\alpha}{\text{—}} y = s_\alpha x$ $\alpha \in R^+$ pos. roots of $\mathfrak{g}^\vee$

Moment graphs

Fix a lattice of finite rank: $X \simeq \mathbb{Z}^r$, $r < \infty$.

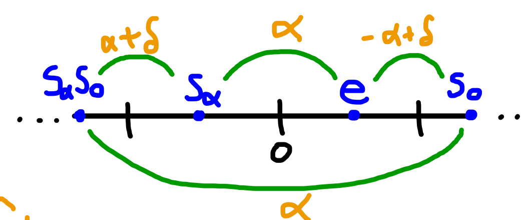
A moment graph $\mathcal{G}$ on X is a graph whose edges are labelled by non-zero elements of X.

Example ② $\hat{\mathfrak{g}} \supset \hat{\mathfrak{b}} \supset \hat{\mathfrak{h}}$ affine Kac-Moody alg. $/\mathbb{C}$, Borel, Cartan

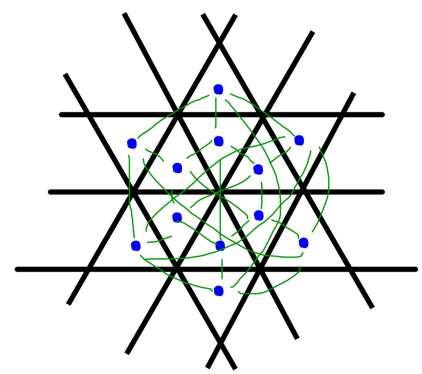
affine Bruhat graph $\hat{\mathcal{G}}$ on $\mathbb{Z}\hat{R} = \mathbb{Z}R \oplus \mathbb{Z}\delta$

root lattice of $\hat{\mathfrak{g}}^\vee$

$\widehat{\mathfrak{sl}}_2$



$\widehat{\mathfrak{sl}}_3$



vertices $\leftrightarrow$ affine Weyl gp $\hat{W}$ $\leftrightarrow$ set of alcoves $\mathcal{A}$

labelled edges $\leftrightarrow$ $x \overset{\alpha}{—} y = s_\alpha x$ $\quad \alpha \in \hat{R}^+$

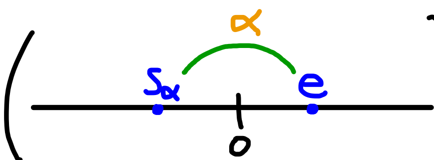
pos. roots of $\hat{\mathfrak{g}}^\vee$

The structure algebra of a moment graph

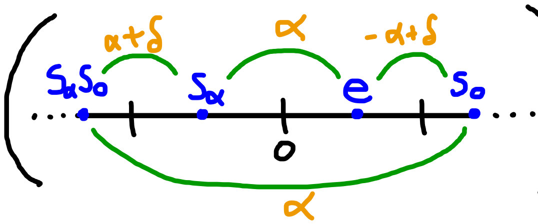
Let $S = \mathrm{Sym}(X \otimes_{\mathbb{Z}} k)$, k field $= \bar{k}$ char $k \neq 2$

The structure algebra of a moment graph $\mathcal{G}$ on X is:

$$\mathcal{Z} = \mathcal{Z}(\mathcal{G}) = \Big\{ (z_x) \in \bigoplus_{\substack{x \\ \text{vertex}}} S \;\Big|\; z_x \equiv z_y \ (\mathrm{mod}\ \alpha) \text{ for any } x \overset{\alpha}{\text{---}} y \text{ edge} \Big\}$$

Example ① $\mathcal{Z}\Big($ 

$\Big) \ni (271 + 3\alpha,\ 271 - 59\alpha)$

Example ② $\mathcal{Z}\Big($ 

$\Big) \ni (\ldots, \delta, -\alpha, \alpha, \delta, \ldots)$

BRUHAT ORDER

BRUHAT ORDER

Why consider these algebras?

- $\hat{T}^\vee := T^\vee \times \mathbb{C}^\times \curvearrowright G^\vee((t))/I^\vee$ $\quad S \simeq H^\bullet_{\hat{T}^\vee}(pt, k)$, $\quad \hat{Z} \underset{\substack{\text{graded}\\ S-\text{mods}}}{\cong} H^\bullet_{\hat{T}^\vee}\left(G^\vee((t))/I^\vee, k\right)$

- $\hat{Z}_{\tilde{s}} \cong \underbrace{Z}_{\text{center}}(\underbrace{\mathcal{O}}_{\substack{\text{deformed}\\ \text{cat } \mathcal{O} \text{ for } \mathfrak{g}}}{}_{\tilde{s}, \underbrace{\sigma}_{\text{regular wt}}}) \cong Z(\mathcal{O}^{VF}_{\tilde{s},\sigma})$ (Fiebig) $\quad$ (char 0)

  $\mathcal{O}^{VF}$: modules with Verma flag

Recall: (dual) Verma $\Delta(\underbrace{w\cdot\sigma}_{\text{antidominant}}) \simeq \underbrace{\mathbb{C}_{X_w}}_{0\text{-extension}}$

VS

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$\frac{\infty}{2}$-BRUHAT ORDER

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- $\hat{T}^\vee := T^\vee \times \mathbb{C}^\times \curvearrowright G^\vee((t))/N^\vee((t))T^\vee[[t]]$ $\quad S \simeq H^\bullet_{\hat{T}^\vee}(pt, k)$, $\quad \hat{Z} \underset{\substack{\text{graded}\\ S-\text{mods}}}{\overset{\text{Feigin-Frenkel}}{\cong}} H^\bullet_{\hat{T}^\vee}\left(G^\vee((t))/N^\vee((t))T^\vee[[t]], k\right)$

- $\hat{Z}_{\tilde{s}} \cong Z(\mathcal{O}^{WF}_{\tilde{s}, \underbrace{\sigma}_{\text{antidominant regular wt}}})$ (Arakawa - L, 2015) $\quad$ (char 0)

  $\mathcal{O}^{WF}$: modules with Wakimoto flag

Recall: Wakimoto $W(w\cdot\sigma) \simeq \underbrace{\mathbb{C}_{X^{\frac{\infty}{2}}_w}}_{0\text{-extension}}$

<u>Rmk</u> Can think of it as a "limit" of a Verma:

(Arakawa) $W(w\cdot\sigma) = \lim\limits_{\overrightarrow{\gamma \in \mathcal{C}^-}} T_\gamma \Delta(w\cdot\sigma)$

$\mathcal{Z}$-modules with filtrations

Let $(\Lambda, \leq)$ be a poset.

$\mathcal{Z}\text{-mod}^{(\Lambda,\leq)}$ = $\mathcal{Z}$-modules M which are free and fin. gen'd as S-mods

+ collection of submods $(M_I)_{I \subseteq \Lambda \text{ ideal}}$ with

$$M_I \subset M_J \ \forall I \subset J, \ M_{I \cup J} = M_I + M_J, \ M_{I \cap J} = M_I \cap M_J \ \forall I, J$$

(+ some more technical assumptions)

$$\& \ M_{[x]} = M_{\{\geq x\}} / M_{\{> x\}} \text{ is a free } S\text{-mod } \forall x \in \Lambda$$

Example② $\widehat{\mathcal{Z}}$ structure alg. of an affine Bruhat graph $\widehat{\mathcal{G}} = \mathcal{G}(\hat{\mathfrak{g}}, \hat{\mathfrak{b}}, \hat{\mathfrak{h}})$

$$(\Lambda, \leq) = (\underset{\widehat{W}}{\text{Weyl gp}}, \underset{\leqslant}{\text{Bruhat order}})$$

Consider in $\widehat{\mathcal{Z}}\text{-mod}^{(\widehat{W}, \leq)}$ the subcategory $\mathcal{V}$ consisting of modules M with the property that any $(z_x) \in \widehat{\mathcal{Z}}$ acts on $M_{[x]}$ via z_x.

Thm (Fiebig, 03) $\mathcal{V}$ is equivalent to the category $\mathcal{O}^{VF}_{\tilde{S}, \sigma}$

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(+ some more technical assumptions)

& $M_{[x]} = M_{\{\geq x\}} / M_{\{>x\}}$ is a free S-mod $\forall x \in \Lambda$

Example ②' $\widehat{\mathcal{Z}}$ structure alg. of an affine Bruhat graph $\widehat{\mathcal{G}} = \mathcal{G}(\hat{\mathfrak{g}}, \hat{\mathfrak{b}}, \hat{\mathfrak{h}})$

$$(\Lambda, \leq) = \left(\underset{\widehat{W}}{\text{Weyl gp}}, \; \underset{\preccurlyeq}{\tfrac{\infty}{2}\text{- Bruhat order}} \right)$$

Consider in $\widehat{\mathcal{Z}}\text{-mod}^{(\widehat{W}, \preccurlyeq)}$ the subcategory $\mathcal{V}^{\frac{\infty}{2}}$ consisting of modules M with the property that any $(z_x) \in \widehat{\mathcal{Z}}$ acts on $M_{[x]}$ via z_x.

Thm (Arakawa-L, 15) $\mathcal{V}^{\frac{\infty}{2}}$ is equivalent to the category $\mathcal{O}^{WF}_{\tilde{S},\sigma}$

An exact structure on $\mathcal{Z}\text{-mod}^{(\Lambda,\leq)}$

A sequence $M \to L \to N$ of objs in $\mathcal{Z}\text{-mod}^{(\Lambda,\leq)}$ is <u>exact</u> if

$$0 \to M_{\mathcal{I}} \to L_{\mathcal{I}} \to N_{\mathcal{I}} \to 0 \text{ is a SES of } S\text{-mods } \forall \underset{\text{ideal}}{\mathcal{I} \subset \Lambda}$$

$\rightsquigarrow$ it makes sense to talk about projectives!

<u>Example</u> 2 (Fiebig, 06) For any $w \in \widehat{W}$ there exists a unique indecomposable projective object $P_w \in \mathcal{V} \subseteq \widehat{\mathcal{Z}}\text{-mods}^{(\widehat{W},\leq)}$ with $(P_w)_{[x]} = 0$ unless $x \leq y$ and $(P_w)_{[w]} \cong S$

(char $k=0$) Braden-MacPherson : $(P_w)_{[x]} \simeq \mathbb{H}^{\bullet}_{\widehat{T}^\vee}(\overline{X}_w)_x \langle ? \rangle$ some shift in the grading

An exact structure on $\mathcal{Z}\text{-mod}^{(\Lambda,\leq)}$

A sequence $M \to L \to N$ of objs in $\mathcal{Z}\text{-mod}^{(\Lambda,\leq)}$ is <u>exact</u> if

$$0 \to M_{\mathcal{I}} \to L_{\mathcal{I}} \to N_{\mathcal{I}} \to 0 \text{ is a SES of } S\text{-mods } \forall \underset{\text{ideal}}{\mathcal{I} \subset} \Lambda$$

$\rightsquigarrow$ it makes sense to talk about projectives!

<u>Example</u> 2' (Fiebig, 09) For any $w \in \widehat{W}$ there exists a unique indecomposable projective object $P_w^{\frac{\infty}{2}} \in \mathcal{V}^{\frac{\infty}{2}} \subseteq \widehat{\mathcal{Z}}\text{-mods}^{(\widehat{W},\preceq)}$ with $(P_w^{\frac{\infty}{2}})_{[x]} = 0$ unless $x \preceq y$ and $(P_w^{\frac{\infty}{2}})_{[w]} \cong S$

(char $k = 0$) L.: $(P_w^{\frac{\infty}{2}})_{[x]} \simeq \mathbb{H}^{\bullet}_{\widehat{T}^{\vee}}(\overline{X}_w^{\frac{\infty}{2}})_x \langle ? \rangle$ some shift in the grading

Hecke modules

BRUHAT ORDER VS $\frac{\infty}{2}$-BRUHAT ORDER

Affine Hecke algebra $\widehat{\mathcal{H}} = \bigoplus_{x \in \widehat{W}} \mathbb{Z}[v^{\pm 1}] H_x$ (std basis elt) $\quad H_x \cdot \underbrace{(H_s + v)}_{\underline{H}_s} = \begin{cases} H_{xs} + v H_x & xs > x \\ H_{xs} + v^{-1} H_x & x < xs \end{cases}$

$\overline{}: \widehat{\mathcal{H}} \longrightarrow \widehat{\mathcal{H}}$ inv. $\quad H_x \mapsto H_{x^{-1}}^{-1} \quad v^{\pm 1} \mapsto v^{\mp 1}$

Thm (Kazhdan-Lusztig 79) For any $w \in \widehat{W}$ there exists a unique $\underline{H}_w \in \widehat{\mathcal{H}}$ s.t.

$\bullet\ \overline{\underline{H}_w} = \underline{H}_w \qquad \bullet\ \underline{H}_w = H_w + \sum_{x < w} h_{x,w} H_x, \quad h_{x,w} \in v\mathbb{Z}[v]$

KL-poly

Periodic module $\mathcal{P} = \bigoplus_{A \in \mathcal{A}} \mathbb{Z}[v^{\pm 1}] A \circlearrowleft \widehat{\mathcal{H}} \qquad A \cdot \underbrace{(H_s + v)}_{\underline{H}_s} = \begin{cases} As + vA & As > A \\ As + v^{-1} A & A < As \end{cases}$

$\cup$

$\mathcal{P}^\circ = \left\langle E_\lambda := \sum v^{??} (w(A_0^-) + \lambda) \,\middle|\, \lambda \in X \right\rangle$

Thm (Lusztig, 80) There exists an involution $\overline{}: \mathcal{P}^\circ \longrightarrow \mathcal{P}^\circ$ s.t.

$\bullet\ \overline{(p \cdot h)} = \overline{p} \cdot \overline{h} \qquad \bullet\ \overline{E_\lambda} = E_\lambda$

For any $A \in \mathcal{A}$ there exists a unique $\underline{D}_A \in \mathcal{P}^\circ$ s.t. $\bullet\ \overline{\underline{D}_A} = \underline{D}_A$

$\bullet\ \underline{D}_A = A + \sum_{B \neq A} p_{B,A} B \qquad p_{B,A} \in v\mathbb{Z}[v]$

Lusztig's periodic poly

Multiplicity formulae & polynomials

BRUHAT ORDER | BRUHAT ORDER

affine KL-conjecture
(negative level)

$$\left[\Delta(x\cdot\sigma) : L(w\cdot\sigma)\right] = h_{x,w}(1)$$

Verma — simple of hw $w\cdot\sigma$

σ antidominant regular

VS | VS

$\frac{\infty}{2}$-BRUHAT ORDER | $\frac{\infty}{2}$-BRUHAT ORDER

(G_1T-version of)
Lusztig's conjecture

$$\left[Z(x\cdot_p 0) : L_1(w\cdot_p 0)\right] = p_{A_x,A_w}(1)$$

Baby Verma — unique simple quot of $Z(w\cdot_p 0)$

Feigin-Frenkel-Lusztig conjecture

$$\left[\bar{\Delta}(x\cdot\sigma) : L(w\cdot\sigma)\right] = p_{A_x,A_w}(1)$$

restricted Verma

σ regular, dominant + further regularity assumptions

From $\mathcal{V}$ to the Hecke algebra

$\mathbb{Z}$-grading on S given by $S_{(2)} = X \otimes_{\mathbb{Z}} k$,

$\rightsquigarrow$ $\mathbb{Z}$-grading on all S-mods we consider.

$$M \in \hat{\mathcal{Z}}\text{-mod}^{(\hat{W}, \leq)} \Longrightarrow M_{[x]} \simeq \bigoplus S\langle n_i \rangle \quad n_i \in \mathbb{Z}$$

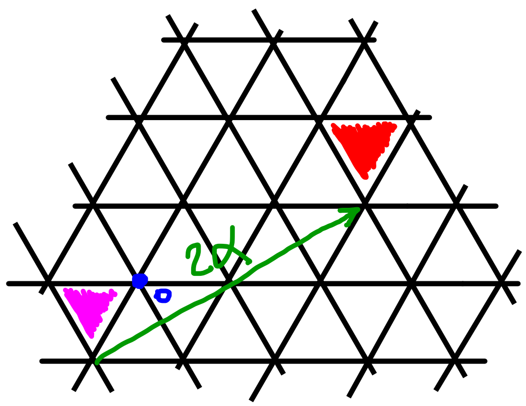
$$\begin{tikzcd}
{[\mathcal{V}]} \arrow[d, "h"] & \ni [M] \arrow[d, mapsto] & {[P_w]} \arrow[d, mapsto] \\
\hat{\mathcal{H}} = \bigoplus_{x \in \hat{W}} \mathbb{Z}[v^{\pm 1}] H_x & \ni \sum_{x \in \hat{W}} v^{\ell(x)} \underline{\mathrm{rk}}\, M_{[x]} H_x & \underline{H}_w = H_w + \sum_{x \in \hat{W}} h_{x,w} H_x
\end{tikzcd}$$

$\mathrm{char}\, k = 0 \iff [\Delta(x \cdot \sigma) : L(w \cdot \sigma)] = h_{x,w}(1) \quad \forall x \in \hat{W}$

$$\underline{\mathrm{rk}} \bigoplus S\langle n_i \rangle = \sum v^{-n_i} \in \mathbb{Z}[v^{\pm 1}]$$

The category $\mathcal{C}$

Since for any alcove $A \in \mathcal{A}$ there exists a unique alcove $\bar{A}$ with the origin in its closure and a unique element $\gamma \in \mathbb{Z}R$ in the coroot lattice such that $A = \bar{A} + \gamma$, we can define



$$\mathcal{A} \longrightarrow W \quad \text{Weyl gp} \left(\longleftrightarrow \begin{array}{l} \text{alcoves with } 0 \\ \text{in their closure} \end{array} \right)$$

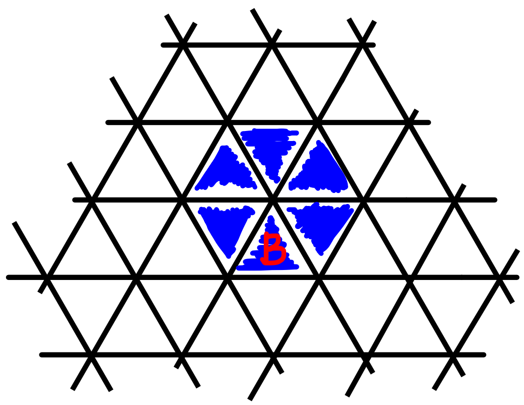
$$A \longmapsto \bar{A}$$

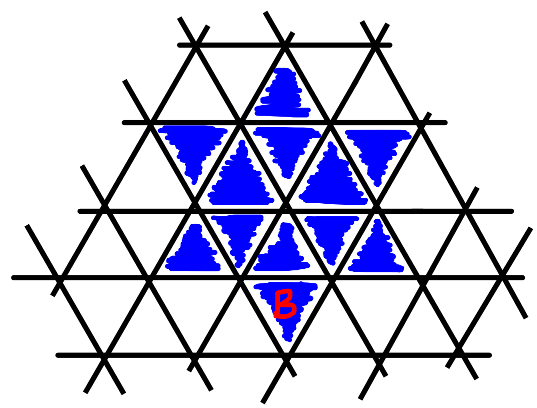
Let $\mathcal{Z}$ be again the structure algebra of $\mathcal{G} = \mathcal{G}(\mathfrak{g}, \mathfrak{b}, \mathfrak{h})$.

$\mathcal{C}$ is the full subcategory of $\mathcal{Z}\text{-mods}^{(\mathcal{A},\leq)}$ consisting of objects $(M, (M_I))$ such that $(z_x)_{x \in W} \in \mathcal{Z}$ acts on $M_{[A]}$ via $z_{\bar{A}}$.

Projective objects in $\mathcal{C}$

<u>Thm</u> (Fiebig-L) For any alcove $B \in \mathcal{A}$ there exists a unique indecomposable projective object P_B in $\mathcal{C}$ with the property that $(P_B)_{[A]} = 0$ unless $A \preceq B$ and $(P_B)_{[B]} \cong S$.





<u>Thm</u>. (Fiebig-L, 15) If char $k = 0$ or char $k \gg 0$ $\underline{ch}(P_B)_{[A]} = p_{A,B}$

From $\mathcal{C}$ to the periodic module

$\mathbb{Z}$-grading on S given by $S_{(2)} = X \otimes_{\mathbb{Z}} \Bbbk$,

$\rightsquigarrow$ $\mathbb{Z}$-grading on all S-mods we consider.

$$M \in \mathbb{Z}\text{-mod}^{(\mathcal{A},\leq)} \Longrightarrow M_{[A]} \simeq \bigoplus S\langle n_i \rangle \quad n_i \in \mathbb{Z}$$

$$\begin{array}{ccccc}
[\mathcal{C}] & \ni & [M] & & [P_B] \\
\Big\downarrow h & & \Big\downarrow & \text{char}\, k = 0 \atop \gg 0 & \Big\downarrow \\
\mathcal{P} = \bigoplus_{A \in \mathcal{A}} \mathbb{Z}[v^{\pm 1}]A & \ni & \sum_{A \in \mathcal{A}} v^{??} \underline{\mathrm{rk}}\, M_{[A]}\, A & & \underline{D}_B = B + \sum_{A \in \mathcal{A}} p_{A,B}\, A
\end{array}$$

$$\underline{\mathrm{rk}} \bigoplus S\langle n_i \rangle = \sum v^{-n_i} \in \mathbb{Z}[v^{\pm 1}]$$

From $\mathcal{V}$ to the Hecke algebra

$\mathbb{Z}$-grading on S given by $S_{(2)} = X \otimes_{\mathbb{Z}} k$,

$\rightsquigarrow$ $\mathbb{Z}$-grading on all S-mods we consider.

$$M \in \hat{Z}\text{-mod}^{(\hat{W}, \leq)} \implies M_{[x]} \simeq \bigoplus S\langle n_i \rangle \quad n_i \in \mathbb{Z}$$

$$\begin{array}{ccc} [\mathcal{V}] & \ni [M] & [P_w] \\ \downarrow h & \downarrow & \downarrow \\ \hat{\mathcal{H}} = \bigoplus_{x \in \hat{W}} \mathbb{Z}[v^{\pm 1}] H_x & \ni \sum_{x \in \hat{W}} v^{\ell(x)} \underline{\mathrm{rk}}\, M_{[x]} H_x & \underline{H}_w = H_w + \sum_{x \in \hat{W}} h_{x,w} H_x \end{array}$$

$$\text{char } k = 0 \iff [\Delta(x \cdot \sigma) : L(w \cdot \sigma)] = h_{x,w}(1) \quad \forall x \in \hat{W}$$

$$\underline{\mathrm{rk}} \bigoplus S\langle n_i \rangle = \sum v^{-n_i} \in \mathbb{Z}[v^{\pm 1}]$$

From $\mathcal{C}$ to the periodic module

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$$\begin{tikzcd}
{[\mathcal{C}]} \arrow[d, "h"] & \ni [M] \arrow[d, mapsto] & {[P_B]} \arrow[d, mapsto, "\operatorname{char} k = p > 0"'] \\
\mathcal{P} = \bigoplus_{A \in \mathcal{A}} \mathbb{Z}[v^{\pm 1}]A & \ni \sum_{A \in \mathcal{A}} v^{??}\, \underline{\mathrm{rk}}\, M_{[A]}\, A & \underline{D}_B = B + \sum_{A \in \mathcal{A}} p_{A,B}\, A
\end{tikzcd}$$

$$\implies [Z(x \cdot_p 0) : L(w \cdot_p 0)] = p_{A,B}(1)$$

$$A = xA_0^+, \quad B = wA_0^+ \quad \forall x \in \widehat{W}$$

$$\underline{\mathrm{rk}} \bigoplus S\langle n_i \rangle = \sum v^{-n_i} \in \mathbb{Z}[v^{\pm 1}]$$

From $\mathcal{C}$ to the periodic module

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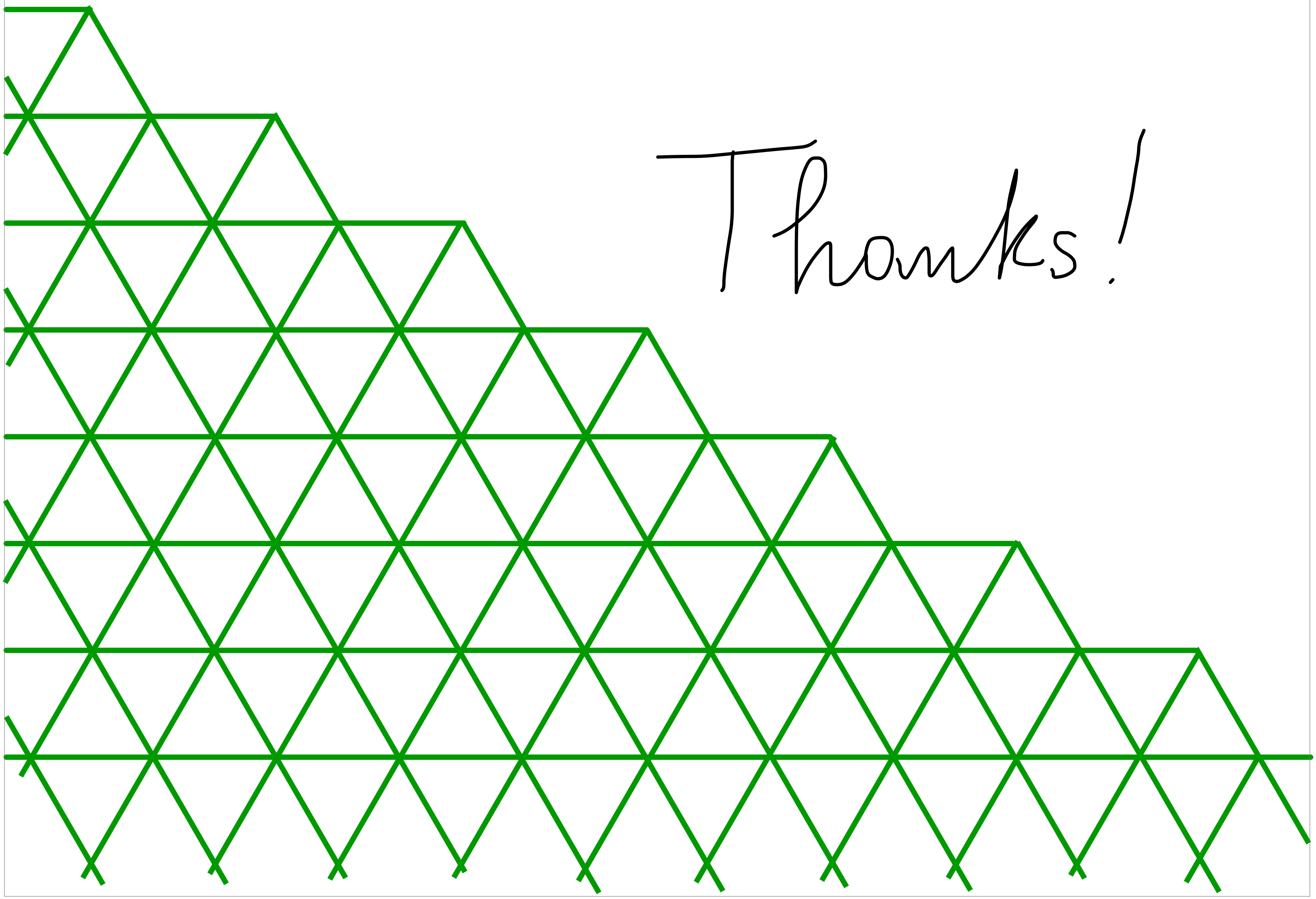
$$\begin{tikzcd}
{[\mathcal{C}]} \arrow[d, "h"] & \ni [M] \arrow[d, mapsto] & {[P_B]} \arrow[d, mapsto] \arrow[r, leftrightsquigarrow, "??"] & {[\bar{\Delta}(x\cdot\sigma) : L(w\cdot\sigma)] = p_{A,B}(1)} \\
P = \bigoplus_{A\in\mathcal{A}} \mathbb{Z}[v^{\pm 1}]A & \ni \sum_{A\in\mathcal{A}} v^{??}\, \underline{rk}\, M_{[A]}\, A & \underline{D}_B = B + \sum_{A\in\mathcal{A}} p_{A,B}\, A & A = xA_0^+,\ B = wA_0^+ \quad \forall x \in \widehat{W}
\end{tikzcd}$$

$$\underline{rk} \bigoplus S\langle n_i\rangle = \sum v^{-n_i} \in \mathbb{Z}[v^{\pm 1}]$$



Thanks!