Local-global conjectures in the representation theory of finite groups

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Philosophy

G a finite group, p a prime

Sylow p-subgroups $P \leq G$ and their normaliser $N_G(P)$
control aspects of the structure of G.

Theorem (Brauer–Suzuki)

Assume that a Sylow 2-subgroup of G is quaternion. Then G is not simple.

In general, for $D \leq G$ any p-subgroup, consider *local subgroups* $N_G(D)$.

Theorem (Frobenius)

Assume that $N_G(D)/C_G(D)$ is a p-group for every p-subgroup $1 \neq D$. Then G has a normal subgroup $N \lhd G$ with $|G : N| = |P|$.

Similarly, representation theory should be controlled by local data.

Characters of finite groups

G finite group, K a field

Irreducible representations

$$\rho : G \to \mathrm{GL}_n(K)$$

are determined by their *characters*

$$\chi : G \to K, \quad g \mapsto \mathrm{tr}(\rho(g)).$$

G finite $\Longrightarrow$ has only *finitely many* irreducible characters over K

$\mathrm{Irr}(G)$ = characters of the irreducible complex representations of G

$\chi \in \mathrm{Irr}(G)$: *character degree* $\chi(1)$ (dimension of representation)

Frobenius: $\chi(1)$ divides $|G|$, $\quad \sum_\chi \chi(1)^2 = |G|$.

Further arithmetic properties?

The McKay conjecture

Example

$G = \mathfrak{S}_5$: $\{\chi(1) \mid \chi \in \mathrm{Irr}(G)\} = \{1, 1, 4, 4, 5, 5, 6\}$, and for

$p = 2$: $\{\chi(1) \mid \chi \in \mathrm{Irr}(N_G(P_2))\} = \{1, 1, 1, 1, 2\}$,

$p = 3$: $\{\chi(1) \mid \chi \in \mathrm{Irr}(N_G(P_3))\} = \{1, 1, 1, 1, 2, 2\}$,

$p = 5$: $\{\chi(1) \mid \chi \in \mathrm{Irr}(N_G(P_5))\} = \{1, 1, 1, 1, 4\}$.

For p a prime let

$$\mathrm{Irr}_0(G) = \{\chi \in \mathrm{Irr}(G) \mid \chi(1) \not\equiv 0 \pmod p\}$$

Conjecture (McKay, 1972)

G finite group, p prime, $P \leq G$ a Sylow p-subgroup of G. Then:

$$|\mathrm{Irr}_0(G)| = |\mathrm{Irr}_0(N_G(P))|.$$

Thus: global information $\mathrm{Irr}_0(G)$ controlled by local data $\mathrm{Irr}_0(N_G(P))$.

The McKay conjecture, II

Theorem (Isaacs–M.–Navarro (2007))

The McKay conjecture holds for all groups, if all finite non-abelian simple groups satisfy a certain stronger inductive condition (iMcK).

Condition (iMcK) for a simple group S involves

1. the *universal central extension* $G \twoheadrightarrow S$ of S,
2. a bijection $\mathrm{Irr}_0(G) \longrightarrow \mathrm{Irr}_0(N_G(P))$, which is
3. $\mathrm{Aut}(S)$-equivariant, and
4. extension properties of characters in $\mathrm{Irr}_0(G)$, $\mathrm{Irr}_0(N_G(P))$.

Example

If $\mathrm{Out}(S) = 1$ then: (iMcK) satisfied for $S \iff$
all central extension $G \twoheadrightarrow S$ satisfy ordinary McKay conjecture.

The McKay conjecture, III

The condition (iMcK) has so far been proved for

1. the sporadic simple groups
2. the alternating groups
3. 9 out of 16 series of groups of Lie type (Cabanes–Späth ('13))
4. groups of Lie type for p the 'defining prime' (Maslowski, Späth ('12))

For a complete solution the character theory of simple groups of Lie type has to be developed further.

Open problem

Understand the action of $\mathrm{Aut}(S)$ *on* $\mathrm{Irr}(S)$ *(S of Lie type).*

Blocks

G finite group, p prime, $\mathcal{O} \geq \mathbb{Z}_p$ large enough $\Longrightarrow$

$$\mathcal{O}G = B_1 \oplus \ldots \oplus B_r$$

minimal 2-sided ideals: *Brauer p-blocks*, and

$$\mathrm{Irr}(G) = \mathrm{Irr}(B_1) \sqcup \ldots \sqcup \mathrm{Irr}(B_r).$$

Brauer: To a p-block B are attached

- a *defect group* D (a p-subgroup of G),
- a *Brauer corresponding block* b of $N_G(D)$.

Example

- $p \nmid |G| \Longrightarrow |\mathrm{Irr}(B)| = 1$ and $D = 1$ for all blocks,
- $|G| = p^a \Longrightarrow$ just one block, with $D = G$,
- always: one block with $1_G \in \mathrm{Irr}(B)$, and D a Sylow p-subgroup, the *principal block*.

The Alperin–McKay conjecture

B a p-block with defect group D, then set

$$\mathrm{Irr}_0(B) = \{\chi \in \mathrm{Irr}(B) \mid \chi(1)_p = |G:D|_p\} \quad (\textit{height 0 characters}).$$

Example

If D is Sylow p-subgroup $\Longrightarrow$

$$\mathrm{Irr}_0(B) = \{\chi \in \mathrm{Irr}(B) \mid \chi(1)_p = 1\} = \mathrm{Irr}_0(G) \cap \mathrm{Irr}(B).$$

McKay conjecture has refinement to block-wise version:

Conjecture (Alperin–McKay, 1976)

Let B be a p-block of G with Brauer correspondent b. Then

$$|\mathrm{Irr}_0(B)| = |\mathrm{Irr}_0(b)|.$$

Again: global data are predicted by local data

The Alperin–McKay conjecture, II

Proved for

- p-solvable groups
- symmetric groups,...

Theorem (Späth (2013))

The Alperin–McKay conjecture holds for all groups, if all finite non-abelian simple groups satisfy a certain stronger inductive condition (iAM).

Again, (iAM) for a simple group S concerns

- universal central extension of S,
- an $\mathrm{Aut}(S)$-equivariant bijection $\mathrm{Irr}_0(B) \to \mathrm{Irr}_0(b)$,
- extension properties of characters in $\mathrm{Irr}_0(B)$, $\mathrm{Irr}_0(b)$.

Brauer's height 0 conjecture

When is $\mathrm{Irr}_0(B) = \mathrm{Irr}(B)$ (all characters of B have the same height)?

Example

$G = \mathfrak{S}_5$, B principal p-block

$p = 5$: $\{\chi(1) \mid \chi \in \mathrm{Irr}(B)\} = \{1, 1, 4, 4, 6\}$, 5-Sylow abelian

$p = 3$: $\{\chi(1) \mid \chi \in \mathrm{Irr}(B)\} = \{1, 4, 5\}$, 3-Sylow abelian

$p = 2$: $\{\chi(1) \mid \chi \in \mathrm{Irr}(B)\} = \{1, 1, 5, 5, 6\}$, 2-Sylow non-abelian.

Conjecture (Brauer, 1955)

Let B be a p-block of G with defect group D. Then:

$$\mathrm{Irr}_0(B) = \mathrm{Irr}(B) \quad \Longleftrightarrow \quad D \text{ abelian}.$$

The local structure (of D) should determine arithmetic properties of $\mathrm{Irr}(B)$.

Brauer's height 0 conjecture, II

Theorem (Berger–Knörr (1988))

The '$\Longleftarrow$'-direction of BHZ holds, if it holds for all quasi-simple groups.

G *quasi-simple* $:\Longleftrightarrow G/Z(G)$ simple, $G = [G, G]$

Theorem (Kessar–M. (2013))

The '$\Longleftarrow$'-direction of BHZ holds (for all quasi-simple, so) for all groups.

Ingredient: classification of all blocks of finite simple groups of Lie type.

Theorem (Navarro–Späth (2012))

The '$\Longrightarrow$'-direction of BHZ holds, if all finite non-abelian simple groups satisfy the inductive (iAM) condition.

The Alperin weight conjecture

For representation $G \to \mathrm{GL}_n(K)$ over field K of characteristic $p > 0$ one defines *Brauer character* $\varphi : G_{p'} \to \mathbb{C}$.

$\mathrm{IBr}(G)$ = Brauer characters of the irreducible representations of G in characteristic p

Have $\quad \mathrm{IBr}(G) = \mathrm{IBr}(B_1) \sqcup \ldots \sqcup \mathrm{IBr}(B_r)$.

Conjecture (Alperin, 1980)

B a p-block of G with abelian defect group, Brauer correspondent b. Then

$$|\mathrm{IBr}(B)| = |\mathrm{IBr}(b)|.$$

(There is a version for general defect groups, involving *weights*.)

Again: Global information on $\mathrm{IBr}(G)$ in characteristic $p > 0$ ist controlled by local data on $\mathrm{IBr}(N_G(D))$.

The Alperin weight conjecture, II

Theorem (Navarro–Tiep (2011); Späth (2013))

The Alperin weight conjecture holds for all groups, if all finite non-abelian simple groups satisfy a certain stronger inductive condition (iBAW).

The condition (iBAW) has been shown for

1. sporadic groups (An–Dietrich ('12))
2. alternating groups (Alperin–Fong, Michler–Olsson ('91), M. ('14))
3. all groups of Lie type and p the defining prime (Späth ('13))
4. groups with cyclic defect group (Koshitani–Späth ('14))

For a complete solution need more information on groups S of Lie type:

Open problem

Understand the decomposition matrix between $\mathrm{Irr}(S)$ *and* $\mathrm{IBr}(S)$.

The Alperin weight conjecture, III

For $\chi \in \mathrm{Irr}(G)$ let χ° the restriction to $G_{p'}$. Then

$$\chi^\circ = \sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi,\varphi}\,\varphi \qquad \text{for suitable } d_{\chi,\varphi} \geq 0.$$

$D = (d_{\chi,\varphi})$ is the *decomposition matrix*.

Theorem (Koshitani–Späth (2013))

The Alperin weight conjecture holds for all blocks with abelian defect, if all finite non-abelian simple groups S satisfy the (iAM) condition, and

- *the decomposition matrix of S has upper triangular shape such that*
- *the characters in the triangular part are* $\mathrm{Aut}(S)$*-invariant.*

Thus, the inductive condition (iAM) is relevant for *all three*:
the Alperin–McKay, Brauer height 0, and Alperin weight conjecture.

Simple groups of Lie type

All inductive conditions have been shown to hold for sporadic and alternating groups.
So 'only' groups of Lie type remain.

For $\mathbb{F}_q$ finite field, have classical groups

$$\mathrm{PSL}_n(q) := \mathrm{SL}_n(\mathbb{F}_q)/Z(\mathrm{SL}_n(\mathbb{F}_q)),\ \mathrm{PSp}_{2n}(q),\ \mathrm{SO}_{2n+1}(q)', \ldots$$

In fact, for every simple complex Lie algebra there exists a simple group of Lie type over $\mathbb{F}_q$, i.e., for types A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2.

Additionally, twisted types, like unitary groups

$$\mathrm{PSU}_n(q) = \mathrm{SU}_n(q)/Z(\mathrm{SU}_n(q)),$$

Steinberg triality groups, Suzuki groups and Ree groups.

In total, 6 doubly infinite series of classical groups and 10 further series of exceptional simple groups of Lie type.

Proving the inductive conditions

'Only' need to deal with simple groups of Lie type.

Philosophy:

1) the character theory of groups of Lie type is 'similar' to that of $\mathrm{SL}_n(q)$ (Lusztig's theory),
2) the character theory of $\mathrm{SL}_n(q)$ is 'similar' to that of $\mathrm{GL}_n(q)$ (Clifford theory, Späth's Lemma),
3) for $\mathrm{GL}_n(q)$ 'sufficient' to look at unipotent characters(Bonnafé–Rouquier Morita-equivalences),
4) the character theory of $\mathrm{GL}_n(q)$ is 'controlled by' that of its Weyl group, the symmetric group $\mathfrak{S}_n$ (generalised Harish-Chandra theory),
5) for $\mathfrak{S}_n$ 'everything' can be reduced to combinatorics of partitions (hooks, cores, degree formula,...).

Example: unipotent blocks of $\mathrm{GL}_n(q)$

$G = \mathrm{GL}_n(q)$, $p \nmid q$. Consider *unipotent characters*

$$\mathrm{Uch}(G) := \{\text{constituents of } \mathrm{Ind}_B^G(1)\} \subseteq \mathrm{Irr}(G)$$

(permutation character on Borel subgroup $B \leq G$).

Theory of Hecke algebras: $\mathrm{Uch}(G) \overset{1-1}{\longleftrightarrow} \{\lambda \vdash n\}, \quad \chi_\lambda \leftrightarrow \lambda.$

Important parameter: $d := d_p(q) :=$ order of q modulo p

Theorem (Fong–Srinivasan (1982))

$\chi_\lambda, \chi_\mu \in \mathrm{Uch}(G)$ *lie in same p-block* $\iff$ d-*core*$(\lambda) = d$-*core*(μ)

Note: generic, combinatorial description, does not mention p or q

Example

$p = 2 \Rightarrow d = 1$; all 1-cores are trivial$\Longrightarrow$ $\mathrm{Uch}(G)$ lies in a single 2-block

The bijection for $p|(q-1)$ in $\mathrm{GL}_n(q)$

Example

Consider (iMcK) for primes p dividing $q-1$ for $\mathrm{GL}_n(q)$, $p > n$.

Global situation

Fong–Srinivasan: $\mathrm{Uch}(G)$ lies in single p-block B, and

$$\mathrm{Irr}_0(B) \supset \mathrm{Uch}(G) \stackrel{1-1}{\longleftrightarrow} \mathcal{H}(\mathfrak{S}_n) \stackrel{1-1}{\longleftrightarrow} \{\lambda \vdash n\}.$$

Local situation

Sylow p-subgroup lies in maximal torus $T = \{\mathrm{diag}(a_1, \ldots, a_n)\}$, with normaliser $N_G(P) = T.\mathfrak{S}_n$.

Then

$$\mathrm{Irr}_0(N_G(P)) \supset \mathrm{Irr}_0(\mathfrak{S}_n) = \mathrm{Irr}(\mathfrak{S}_n) \stackrel{1-1}{\longleftrightarrow} \{\lambda \vdash n\}.$$

Still need to worry about the 'other' characters...

Blocks and Lusztig induction

More generally: for G of Lie-type, p-blocks are described *geometrically*: by constituents of Lusztig induced characters $R_L^G(\lambda)$, where

- $L \leq G$ d-split Levi subgroup,
- $\lambda \in \mathrm{Irr}(L)$ d-cuspidal.

(Fong–Srinivasan ('80s), Broué–M.–Michel ('93), Cabanes–Enguehard ('90s), Bonnafé–Rouquier ('03), Kessar–M. ('13))

(Lusztig induction is defined by ℓ-adic cohomology of certain varieties attached to algebraic group of G.)

Leads to combinatorial description of p-blocks in terms of hooks, cohooks of Lusztig symbols, via associated cyclotomic Hecke algebras.

Further directions

Structural explanations for the required bijections?

Conjecture (Broué, 1989)

B a p-block of G with abelian defect group D, Brauer correspondent b. Then: B-mod is derived equivalent to b-mod.

Broué: Implies all previous conjectures *when D abelian*.

Proved for several classes of groups, but no reduction so far.

Open problem

Find right generalisation to non-abelian defect.

Characters of positive height

Alperin–McKay conjecture, BHZ: only about characters of height zero.

What positive heights occur when defect non-abelian?

Two answers: Dade's conjecture (see talk of Späth); and very recently:

Conjecture (Eaton–Moreto, 2013)

B a p-block of G with defect group D. Then $mh(B) = mh(D)$.

Here $\mathrm{mh}(B)$ is the smallest non-zero height in the block B.

Theorem (Brunat–Malle (2014))

The Eaton–Moreto conjecture holds for the principal blocks of all quasi-simple groups.

But no reduction to simple groups, yet.