Langlands Duality and T-Duality

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mostly joint work with Ulrich Bunke

Theorem

Theorem (Daenzer-van Erp, Bunke-N.)

G: compact Lie group, without B,C factors
G^L: Langlands dual Lie group
Then G and G^L are (topologically) T-dual to each other.

Dual Tori

T: Torus (abelian, compact, connected Lie group)

$$T \cong V/\Gamma \cong U(1)^n.$$

$\hat{T}$: dual torus

$$\hat{T} := \mathrm{Hom}\big(H_1 T, U(1)\big) \cong V^*/\Gamma^* \cong U(1)^n$$

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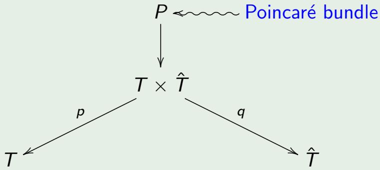
Example



$c_1(P) = \sum_{i=1}^n \theta_i \cup \hat{\theta}_i$

$\theta_i \in H^1(T)\, \hat{\theta}_i \in H^1(\hat{T})$

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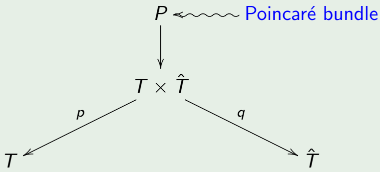

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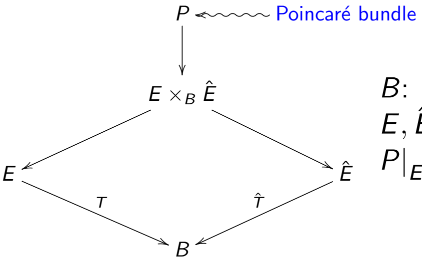
Isomorphisms

$K^*(T) \cong K^{*-\dim \hat{T}}(\hat{T})$ $\qquad \alpha \mapsto q_!(p^*\alpha \cup [P])$

$HP^*(T, \mathbb{Q}) \cong HP^{*-\dim \hat{T}}(\hat{T}, \mathbb{Q})$ $\qquad \alpha \mapsto q_!(p^*\alpha \cup \mathrm{ch}[P])$

$D(\mathrm{Sky}\, T) \cong D(\mathrm{Loc}\, \hat{T})$ $\qquad \alpha \mapsto q_*(p^*\alpha \otimes P)$

Parametrised version: T-Duality



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$E, \hat{E}$:	principal bundles
$P\|_{E_b \times \hat{E}_b}$:	Poincaré bundle.

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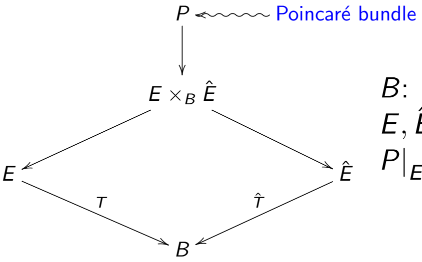

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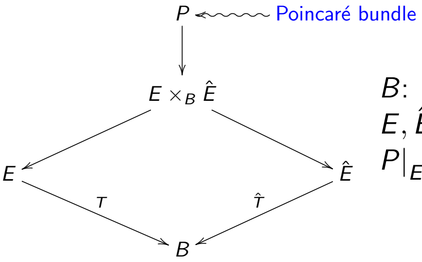

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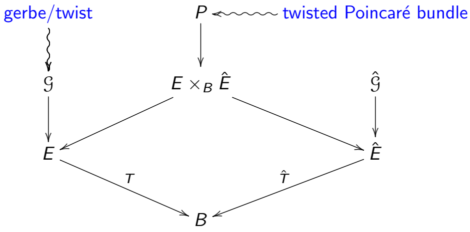
Solution: Allow twisted Poincaré bundle.



Twists and twisted vector bundles

- A twist over M is a $U(1)$-gerbe $\mathcal{G} \to M$
  (classified by $\check{H}^2(M, U(1)) \cong H^3(M, \mathbb{Z})$)
- For simplicity: open covering $\{U_i\}$ + cech 2-cocycle

$$g_{ijk} : U_i \cap U_j \cap U_k \to U(1)$$

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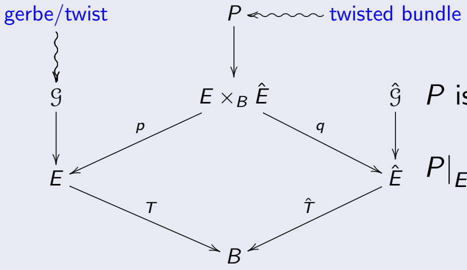
- A twisted vector bundle is given by
  1. vector bundles $L_i \to U_i$
  2. transition functions $\varphi_{ij} : L_i|_{U_i} \to L_j|_{U_j}$
  3. s.t. $\varphi_{ik} = g_{ijk} \cdot \varphi_{jk} \circ \varphi_{ij}$
- Twisted K-theory $K^0_{\mathcal{G}}(M)$: Grothendieck group of twisted vector bundles

T-Duality

Definition (Mathai et al., Bunke-Schick,...)

A T-Duality is a diagram



P is $p^*\mathcal{G} - q^*\hat{\mathcal{G}}$-twisted

$P|_{E_b \times \hat{E}_b}$ is Poincaré-bundle.

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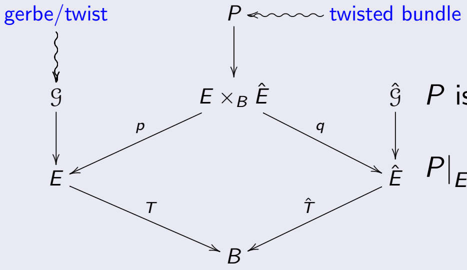

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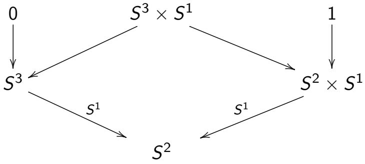
Warning

$(E, \mathcal{G})$ *does not determine* $(\hat{E}, \hat{\mathcal{G}})$.

Examples

1. $(S^3, 0) \sim_T (S^2 \times S^1, 1)$



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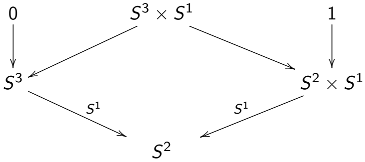

2. $(S^5, 0) \sim_T (\mathbb{C}P^2 \times S^1, c_1 \cup \theta)$
3. $(E, 0) \sim_T (B \times \hat{T}, \sum_{i=1}^n c_i E \cup \hat{\theta}_i)$

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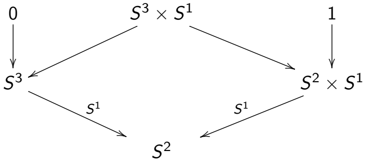

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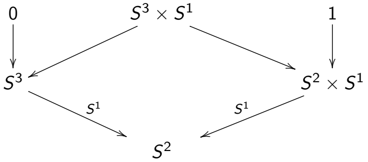

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6. $(SU(2), 2) \sim_T (SO(3), 1)$

Physical interpretation/motivation: $(E, T) \sim_T (\hat{E}, \hat{T})$
$\rightsquigarrow$ WZW models with target E and $\hat{E}$ are equivalent

Consequences

Theorem

For $(E, \mathcal{G}) \sim_T (\hat{E}, \hat{\mathcal{G}})$ *we have isomorphism*

$$K^*_{\mathcal{G}}(E) \xrightarrow{\simeq} K^{*-\dim \hat{T}}_{\hat{\mathcal{G}}}(\hat{E}) \qquad \text{(Bunke-Schick)}$$

$$HP^*_{\mathcal{G}}(E, \mathbb{Q}) \xrightarrow{\simeq} HP^{*-\dim \hat{T}}_{\hat{\mathcal{G}}}(\hat{E}, \mathbb{Q}) \qquad \text{(Mathai-Rosenberg)}$$

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- $K^0_{[1]}(S^2 \times S^1) \cong K^1(S^3) \cong \mathbb{Z}$
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Theorem

$$D(\mathrm{Sky}_{\mathcal{G}}\, T) \xrightarrow{\sim} D(\mathrm{Loc}_{\hat{\mathcal{G}}}\, \hat{T}) \qquad \text{(Ruderer)}$$

$$\widehat{K}^*_{\mathcal{G}}(E) \xrightarrow{\sim} \widehat{K}^{*-\dim \hat{T}}_{\hat{\mathcal{G}}}(\hat{E}) \qquad \text{(Kahle-Valentino)}$$

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Corollary

$K^{*+\mathcal{G}}(G) \cong K^{*-\mathrm{rank}\,G+\hat{\mathcal{G}}}(G^L)$

Sketch of Proof

General Method: $E \to B$ torus $T = V/\Gamma$ bundle, Leray-Serre:

$$H^p(B) \otimes \Lambda^q(\Gamma^*) \Rightarrow H^{p+q}(E)$$

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Lemma

- $(E, \mathcal{G})$ *admits* T*-dual* $\Leftrightarrow [\mathcal{G}] \in F^2 H^3(E)$
- $\left\{ T\text{-duals for } (\hat{E}, \hat{\mathcal{G}}) \right\}$
  $\overset{1-1}{\longleftrightarrow} \left\{ \text{Representatives } (e, f) \text{ for } [\mathcal{G}] \text{ in } H^3 B \oplus H^2 B \otimes \Gamma^* \right\}$
- $c_i \hat{E} = f(\theta_i)$

Sketch of Proof II

- G compact Lie, $T \subset G$ maximal torus
- $R \subset t^*$ roots, $C \subset t$ coroots
- $\Gamma^* = \mathrm{Hom}(T, U(1)) = H^1 T$ weight lattice
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  (aside: $\pi_1(G) = \Gamma / C$, $ZG = (\Gamma^* / R)^*$)
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Facts

- *$H^*(B)$ torsion free, even*

$$H^2 B = \Gamma_C^* \qquad H^4 B = \mathrm{Sym}^2(\Gamma_C)_W^*$$

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Facts

- *$H^*(B)$ torsion free, even*

  $$H^2 B = \Gamma^*_C \qquad H^4 B = \mathrm{Sym}^2(\Gamma_C)^*_W$$
- *If G has no B, C factors*
  1. $\exists f : \mathfrak{t} \xrightarrow{\sim} \mathfrak{t}^*$ *with* $f|_C : C \xrightarrow{\sim} R$.
  2. $\exists G/T \xrightarrow{\sim} G^L/T^L$

Sketch of Proof III

Leray Serre for $G \to G/T$ E_2-page: $H^p(B) \otimes \Lambda^q(\Gamma^*)$

$\Lambda^3\Gamma^*$	0	$\bullet$	0	$\bullet$
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$\bullet$	0	$\Gamma_C^* \otimes \Gamma^*$	0	$\bullet$
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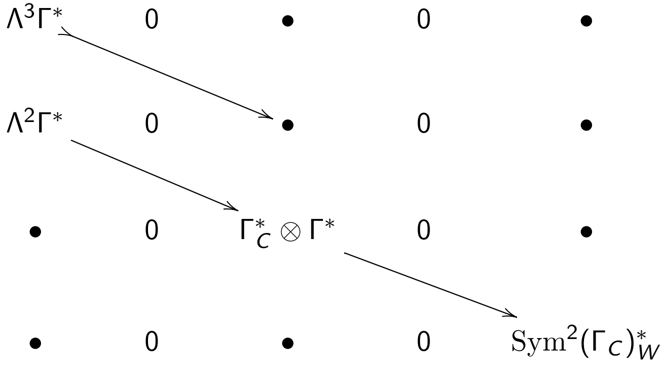
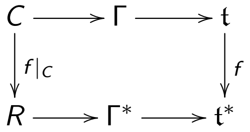


Take element in $\Gamma_C^* \otimes \Gamma^* = \mathrm{Hom}(\Gamma_C, \Gamma^*)$ induced by



Questions

- G simple, simply connected
  $H^3(G, \mathbb{Z}) = \mathbb{Z} =$ Cohomology of sequence

  $$\Lambda^2\Gamma^* \to \Gamma^* \otimes \Gamma^* \to \mathrm{Sym}^2(\Gamma_C)^*_W$$

  Level of $f \in \Gamma_* \otimes \Gamma_*$?
- Why is T-dual of $G \to G/T$ a group?
- Relation to geometric Langlands correspondences?

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Proposition

Every cohomology theory E^* which admits T-Duality isomorphisms can be expressed in terms of complex K-theory. More precisely: E is a KU-module spectrum.

Proof uses Snaith's Theorem $\Sigma^\infty_+ K(\mathbb{Z},2)[b^{-1}] \simeq KU$.

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Corollary

There is an equivalence of categories between

$$\{\textit{Cohomology theories with } T\textit{-Duality isomorphisms}\}$$

and

$$D\big(\mathbb{Z}[b, b^{-1}]\big) \quad \leftarrow \textit{derived category}\ , |b| = 2$$

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$$H^3(M) \cong [M, K(\mathbb{Z},3)] \cong [M, \mathrm{Spin}_\infty] \to [M, O_\infty]$$

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Assume $(E, \mathcal{G}) \sim_T (\hat{E}, \hat{\mathcal{G}})$ *over* B, B *stably framed,* $\dim E \leq 6$. *Then*

$$E_{[\mathcal{G}]} = \hat{E}_{[\hat{\mathcal{G}}]} \qquad \in \Omega_*^{fr}$$

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Example

$(SU(2), 2) \sim_T (SO(3), 1)$ over S^2
$\Omega_3^{fr} \cong \pi_3^{st} \cong \mathbb{Z}/24\nu$
$SU(2)_2 = \nu + 2\nu$
$SO(3)_1 = 2v + v$

Generalization

- $\mathcal{E} \to \mathrm{Spec} R$ elliptic curve with certain properties
  $\rightsquigarrow$ elliptic cohomology theory
- 'Universal' elliptic curve over $\overline{\mathcal{M}}_{ell}$
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Conjecture (N.)

For $(E, \mathcal{G}) \sim_T (\hat{E}, \hat{\mathcal{G}})$ over B, the classes

$$\int_{E/B} \sigma(\mathcal{G}) \qquad \textit{and} \qquad \int_{\hat{E}/B} \sigma(\hat{\mathcal{G}})$$

in $\mathrm{tmf}^{-\dim T}(B)$ agree.