Spherical varieties: interactions with representation theory and generalizations
Part II

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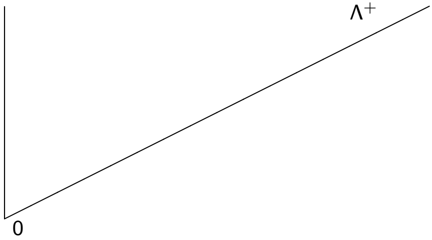

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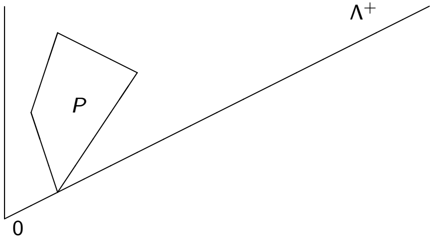

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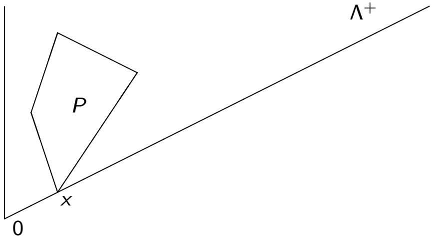

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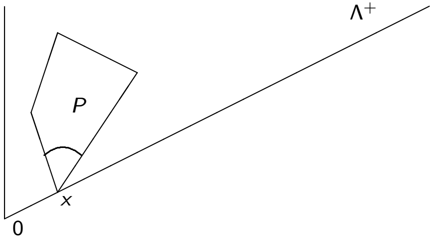

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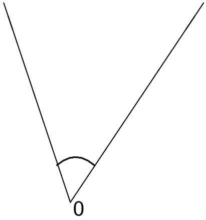

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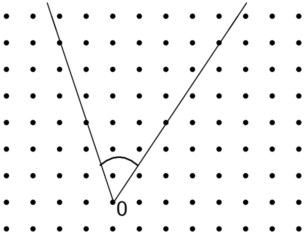

(lattice determined by the principal isotropy)

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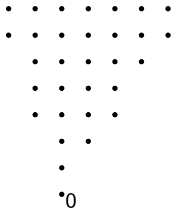

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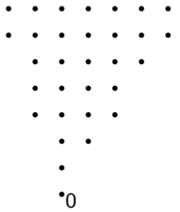

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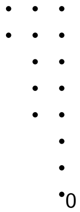

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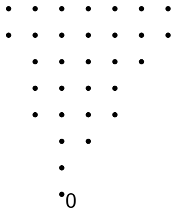

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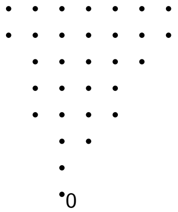

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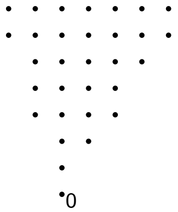

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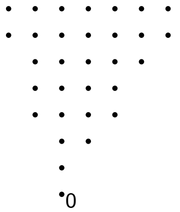

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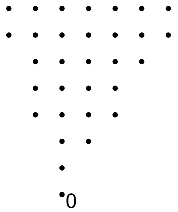

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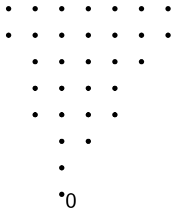

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The description of H rests ultimately on a list of ~ 30 basic cases.

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B = Borel subgroup of G

Goal: describe the finite set

$$\mathfrak{B}(X) = \{B\text{-orbits of } X\}$$

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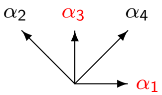
B-orbits: example

$G = \mathrm{Sp}(4)$
$H = TU_{\alpha_2}U_{\alpha_4}$



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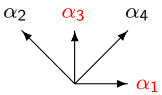

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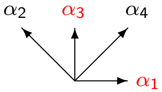

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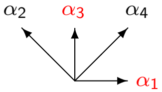

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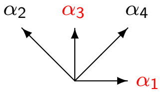

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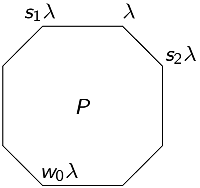

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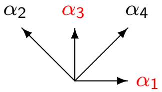

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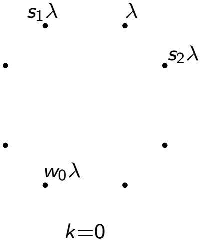

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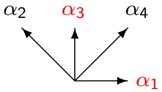
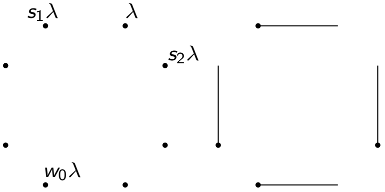


Subpolytopes: $w\lambda + (P \cap w(\mathbb{Q}\alpha_{i_1} + \ldots + \mathbb{Q}\alpha_{i_k}))$ provided $\alpha_{i_j} \in \Psi$ and $w(\alpha_{i_j}) < 0$ for all j.



k=0

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B-orbits: example

$$G = \mathrm{Sp}(4)$$
$$H = TU_{\alpha_2}U_{\alpha_4}$$
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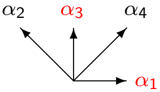

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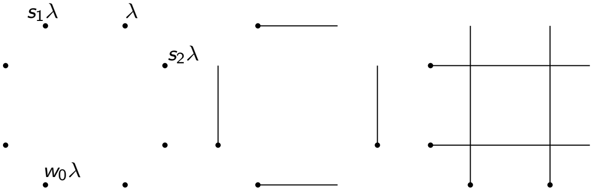

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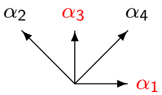

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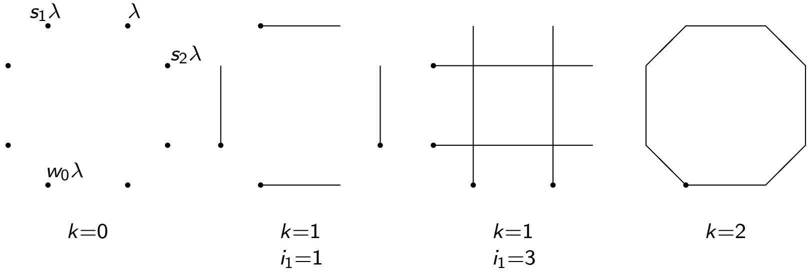

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Project: Classify log-homogeneous varieties using the Luna-Vust invariants (as a G-variety) $+\Sigma_A(X)$.

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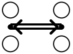

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