Counting representations of free groups

M. Reineke

BU Wuppertal

10 March 2015, Bad Honnef

(joint with S. Mozgovoy, arXiv:1402.6923)

Motivation 1

Fact

For a finite group Γ*,*

$$\#\{\mathbb{C}\text{-}irreps\ of\ \Gamma\}/\simeq\ =\ \#\{conj.\ classes\ of\ \Gamma\}.$$

Motivation 1

Fact

For a finite group Γ*,*

$$\#\{\mathbb{C}\text{-irreps of }\Gamma\}/\simeq\; = \;\#\{\text{conj. classes of }\Gamma\}.$$

Question: What about infinite groups?

Motivation 1

Fact

For a finite group Γ,

$$\#\{\mathbb{C}\text{-irreps of } \Gamma\}/\simeq \; = \; \#\{\text{conj. classes of } \Gamma\}.$$

Question: What about infinite groups?

Leads to concepts (*for certain classes of groups*) like

- subgroup growth: study the sequence

$$(\#\{\text{subgroups of index } n\})_n,$$

- representation growth: study the sequence

$$(\#\{\text{irreps of dimension } n\}/\simeq)_n.$$

Motivation 2

Fact

For a finite group,

$$\#\{irreps\}/\simeq \;=\; \#\{conj.\ classes\}.$$

Question: What about infinite groups?

Motivation 2

Fact

For a finite group,

$$\#\{irreps\}/\simeq \;=\; \#\{conj.\ classes\}.$$

Question: What about infinite groups?

Today: free groups in $m \geq 1$ generators. Fix such m.

Definition

$$F_m = \underbrace{\mathbb{Z} * \ldots * \mathbb{Z}}_{m-\text{times}}.$$

Motivation 2

Fact

For a finite group,

$$\#\{irreps\}/\simeq \; = \; \#\{conj.\ classes\}.$$

Question: What about infinite groups?

Today: free groups in $m \geq 1$ generators. Fix such m.

Definition

$$F_m = \underbrace{\mathbb{Z} * \ldots * \mathbb{Z}}_{m-\text{times}}.$$

Reason: F_m homologically quite simple:

Motivation 2

Fact

For a finite group,

$$\#\{\textit{irreps}\}/\simeq \; = \; \#\{\textit{conj. classes}\}.$$

Question: What about infinite groups?

Today: free groups in $m \geq 1$ generators. Fix such m.

Definition

$$F_m = \underbrace{\mathbb{Z} * \ldots * \mathbb{Z}}_{m-\text{times}}.$$

Reason: F_m homologically quite simple:

Fact

$$G \text{ finite} : \mathrm{gldim}\, \mathbb{C}G = 0; \quad \mathrm{gldim}\, \mathbb{C}F_m = 1.$$

Subgroup growth for free groups

We will consider numbers of subgroups and "numbers" of completely reducible representations of F_m.

We will consider numbers of subgroups and "numbers" of completely reducible representations of F_m.

Definition

$$s_n = \#\{\text{index } n \text{ subgroups of } F_m\}$$

Subgroup growth for free groups

We will consider numbers of subgroups and "numbers" of completely reducible representations of F_m.

Definition

$$s_n = \#\{\text{index } n \text{ subgroups of } F_m\}$$

Theorem (M. Hall, 1949)

$$\sum_{n\geq 1} s_n \frac{t^n}{n} = \log\left((\sum_{n\geq 0} (n!)^{m-1} t^n \right).$$

We will consider numbers of subgroups and "numbers" of completely reducible representations of F_m.

Definition

$$s_n = \#\{\text{index } n \text{ subgroups of } F_m\}$$

Theorem (M. Hall, 1949)

$$\sum_{n\geq 1} s_n \frac{t^n}{n} = \log\left((\sum_{n\geq 0} (n!)^{m-1} t^n \right).$$

Exercise: Quickly check for $m = 1$.

First main result

Approach: To get finite quantities:

Approach: To get finite quantities:

Count isomorphism classes of representations of F_m over finite fields $\mathbb{F}_q$.

First main result

Approach: To get finite quantities:

Count isomorphism classes of representations of F_m over finite fields $\mathbb{F}_q$. Completely reducible (=semisimple) representations give best results.

First main result

Approach: To get finite quantities:

Count isomorphism classes of representations of F_m over finite fields $\mathbb{F}_q$. Completely reducible (=semisimple) representations give best results.

Definition

$$a_n(q) = \#\{n\text{-dimensional } \textit{completely reducible } \mathbb{F}_q\text{-reps of } F_m\}/\simeq .$$

First main result

Approach: To get finite quantities:

Count isomorphism classes of representations of F_m over finite fields $\mathbb{F}_q$. Completely reducible (=semisimple) representations give best results.

Definition

$$a_n(q) = \#\{n\text{-dimensional } \textit{completely reducible } \mathbb{F}_q\text{-reps of } F_m\}/\simeq .$$

Theorem

$a_n(q)$ behaves polynomially in q, i.e.

$$a_n(q) \in \mathbb{Z}[q],$$

First main result

Approach: To get finite quantities:

Count isomorphism classes of representations of F_m over finite fields $\mathbb{F}_q$. Completely reducible (=semisimple) representations give best results.

Definition

$a_n(q) = \#\{n\text{-dimensional } \textit{completely reducible } \mathbb{F}_q\text{-reps of } F_m\}/\simeq .$

Theorem

$a_n(q)$ behaves polynomially in q, i.e.

$$a_n(q) \in \mathbb{Z}[q],$$

and there is an explicit formula for $a_n(q)$.

First main result

Approach: To get finite quantities:

Count isomorphism classes of representations of F_m over finite fields $\mathbb{F}_q$. Completely reducible (=semisimple) representations give best results.

Definition

$$a_n(q) = \#\{n\text{-dimensional } \textit{completely reducible } \mathbb{F}_q\text{-reps of } F_m\}/\simeq .$$

Theorem

$a_n(q)$ behaves polynomially in q, i.e.

$$a_n(q) \in \mathbb{Z}[q],$$

and there is an explicit formula for $a_n(q)$.

We need some ingredients to state this explicit formula.

Ingredients for the explicit formula 1

Definition

Formal power series ring $R = \mathbb{Q}(q)[[t]]$ with maximal ideal $\mathfrak{m} = t\mathbb{Q}(q)[[t]]$;

Ingredients for the explicit formula 1

Definition

Formal power series ring $R = \mathbb{Q}(q)[[t]]$ with maximal ideal $\mathfrak{m} = t\mathbb{Q}(q)[[t]]$; thus $1 + \mathfrak{m} \subset R^*$.

Ingredients for the explicit formula 1

Definition

Formal power series ring $R = \mathbb{Q}(q)[[t]]$ with maximal ideal $\mathfrak{m} = t\mathbb{Q}(q)[[t]]$; thus $1 + \mathfrak{m} \subset R^*$.

Definition

A certain q-hypergeometric function

$$F(t) = \sum_{n \geq 0} ((q^n - 1) \cdot \ldots \cdot (q - 1))^{m-1} t^n \in \mathbb{Q}(q)[[t]].$$

Definition

Formal power series ring $R = \mathbb{Q}(q)[[t]]$ with maximal ideal $\mathfrak{m} = t\mathbb{Q}(q)[[t]]$; thus $1 + \mathfrak{m} \subset R^*$.

Definition

A certain q-hypergeometric function

$$F(t) = \sum_{n\geq 0} ((q^n - 1) \cdot \ldots \cdot (q-1))^{m-1} t^n \in \mathbb{Q}(q)[[t]].$$

Definition

Shift operator T on R: the $\mathbb{Q}(q)$-linear operator with

$$T(t^n) = q^{(1-m)\binom{n}{2}} t^n.$$

Ingredients for the explicit formula 2

Definition

Plethystic operations:

Ingredients for the explicit formula 2

Definition

Plethystic operations:

- $\mathrm{Exp} : \mathfrak{m} \to 1 + \mathfrak{m}$ unique operator such that

Definition

Plethystic operations:

- $\mathrm{Exp} : \mathfrak{m} \to 1 + \mathfrak{m}$ unique operator such that

$$\mathrm{Exp}(f + g) = \mathrm{Exp}(f) \cdot \mathrm{Exp}(g)$$

Ingredients for the explicit formula 2

Definition

Plethystic operations:

- $\mathrm{Exp} : \mathfrak{m} \to 1 + \mathfrak{m}$ unique operator such that

$$\mathrm{Exp}(f+g) = \mathrm{Exp}(f) \cdot \mathrm{Exp}(g) \text{ and } \mathrm{Ext}(q^i t^n) = \frac{1}{1 - q^i t^n}$$

Definition

Plethystic operations:

- $\mathrm{Exp} : \mathfrak{m} \to 1 + \mathfrak{m}$ unique operator such that

$$\mathrm{Exp}(f+g) = \mathrm{Exp}(f) \cdot \mathrm{Exp}(g) \text{ and } \mathrm{Ext}(q^i t^n) = \frac{1}{1 - q^i t^n}$$

- $\mathrm{Log} : 1 + \mathfrak{m} \to \mathfrak{m}$ inverse to Exp,

Ingredients for the explicit formula 2

Definition

Plethystic operations:

- $\mathrm{Exp} : \mathfrak{m} \to 1 + \mathfrak{m}$ unique operator such that

$$\mathrm{Exp}(f+g) = \mathrm{Exp}(f) \cdot \mathrm{Exp}(g) \text{ and } \mathrm{Ext}(q^i t^n) = \frac{1}{1 - q^i t^n}$$

- $\mathrm{Log} : 1 + \mathfrak{m} \to \mathfrak{m}$ inverse to Exp,
- $\mathrm{Pow} : (1 + \mathfrak{m}) \times R \to 1 + \mathfrak{m}$ defined by

$$\mathrm{Pow}(f, g) = \mathrm{Exp}(g \cdot \mathrm{Log}(f)).$$

Definition

Plethystic operations:

- $\mathrm{Exp} : \mathfrak{m} \to 1 + \mathfrak{m}$ unique operator such that

$$\mathrm{Exp}(f+g) = \mathrm{Exp}(f) \cdot \mathrm{Exp}(g) \text{ and } \mathrm{Ext}(q^i t^n) = \frac{1}{1 - q^i t^n}$$

- $\mathrm{Log} : 1 + \mathfrak{m} \to \mathfrak{m}$ inverse to Exp,
- $\mathrm{Pow} : (1 + \mathfrak{m}) \times R \to 1 + \mathfrak{m}$ defined by

$$\mathrm{Pow}(f, g) = \mathrm{Exp}(g \cdot \mathrm{Log}(f)).$$

To summarize, we have three ingredients: explicit series **F**, shift operator **T**, plethystic power **Pow**.

First main result - precise formulation

Definition

$a_n(q) = \#\{n\text{-dimensional } \textit{completely reducible } \mathbb{F}_q\text{-reps of } F_m\}/\simeq .$

Theorem

$$a_n(q) \in \mathbb{Z}[q],$$

First main result - precise formulation

Definition

$a_n(q) = \#\{n\text{-dimensional } \textit{completely reducible } \mathbb{F}_q\text{-reps of } F_m\}/\simeq.$

Theorem

$$a_n(q) \in \mathbb{Z}[q],$$

more precisely

$$\sum_{n\geq 0} a_n(q)t^n = \mathrm{Pow}(T^{-1}F(t)^{-1}, 1-q).$$

First main result - precise formulation

Definition

$a_n(q) = \#\{n\text{-dimensional } \textit{completely reducible } \mathbb{F}_q\text{-reps of } F_m\}/\simeq .$

Theorem

$$a_n(q) \in \mathbb{Z}[q],$$

more precisely

$$\sum_{n\geq 0} a_n(q)t^n = \mathrm{Pow}(T^{-1}F(t)^{-1}, 1-q).$$

Remark: This formula conceals a double/triple recursion.

First main result - precise formulation

Definition

$$a_n(q) = \#\{n\text{-dimensional } \textit{completely reducible } \mathbb{F}_q\text{-reps of } F_m\}/\simeq .$$

Theorem

$$a_n(q) \in \mathbb{Z}[q],$$

more precisely

$$\sum_{n\geq 0} a_n(q)t^n = \mathrm{Pow}(T^{-1}F(t)^{-1}, 1-q).$$

Remark: This formula conceals a double/triple recursion.

Proof: Later.

Examples

$$a_1(q) = (q-1)^m,$$

Examples

$$a_1(q) = (q-1)^m,$$

$$a_n(q) = q^n - q^{n-1} \text{ for } m = 1,$$

Examples

Examples

$$a_1(q) = (q-1)^m,$$

$$a_n(q) = q^n - q^{n-1} \text{ for } m = 1,$$

$$a_2(q) = (q-1)^m \cdot (q^{m-1}(q-1)^{m-1}\left((q+1)^{m-1} - 1\right) +$$
$$+\frac{1}{2}q\left((q+1)^{m-1} + (q-1)^{m-1}\right)).$$

Second main result

Theorem

Positivity around $q = 1$:

Second main result

Theorem

Positivity around $q = 1$:

$$a_n(q) \in \mathbb{N}[q-1].$$

Second main result

Theorem

Positivity around $q = 1$:

$$a_n(q) \in \mathbb{N}[q-1].$$

For $m \geq 2$, lowest term formula

$$a_n(q) = \varphi(n)n^{m-2} \cdot (q-1)^n +$$

Second main result

Theorem

Positivity around $q = 1$:

$$a_n(q) \in \mathbb{N}[q-1].$$

For $m \geq 2$, lowest term formula

$$a_n(q) = \varphi(n) n^{m-2} \cdot (q-1)^n + O((q-1)^{n+1}),$$

Second main result

Theorem

Positivity around $q = 1$:

$$a_n(q) \in \mathbb{N}[q-1].$$

For $m \geq 2$, lowest term formula

$$a_n(q) = \varphi(n)n^{m-2} \cdot (q-1)^n + O((q-1)^{n+1}),$$

where φ is Euler's totient function.

Second main result

Theorem

Positivity around $q = 1$:

$$a_n(q) \in \mathbb{N}[q-1].$$

For $m \geq 2$, lowest term formula

$$a_n(q) = \varphi(n)n^{m-2} \cdot (q-1)^n + O((q-1)^{n+1}),$$

where φ is Euler's totient function.

Thus, the Taylor coefficients of the $a_n(q)$ around $q = 1$ count something intrinsic to F_m.

Second main result

Theorem

Positivity around $q = 1$:

$$a_n(q) \in \mathbb{N}[q-1].$$

For $m \geq 2$, lowest term formula

$$a_n(q) = \varphi(n)n^{m-2} \cdot (q-1)^n + O((q-1)^{n+1}),$$

where φ is Euler's totient function.

Thus, the Taylor coefficients of the $a_n(q)$ around $q = 1$ count something intrinsic to F_m. **But what do they count?**

Theorem

Positivity around $q = 1$:

$$a_n(q) \in \mathbb{N}[q-1].$$

For $m \geq 2$, lowest term formula

$$a_n(q) = \varphi(n)n^{m-2} \cdot (q-1)^n + O((q-1)^{n+1}),$$

where φ is Euler's totient function.

Thus, the Taylor coefficients of the $a_n(q)$ around $q = 1$ count something intrinsic to F_m. **But what do they count?** We don't know.

Second main result

Theorem

Positivity around $q = 1$:

$$a_n(q) \in \mathbb{N}[q-1].$$

For $m \geq 2$, lowest term formula

$$a_n(q) = \varphi(n)n^{m-2} \cdot (q-1)^n + O((q-1)^{n+1}),$$

where φ is Euler's totient function.

Thus, the Taylor coefficients of the $a_n(q)$ around $q = 1$ count something intrinsic to F_m. **But what do they count?** We don't know.

(Proof is purely combinatorial in terms of first theorem.)

Definition

Adams operators ψ_n:

Definition

Adams operators ψ_n: the $\mathbb{Q}$-linear operator on R such that

$$\psi_n(q^i) = q^{ni}, \quad \psi_n(t^d) = t^{nd};$$

Definition

Adams operators ψ_n: the $\mathbb{Q}$-linear operator on R such that

$$\psi_n(q^i) = q^{ni}, \quad \psi_n(t^d) = t^{nd};$$

$$\Psi = \sum_{n \geq 1} \frac{1}{n} \psi_n.$$

Relation to subgroup growth

Definition

Adams operators ψ_n: the $\mathbb{Q}$-linear operator on R such that

$$\psi_n(q^i) = q^{ni}, \quad \psi_n(t^d) = t^{nd};$$

$$\Psi = \sum_{n \geq 1} \frac{1}{n} \psi_n.$$

Theorem

$$\Psi\left(\frac{1}{q-1}\mathrm{Log}\left(\sum_{n\geq 0} a_n(q)t^n\right)\right)\Bigg|_{t=\frac{x}{(q-1)^{m-1}},\,q=1} = \sum_{n\geq 0} s_n \frac{x^n}{n}$$

Definition

Adams operators ψ_n: the $\mathbb{Q}$-linear operator on R such that

$$\psi_n(q^i) = q^{ni}, \quad \psi_n(t^d) = t^{nd};$$

$$\Psi = \sum_{n\geq 1} \frac{1}{n}\psi_n.$$

Theorem

$$\Psi\left(\frac{1}{q-1}\mathrm{Log}\left(\sum_{n\geq 0} a_n(q)t^n\right)\right)\Bigg|_{t=\frac{x}{(q-1)^{m-1}},q=1} = \sum_{n\geq 0} s_n\frac{x^n}{n}$$

This means: the representation growth knows the subgroup growth.

Geometric interpretation of previous results:

Geometric interpretation of previous results:

Definition

Γ a finitely generated group, G a reductive group:

Geometric interpretation of previous results:

Definition

Γ a finitely generated group, G a reductive group:

$\mathrm{Hom}(\Gamma, G)$ affine variety with G-action,

Character varieties

Geometric interpretation of previous results:

Definition

Γ a finitely generated group, G a reductive group:

$\mathrm{Hom}(\Gamma, G)$ affine variety with G-action,

$$X_\Gamma(G) := \mathrm{Hom}(\Gamma, G)//G$$

character variety.

Geometric interpretation of previous results:

Definition

Γ a finitely generated group, G a reductive group:

$\mathrm{Hom}(\Gamma, G)$ affine variety with G-action,

$$X_\Gamma(G) := \mathrm{Hom}(\Gamma, G)//G$$

character variety.

Here:

$$X_{F_m}(\mathrm{GL}_n(\mathbb{C})) = \mathrm{GL}_n(\mathbb{C})^m//\mathrm{GL}_n(\mathbb{C}),$$

Geometric interpretation of previous results:

Definition

Γ a finitely generated group, G a reductive group:

$\mathrm{Hom}(\Gamma, G)$ affine variety with G-action,

$$X_\Gamma(G) := \mathrm{Hom}(\Gamma, G)//G$$

character variety.

Here:

$$X_{F_m}(\mathrm{GL}_n(\mathbb{C})) = \mathrm{GL}_n(\mathbb{C})^m//\mathrm{GL}_n(\mathbb{C}),$$

$\mathrm{GL}_n(\mathbb{C})$ acting on m copies of itself via simultaneous conjugation.

Character varieties - examples

Examples

$$X_{F_m}(\mathbb{C}^*) \simeq (\mathbb{C}^*)^m,$$

Character varieties - examples

Examples

$$X_{F_m}(\mathbb{C}^*) \simeq (\mathbb{C}^*)^m,$$

$$X_{F_1}(\mathrm{GL}_n(\mathbb{C})) = \mathrm{GL}_n(\mathbb{C})//\mathrm{GL}_n(\mathbb{C}) \simeq \mathbb{A}^n \setminus \mathbb{A}^{n-1},$$

Character varieties - examples

Examples

$$X_{F_m}(\mathbb{C}^*) \simeq (\mathbb{C}^*)^m,$$

$$X_{F_1}(\mathrm{GL}_n(\mathbb{C})) = \mathrm{GL}_n(\mathbb{C})//\mathrm{GL}_n(\mathbb{C}) \simeq \mathbb{A}^n \setminus \mathbb{A}^{n-1},$$

$$X_{F_2}(\mathrm{GL}_2(\mathbb{C})) \simeq \mathbb{A}^5 \setminus \mathbb{A}^3.$$

Implications for character varieties

Theorem

$X_{F_m}(\mathrm{GL}_n(\mathbb{C}))$ *admits model* $\mathcal{X}$ *over* $\mathrm{Spec}(\mathbb{Z})$,

Implications for character varieties

Theorem

$X_{F_m}(\mathrm{GL}_n(\mathbb{C}))$ *admits model* $\mathcal{X}$ *over* $\mathrm{Spec}(\mathbb{Z})$*, that is,*

$$X_{F_m}(\mathrm{GL}_n(\mathbb{C})) \simeq \mathrm{Spec}(\mathbb{C}) \times_{\mathrm{Spec}(\mathbb{Z})} \mathcal{X},$$

Implications for character varieties

Theorem

$X_{F_m}(\mathrm{GL}_n(\mathbb{C}))$ admits model $\mathcal{X}$ over $\mathrm{Spec}(\mathbb{Z})$, that is,

$$X_{F_m}(\mathrm{GL}_n(\mathbb{C})) \simeq \mathrm{Spec}(\mathbb{C}) \times_{\mathrm{Spec}(\mathbb{Z})} \mathcal{X},$$

$$a_n(q) = \#\mathcal{X}(\mathbb{F}_q),$$

Theorem

$X_{F_m}(\mathrm{GL}_n(\mathbb{C}))$ *admits model* $\mathcal{X}$ *over* $\mathrm{Spec}(\mathbb{Z})$, *that is,*

$$X_{F_m}(\mathrm{GL}_n(\mathbb{C})) \simeq \mathrm{Spec}(\mathbb{C}) \times_{\mathrm{Spec}(\mathbb{Z})} \mathcal{X},$$

$$a_n(q) = \#\mathcal{X}(\mathbb{F}_q),$$

$$\chi(X_{F_m}(\mathrm{PGL}_n(\mathbb{C}))) = \varphi(n)n^{m-2}.$$

Implications for character varieties

Theorem

$X_{F_m}(\mathrm{GL}_n(\mathbb{C}))$ admits model $\mathcal{X}$ over $\mathrm{Spec}(\mathbb{Z})$, that is,

$$X_{F_m}(\mathrm{GL}_n(\mathbb{C})) \simeq \mathrm{Spec}(\mathbb{C}) \times_{\mathrm{Spec}(\mathbb{Z})} \mathcal{X},$$

$$a_n(q) = \#\mathcal{X}(\mathbb{F}_q),$$

$$\chi(X_{F_m}(\mathrm{PGL}_n(\mathbb{C}))) = \varphi(n)n^{m-2}.$$

Remark: Positivity property $a_n(q) \in \mathbb{N}[q-1]$ suggests existence of paving of $X_{F_m}(\mathrm{GL}_n(\mathbb{C}))$ by (quotients by finite groups of) tori.

Hall algebra

k a finite field.

Hall algebra

k a finite field.

Definition

$$\mathbf{Rep} = \{\text{isoclasses of fin. dim. } k\text{-reps of } F_m\},$$

Hall algebra

k a finite field.

Definition

$$\mathbf{Rep} = \{\text{isoclasses of fin. dim. } k\text{-reps of } F_m\},$$

$$H((kF_m)) = \{\text{functions } \mathbf{Rep} \to \mathbb{Q}\}$$

as $\mathbf{Q}$-vector space;

Hall algebra

k a finite field.

Definition

$$\mathbf{Rep} = \{\text{isoclasses of fin. dim. } k\text{-reps of } F_m\},$$

$$H((kF_m)) = \{\text{functions } \mathbf{Rep} \to \mathbb{Q}\}$$

as $\mathbf{Q}$-vector space; associative algebra via multiplication

$$(f * g)(V) = \sum_{U \subset V} f(U)g(V/U).$$

Hall algebra

k a finite field.

Definition

$$\mathbf{Rep} = \{\text{isoclasses of fin. dim. } k\text{-reps of } F_m\},$$

$$H((kF_m)) = \{\text{functions } \mathbf{Rep} \to \mathbb{Q}\}$$

as $\mathbf{Q}$-vector space; associative algebra via multiplication

$$(f * g)(V) = \sum_{U \subset V} f(U) g(V/U).$$

Algebra map (integration) $\int : H((kF_m)) \to \mathbb{Q}[[t]]$,

$$\int f = \sum_V \frac{|k|^{(1-m)\binom{\dim V}{2}}}{|\mathrm{Aut}(V)|} f(V) t^{\dim V}.$$

Proof of first theorem

Lemma (Key lemma)

$\mathbf{1} \in H((kF_m))$ *constant function with value* 1*:*

Proof of first theorem

Lemma (Key lemma)

$\mathbf{1} \in H((kF_m))$ *constant function with value* 1*:*

$$\mathbf{1}^{-1} \textit{ supported on completely reducible reps},$$

Proof of first theorem

Lemma (Key lemma)

$\mathbf{1} \in H((kF_m))$ *constant function with value* 1*:*

$$\mathbf{1}^{-1} \text{ supported on completely reducible reps,}$$

$$\mathbf{1}^{-1}(\bigoplus_S S^{m_s}) = \prod_S (-1)^{m_S} |\mathrm{End}(S)|^{\binom{m_S}{2}}.$$

Proof of first theorem

Lemma (Key lemma)

$\mathbf{1} \in H((kF_m))$ constant function with value 1:

$$\mathbf{1}^{-1} \text{ supported on completely reducible reps,}$$

$$\mathbf{1}^{-1}(\bigoplus_S S^{m_s}) = \prod_S (-1)^{m_S} |\mathrm{End}(S)|^{\binom{m_S}{2}}.$$

Corollary (After some calculation)

$$\int \mathbf{1}^{-1} = \mathrm{Pow}\left(\sum_{n \geq 0} a_n(q) t^n, \frac{1}{1-q} \right) \Bigg|_{q=|k|}.$$

Proof of first theorem

Lemma (Key lemma)

$\mathbf{1} \in H((kF_m))$ constant function with value 1:

$$\mathbf{1}^{-1} \text{ supported on completely reducible reps,}$$

$$\mathbf{1}^{-1}(\bigoplus_S S^{m_s}) = \prod_S (-1)^{m_S} |\mathrm{End}(S)|^{\binom{m_S}{2}}.$$

Corollary (After some calculation)

$$\int \mathbf{1}^{-1} = \mathrm{Pow}\left(\sum_{n\geq 0} a_n(q)t^n, \frac{1}{1-q}\right)\Bigg|_{q=|k|}.$$

$$\int \mathbf{1}^{-1} = \left(\int \mathbf{1}\right)^{-1} = T^{-1}F(t)^{-1}\big|_{q=|k|}.$$

The end

This proves the theorem and ends the talk.

The end

This proves the theorem and ends the talk.

Thank you!