Gröbner methods for representations of combinatorial categories

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A motivation: Homological stability

A sequence of spaces or groups with maps

$$X_1 \to X_2 \to \cdots$$

satisfies **homological stability** (in degree i) if the maps

$$\mathrm{H}_i(X_n) \to \mathrm{H}_i(X_{n+1})$$

are isomorphisms for $n \gg 0$.

A classical example: X_n is the braid group B_n on n strands and $X_n \subset X_{n+1}$ is the inclusion by acting on the first n strands.

A non-example: $X_n = \ker(\mathrm{B}_n \to \Sigma_n)$ is the pure braid group: $\mathrm{H}_1(X_n) \cong \mathbf{Z}^{\binom{n}{2}}$.

Idea: enhance homological stability by adding group actions.

For pure braid group, Σ_n acts on $\mathrm{H}_i(X_n)$ and $\mathrm{H}_1(X_n)$ is the permutation representation on 2-element subsets.
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Church–Ellenberg–Farb packaged this as an **FI-module**. Let **FI** be the category of finite sets and injective maps.

For a finite set S, there is a pure braid group PB_S on strands labeled by S and $S \mapsto \mathrm{H}_i(\mathrm{PB}_S)$ is a functor from **FI** to abelian groups and it's *finitely generated*: there is a finite collection of elements so that the smallest subfunctor containing them is the whole functor.

We could replace **FI** by some other category $\mathcal{C}$ and study its representations: functors to an abelian category like abelian groups or vector spaces.

Even for simple-looking categories like **FI**, there isn't much hope in classifying its representations.

The study of its modules is perhaps best viewed from the lens of commutative algebra.

So we might want to understand things such as noetherian properties (does finite generation get inherited by subrepresentations?), Hilbert series, free resolutions, Koszul duality, etc.

Theme of the talk:

- Reduce representation stability problems to commutative algebra
- Reduce algebraic problems to combinatorial problems
- Combinatorial tools: Gröbner bases and formal languages

Joint work with Andrew Snowden and Andy Putman.

Classical examples:

- **Hilbert basis theorem** (Let $\mathbf{k}$ be noetherian. Every ideal in $A = \mathbf{k}[x_1, \ldots, x_n]$ is finitely generated.)

  follows from

  Dickson's lemma (The poset $\mathbf{Z}^n_{\geq 0}$ under termwise comparison contains no infinite antichains.)
- "Finitely generated A-modules have rational Hilbert series"

  follows from

  "Every regular language has a rational generating function"

Some applications of the ideas:

- **Lannes–Schwartz artinian conjecture**: every subfunctor of a finitely generated endofunctor on the category of vector spaces over a finite field is finitely generated
- **Stembridge's conjecture**: Let $g_{\lambda,\mu,\nu}$ be Kronecker coefficient, i.e., tensor product multiplicity of symmetric group representations.

  Then $g_{d\alpha,d\beta,d\gamma} = 1$ for $d \geq 1$ implies $g_{\lambda+d\alpha,\mu+d\beta,\nu+d\gamma}$ is constant for $d \gg 0$.

  ($\alpha = \beta = \gamma = (1)$ is Murnaghan's stability theorem)

Let $\mathcal{C}$ be a small category and let $\mathbf{k}$ be a ring.
A representation of $\mathcal{C}$ is a functor $F\colon \mathcal{C} \to \mathrm{Mod}_{\mathbf{k}}$ where $\mathrm{Mod}_{\mathbf{k}}$ is the category of left $\mathbf{k}$-modules.

Equivalently, let $\mathbf{k}[\mathcal{C}]$ be the algebra over $\mathbf{k}$ with basis given by morphisms between objects in $\mathcal{C}$ and multiplication given by composition (if possible) or 0 (if not possible).
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In our examples, isomorphism classes of $\mathcal{C}$ parametrized by $\mathbf{Z}_{\geq 0}$, so one more definition is a $\mathbf{Z}_{\geq 0}$-graded $\mathbf{k}$-module $M = \bigoplus_{i \geq 0} M_i$ together with operators $M_i \to M_j$ for every morphism $i \to j$ that compose in the obvious way.

If $\mathbf{k}$ is a field, define **Hilbert series**: $\mathrm{H}_M(t) = \sum_{i \geq 0} \dim_{\mathbf{k}}(M_i) t^i$.

- $\mathcal{C} = \mathbf{FI}$ is the category of finite sets and injective maps. So there is one operation $M_i \to M_j$ for each injection $[i] \to [j]$.
- $\mathcal{C} = \mathbf{FS}^{\mathrm{op}}$ is the *opposite* of the category of finite sets and surjective maps. So there is one operation $M_i \to M_j$ for each surjection $[j] \to [i]$.
- $\mathcal{C} = \mathbf{VI}(\mathbf{F}_q)$ is the category of finite-dimensional $\mathbf{F}_q$-vector spaces and injective linear maps. So there is one operation $M_i \to M_j$ for each linear injection $\mathbf{F}_q^i \to \mathbf{F}_q^j$.
- G-sets, weighted sets, colored injections, symplectic vector spaces, etc.

$\mathbf{k}[\mathcal{C}]$-modules have a natural notion of "finite generation" and "submodule". A module M is **noetherian** if every submodule of it is finitely generated.
$\mathbf{k}[\mathcal{C}]$ is **noetherian** if every finitely generated module is noetherian (this forces $\mathbf{k}$ to be noetherian).

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Theorem (Sam–Snowden)

If $\mathbf{k}$ is left noetherian, then $\mathbf{k}[\mathcal{C}]$ is left noetherian for the examples on the last slide.
If $\mathbf{k}$ is a field, f.g. modules have rational Hilbert series.

The proof uses ideas from Gröbner bases which we review next.

$\mathcal{C} = \mathbf{FI}$ was proven first in paper of Church, Ellenberg, Farb, Nagpal (their technique doesn't apply to the other examples).

A **term order** is an ordering of monomials in $\mathbf{k}[x_1, \dots, x_n]$ so that

- no infinite descending chains
- $m < m'$ implies $m''m < m''m'$

For $f \in \mathbf{k}[x_1, \dots, x_n]$ define $\mathrm{init}(f)$ to be its maximal monomial and for an ideal I, let $\mathrm{init}(I)$ be the $\mathbf{k}$-span of $\mathrm{init}(f)$ for $f \in I$.

Basic properties:

- A generating set for $\mathrm{init}(I)$ gives one for I
- For I homogeneous, I and $\mathrm{init}(I)$ have same Hilbert series

So we can reduce problems of noetherianity and rationality of Hilbert series to monomial ideals.

- If I is a monomial ideal with infinite generating set $m_1, m_2, \ldots$ so that no m_i divides any other m_j, then their exponents would be an infinite antichain in $\mathbf{Z}^n_{\geq 0}$; now use Dickson's lemma
- A degree d monomial in n variables can be encoded as a sequence of "stars and bars" with $n-1$ "bars" and d "stars", e.g., $x_2^3 x_3^2 x_5 \leftrightarrow |***|**||*$

  The set of monomials divisible by a given one is a regular language on the alphabet $\{*, |\}$, i.e., can be encoded as the set of walks in a weighted graph, so has a rational generating function.

Back to $\mathcal{C}$-modules: basic idea

For each object i, there is a projective $\mathcal{C}$-module $P(i)$ with

$$P(i)_j = \mathbf{k}[\mathrm{Hom}_{\mathcal{C}}(i, j)].$$

Intuitively, $P(i)_j$ is the set of all operations that get you from M_i to M_j, so for any choice of element $x \in M_i$ one can define a map $P(i) \to M$.

These spaces have distinguished bases, which we should think of as monomials.

Non-trivial torsion automorphisms prevent us from defining term orders:
e.g., if $g^2 = 1$ then $g < 1$ would imply $1 < g$ and vice versa.

Given a functor $\Phi\colon \mathcal{C}' \to \mathcal{C}$, a $\mathbf{k}[\mathcal{C}]$-module M pulls back to a $\mathbf{k}[\mathcal{C}']$-module $\Phi^*(M)$.

Lemma

Suppose that Φ is essentially surjective and that $\Phi^*(M)$ is finitely generated whenever M is finitely generated.
If $\mathbf{k}[\mathcal{C}']$ is noetherian, then $\mathbf{k}[\mathcal{C}]$ is noetherian.

We say that Φ satisfies "Property (F)".

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Proof.

An infinite increasing chain of submodules in M would lead to an infinite increasing chain of submodules of $\Phi^*(M)$. □

Since non-trivial automorphisms in $\mathcal{C}$ are an issue, the solution is to find $\mathcal{C}'$ with no non-trivial automorphisms and a functor $\Phi\colon \mathcal{C}' \to \mathcal{C}$ satisfying property (F).

To check Property (F), just need to check projectives $P_i(j) = \mathbf{k}[\mathrm{Hom}_{\mathcal{C}}(i,j)]$ and it reduces to a combinatorial problem about factoring morphisms:

For every object $x \in \mathcal{C}$, there exist $y_1 \ldots, y_n \in \mathcal{C}'$ and $f_i\colon x \to \Phi(y_i)$ such that for all $y \in \mathcal{C}'$, every $f\colon x \to \Phi(y)$ can be written as $\Phi(g) \circ f_i$ for some $g\colon y_i \to y$

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We illustrate one example. Let **OI** be the category of finite ordered sets and order-preserving injections. The forgetful functor $\Phi\colon \mathbf{OI} \to \mathbf{FI}$ satisfies property (F):
if $x \in \mathbf{FI}$, then $y_1, \dots, y_n$ are all of the different $n = |x|!$ ordered sets on x and the f_i are identity maps.

Well-quasi-orderings and formal languages

Suppose we can define term orderings for representations of a category $\mathcal{C}$. To prove noetherian properties, we need an analog of Dickson's lemma which asserts that an associated poset is a well-quasi-ordering. Formally, this means that for every sequence $x_1, x_2, \dots$ there exists $i < j$ such that $x_i \leq x_j$.

To study Hilbert series of monomial representations, one approach is to put the monomials in bijection with words in a language such that finitely generated representations correspond to well-behaved classes, like regular languages, or unambiguous context-free languages.
Then one can use basic results about their Hilbert series.

These parts are more combinatorial; we skip them.

Let $\mathbf{Vec}(\mathbf{F}_q)$ be the category of finite-dimensional $\mathbf{F}_q$-vector spaces.

Theorem (Putman, Sam, Snowden)

$\mathbf{F}_q[\mathbf{Vec}(\mathbf{F}_q)]$ *is noetherian.*

The inclusion $\mathbf{VI}(\mathbf{F}_q) \to \mathbf{Vec}(\mathbf{F}_q)$ satisfies property (F).
(Recall that **VI** is the subcategory with linear injections.)

This was conjectured by Jean Lannes and Lionel Schwartz. Their interest comes from a connection of these modules with unstable modules over the Steenrod algebra.

This formulation is misleading: in fact one can prove the statement for $\mathbf{k}[\mathbf{Vec}(R)]$ with $\mathbf{k}$ noetherian and R a finite (commutative) ring.

Let $\mathbf{P}(V)$ be the projectivization of a vector space V.
The Segre embedding is the map

$$\begin{aligned} \mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_n) &\to \mathbf{P}(V_1 \otimes \cdots \otimes V_n) \\ ([v_1], \ldots, [v_n]) &\mapsto [v_1 \otimes \cdots \otimes v_n]. \end{aligned}$$

Three ways to get equations that vanish on the image:

1. Reduce from n to $n-1$ by considering the composition

$$\begin{aligned} \mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_{n-1}) \times \mathbf{P}(V_n) &\to \mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_{n-1} \otimes V_n) \\ &\to \mathbf{P}(V_1 \otimes \cdots \otimes V_n) \end{aligned}$$

2. Permute factors
3. Use linear maps $V_i \to V_i'$.

These operations also extend to other things like higher syzygies (call this Tor_i)

This was formalized by Snowden in the notion of a Δ-module.

But 1. and 2. are similar to the operations given by $\mathcal{C} = \mathbf{FS}^{\mathrm{op}}$. This can made rigorous; intuitively the result is:

Theorem (Sam–Snowden)

For each i, there is a finite list of Segre embeddings whose Tor_i groups allow one to build all others under operations 1., 2., and 3.

This was previously shown by Snowden when $\mathbf{k}$ is a field of characteristic 0; relied on classification of polynomial functors.

The combinatorial (Gröbner) approach ends up being simpler and more general.

Let $\mathcal{O}_K$ be the ring of integers in a number field and let $\mathfrak{p} \subset \mathcal{O}_K$ be a prime. Define

$$\mathbf{SL}_n(\mathcal{O}_K, \mathfrak{p}) = \ker(\mathbf{SL}_n(\mathcal{O}_K) \to \mathbf{SL}_n(\mathcal{O}_K/\mathfrak{p})),$$
$$\mathbf{Sp}_{2n}(\mathcal{O}_K, \mathfrak{p}) = \ker(\mathbf{Sp}_{2n}(\mathcal{O}_K) \to \mathbf{Sp}_{2n}(\mathcal{O}_K/\mathfrak{p})).$$

Theorem (Putman–Sam)

$\{\mathbf{SL}_n(\mathcal{O}_K, \mathfrak{p})\}$ *and* $\{\mathbf{Sp}_{2n}(\mathcal{O}_K, \mathfrak{p})\}$ *satisfy representation stability.*

In other words, their homology groups are finitely generated representations over suitably modified versions of the categories $\mathbf{VI}(\mathbf{F}_q)$ where one can prove noetherian results and use Quillen-type arguments to prove these statements.

There are also versions where $\mathbf{SL}_n(\mathcal{O}_K)$ is replaced by the automorphism group of a free group or a mapping class group of an oriented surface.