ASYMPTOTICS OF BRANCHING LAWS AND INVARIANTS BY GLOBAL QUOTIENTS

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Branching laws

- G' complex semisimple Lie group
- (π, V_λ) finite dim. irreducible rep. of G'
- $G \subseteq G'$ semisimple subgroup

Problem

Decompose V into G-irreducibles

$$V_\lambda = \bigoplus_\mu m_{\mu,\lambda} W_\mu,$$

where W_μ is a G-irrep. and

$$m_{\mu,\lambda} = dim(Hom_G(W_\mu, V_\lambda))$$

is its multiplicity.

Borel-Weil:

$$m_{\mu,\lambda} = \dim\,(W_\mu^* \otimes V_\lambda)^G = \dim H^0(G/B \times G'/B', L_{\mu,\lambda})^G.$$

where $L_{\mu,\lambda}$ is a line bundle over $X := G/B \times G'/B'$.

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Can the space of G-invariant sections be replaced by the space of all sections of some line bundle $\widetilde{L_{\mu,\lambda}}$ over some variety $Y_{\mu,\lambda}$, i.e., is there an identity

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Yes (almost): for

$$Y_{\mu,\lambda} := X^{ss}(L_{\mu,\lambda})//G$$

(GIT-quotient), there is a $k \in \mathbb{N}$ such that

$$H^0(G/B \times G'/B', L^m_{k\mu,k\lambda})^G \cong H^0(Y_{\mu,\lambda}, \widetilde{L^m_{k\mu,k\lambda}}), \quad m \in \mathbb{N}.$$

Question

Is there some GIT-quotient Y which works for all line bundles, i.e., that for every (μ, λ) there exists a $k \in \mathbb{N}$ and a line bundle $\widetilde{L_{k\mu,k\lambda}}$ on Y such that

$$H^0(G/B \times G'/B', L^m_{k\mu,k\lambda})^G \cong H^0(Y, \widetilde{L_{k\mu,k\lambda}}^m), \quad m \in \mathbb{N}?$$

VGIT

If $L \to X$ is an ample line bundle, the unstable locus of L (w.r.t. G) is

$$X^{us}(L) := \{x \in X \mid \forall k \forall s \in H^0(X, L^k)^G s(x) = 0\}.$$

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- $R(X, L) := \bigoplus_{k \geq 0} H^0(X, L^k)$ be the section ring of L
- $R(X, L)^G = \bigoplus_{k \geq 0} H^0(X, L^k)^G$ the ring of G-invariants

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Question

How does the quotient $Y(L) := X^{ss}(L)//G \cong Proj(R(X, L)^G)$ depend on L?

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Theorem (VGIT ; Thaddeus ('96), Dolgachev-Hu ('98))

There are only finitely many GIT-equivalence classes C, and hence only finitely many quotients

$$Y(C) := X^{ss}(C)//G.$$

Small unstable locus

Let $L \in C^G(X)$ be an ample line bundle

- w.l.o.g. $R(X, L)^G$ generated in degree one over $\mathbb{C}$ (OK for the purpose of studying $\mathrm{Proj}(R(X, L)^G)$)
- F:=divisorial component of $X^{us}(L) = V(H^0(X, L)^G)$ (counted with multiplicities of irr. components)
- $F = V(s_F), s_F \in H^0(X, \mathcal{O}_X(F))^G$ the defining section
- sections in $H^0(X, L^k)^G$ vanish to order k along F

Lemma

$$X^{ss}(L)//G \cong X^{ss}(L \otimes \mathcal{O}_X(-F))//G$$

Proof.

There is a natural isomorphism of graded rings $\varphi : R(X, L)^G \to R(X, L \otimes \mathcal{O}_X(-F))^G$;

$$\varphi(s) := \frac{s}{s_F^k}, \quad s \in H^0(X, L^k)^G.$$

Hence $\mathrm{Proj}(R(X, L)^G) \cong \mathrm{Proj}(R(X, L \otimes \mathcal{O}_X(-F))^G)$. □

Definition

Let $M^G(X)$ denote closure of the union of the GIT-equivalence classes C in $C^G(X)$ with $\mathrm{codim}(X^{us}(C)) \geq 2$.

Corollary

For big enough $k \in \mathbb{N}$, the multiplicity function $k \mapsto m_{k\mu,k\lambda}$ equals a multiplicity function $k \mapsto m_{k\mu_0,k\lambda_0}$, for a line bundle L_{μ_0,λ_0} with $\mathrm{codim}\, X^{us}(L_{\mu_0,\lambda_0}) \geq 2$.

Descent of line bundles

Remark

If C is a GIT-equivalence class in $M^G(X)$, $\pi : X^{ss}(C) \mapsto Y := X^{ss}(C)//G$, and L is a line bundle on X with $L = \pi^* E$ on $X^{ss}(C)$, for a line bundle $E \to Y$, then the identity

$$H^0(X, L)^G \cong H^0(Y, E)$$

holds by Hartog's theorem.

Question

Which line bundles L on X are of the form $L = \pi^* E$, for a line bundle $E \to Y$? (Which line bundles on X descend to Y?)

Theorem (S. 2014)

a) If the GIT-equivalence class $C \subseteq M^G(X)$ is of full dimension in the cone $M^G(X)$, then for every line bundle $L \in M^G(X)$ there exists $k \in \mathbb{N}$ and a line bundle $E \to Y := Y(C)$ such that

$$L^k = \pi^* E \quad \text{on} \quad X^{ss}(C)$$

and

$$H^0(X, L^{mk})^G \cong H^0(Y, E^m), \quad m \in \mathbb{N}.$$

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b) If $C \subseteq M^G(X)$ is also of full dimension in $C^G(X)$, then

1. $Pic(Y)_{\mathbb{Q}} \cong Pic(X)_{\mathbb{Q}}$ *and* $\overline{Eff}(Y) \cong C^G(X)$,
2. *Y is a Mori dream space.*

Corollary

All multiplicity functions $k \mapsto m_{k\mu,k\lambda}$ describing branching laws from G' to G are given by (for k big enough) dimension functions $k \mapsto h^0(Y, E^k)$ for line bundles E on the quotient Y .

Remark

The theorem also holds when X is a flag variety $X := \widetilde{G}/\widetilde{B}$ and $G \subseteq \widetilde{G}$ is a semisimple subgroup of G.

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Theorem (Tsanov, S. 2015)

If $G \subseteq \widetilde{G}$ is a principal SL_2-subgroup, and every simple factor of $\widetilde{G}$ has at least 5 positive roots, then $M^G(X)$ admits maximal dimensional GIT-equivalence classes C, and hence the quotient $Y = Y(C)$ is a Mori dream space.

Okounkov bodies and asymptotic multiplicities

Theorem (S., 2014)

There exists a closed convex cone $\Delta(Y)$ and a surjective linear map $q : \Delta(Y) \to \overline{Eff}(Y)$ such that for every (μ, λ) with $L_{\mu,\lambda} = \pi^* E$,

- *the leading coefficient of the polynomial $k \mapsto m_{k\mu,k\lambda}$ equals the volume of the slice $q^{-1}(E) \subseteq \Delta(Y)$,*
- *the multiplicities $m_{k\mu,k\lambda}$ are approximately given by counting lattice points in a convex body;*

$$m_{k\mu,k\lambda} \simeq \# \left\{ q^{-1}(E) \cap \frac{1}{k}\mathbb{Z}^r \right\},$$

where $r := \dim q^{-1}(E)$.

Remark

The cone $\Delta(Y)$ is not unique; it depends on a flag

$$\{pt\} = Y_n \subseteq \cdots \subseteq Y_1 \subseteq Y_0 := Y$$

of closed irreducible subvarieties with codim $Y_i = i$;
$\Delta(Y) = \Delta_{Y_\bullet}(Y)$ – the global Okounkov body of Y w.r.t. $Y_\bullet$.

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Conjecture

If $C \subseteq M^G(X)$ is of maximal dimension in $C^G(X)$, so that $Y = Y(C)$ is a MDS, then Y admits a rational polyhedral global Okounkov body.

THE END

THANK YOU FOR YOUR ATTENTION !