Reduction Theorems for local-global conjectures – revisited

Britta Späth

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Overview

Blocks

G finite group, p a prime, $\mathcal{O} \geq \mathbb{Z}_p$

When decomposing $\mathcal{O}G$ into minimal two-sided ideals B_i

$$\mathcal{O}G = B_1 \oplus \cdots \oplus B_s,$$

$B_1, \ldots, B_s$ are called the p-**blocks of** G.
We write $\mathrm{Bl}(G) = \{B_1, \ldots, B_s\}$.

This gives decompositions

$$\mathrm{Irr}(G) = \dot{\bigcup_{B \in \mathrm{Bl}(G)}} \mathrm{Irr}(B) \text{ and } \mathrm{IBr}(G) = \dot{\bigcup_{B \in \mathrm{Bl}(G)}} \mathrm{IBr}(B),$$

where $\mathrm{IBr}(G)$ is the set of isomorphism classes of simple $\overline{\mathbb{F}}_p G$-modules.

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Brauer Correspondents and Block Induction

One associates to $B \in \mathrm{Bl}(G)$:

- **defect group** $D \leq G$, a p-subgroup
- **Brauer correspondent** $B' \in \mathrm{Bl}(\mathrm{N}_G(D))$

For every $B \in \mathrm{Bl}(G)$ with the defect group D:

$$\chi(1)_p |D| \geq |G|_p \text{ for all } \chi \in \mathrm{Irr}(B).$$

Block Induction

For a p-subgroup $Q \leq G$ and $C \in \mathrm{Bl}(\mathrm{N}_G(Q))$, one denotes by C^G a certain block of G. Then C^G is the **(Brauer) induced block**.
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Alperin's weight Conjecture - general

B p-block of G, D defect group of B

A **weight of** $B \in \mathrm{Bl}(G)$ is a pair (Q, ψ), such that

- $Q \leq G$ is a p-group,
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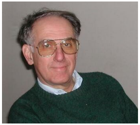

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$|\mathrm{IBr}(B)|$ is the number of G-conjugacy classes of weights of B.

If D is abelian, Alperin's weight conjecture for B states

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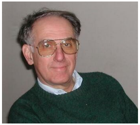

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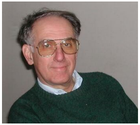

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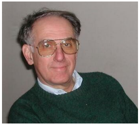

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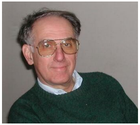

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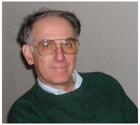

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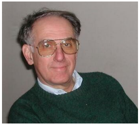

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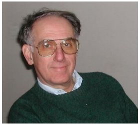

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Reformulation of Alperin's weight conjecture

$\mathcal{C}(G)$:= strictly ascending chains of p-subgroups of G starting in $\{1\}$

$\mathbb{D}$: $\{1\} \lneq D_2 \lneq \ldots \lneq D_r$

$|\mathbb{D}|$ length of $\mathbb{D}$, here r

Theorem (Knörr-Robinson, 1989)

The following are equivalent:

- *Alperin's weight conjecture holds*
- *for every group G and every $B \in \mathrm{Bl}(G)$ with non-trivial defect group*

$$\sum_{\mathbb{D} \in \mathcal{C}(G)/\sim G} (-1)^{|\mathbb{D}|} \sum_{C \in \mathrm{Bl}(\mathrm{N}_G(\mathbb{D})) \text{ with } C^G = B} |\operatorname{Irr}(C)| = 0.$$

The set $\mathcal{C}(G)$ can be replaced by

- the set of chains of radical p-groups
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Numbers of characters with given p-part

Let

$$\begin{array}{lll} \mathrm{Irr}^d(C) & := & \{\chi \in \mathrm{Irr}(C) \mid \chi(1)_p p^d = |G|_p\} \text{ for } C \in \mathrm{Bl}(G) \\ \mathrm{O}_p(G) & & \text{maximal normal } p\text{-subgroup of } G \end{array}$$

Dade's Conjecture (1990)

Let G be a finite group with $\mathrm{O}_p(G) = 1$, $B \in \mathrm{Bl}(G)$ and $d > 0$. Then:

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Results on Dade's Conjecture

Theorem

Dade's Conjecture holds for

- *all sporadic groups (for $p \neq 2$) (An, Conder, Dade, Entz, Hassan, Himstedt, Huang, Kotlica, Murray, O'Brien, Pahlings, Rouquier, Sawabe, Wilson 1990-2010)*
- Sym_n *(An, Olsson, Uno 1995, 1998)*
- *some groups of Lie type (An, Bird, Dade, Himstedt, Huang, Ku, Olsson, Sukizaki, Uno, Yamada 1996 - 2007)*
- *p-solvable groups (Robinson 2000)*
- $G \wr \mathrm{Sym}_n$ *if G satisfies Dade's conjecture (Eaton-Hoefling 2002)*
- *blocks with cyclic defect group (Dade 1996)*
- *unipotent blocks of finite reductive groups (Broué-Malle-Michel Broué-Fong-Srinivasan 1993-2006)*

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- *all sporadic groups (for* $p \neq 2$ *) (An, Conder, Dade, Entz, Hassan, Himstedt, Huang, Kotlica, Murray, O'Brien, Pahlings, Rouquier, Sawabe, Wilson 1990-2010)*
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If Dade's conjecture holds (for all groups), Alperin's weight conjecture and Alperin-McKay conjecture hold (for all groups).

Now we have a reduction theorem.

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Towards the inductive Dade condition – two sets

G a finite group with $O_p(G) = 1$
B block of G with non-trivial defect group, $d \geq 0$

Reformulation of Dade's Conjecture

Dade's conjecture for B is equivalent to $|\overline{\mathfrak{S}_+(B)}| = |\overline{\mathfrak{S}_-(B)}|$, where

$$\mathfrak{S}_\epsilon(B) = \left\{ (\mathbb{D}, \theta) \,\middle|\, \begin{array}{l} \mathbb{D} \in \mathcal{C}(G) \text{ with } (-1)^{|\mathbb{D}|} = \epsilon \\ \text{and } \theta \in \mathrm{Irr}^d(C) \text{ for } C \in \mathrm{Bl}(\mathrm{N}_G(\mathbb{D})) \text{ with } C^G = B \end{array} \right\}$$

for $\epsilon \in \{\pm 1\}$ and $\overline{\mathfrak{S}_\epsilon(B)}$ denotes the set of G-orbits in $\mathfrak{S}_\epsilon(B)$.

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(X, G, θ) is a **character triple** $\iff$ $G \lhd X$, $\theta \in \mathrm{Irr}(G)$ and θ is X-invariant

Definition (Equivalence relation on character triples)

Let $G \lhd X$ and $(\mathbb{D}, \theta), (\mathbb{D}', \theta') \in \mathfrak{S}_+(B) \cup \mathfrak{S}_-(B)$. Then we write

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The inductive Dade condition

S a non-abelian simple group

The inductive Dade condition (iDade) for S

Let $\widehat{S}$ be a group with $\widehat{S}/\operatorname{Z}(\widehat{S}) \cong S$ and $\widehat{S} = [\widehat{S}, \widehat{S}]$, B a block of $\widehat{S}$ with non-normal defect group and let $d \geq 0$. Then there exists some $\operatorname{Aut}(\widehat{S})_B$-equivariant bijection

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Proving the reduction theorem of Dade's Conjecture

- Induction on $|G : Z(G)|$
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- Study of the equivalence relation on character triples

1. Results about the equivalence relation on the character triples apply in a more general context.
2. **New insights** into the proof of the reduction theorems of the McKay, Alperin-McKay and Alperin's weight conjectures (Isaacs-Malle-Navarro 2007 and Navarro-Tiep 2010).

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HERE: How to prove the reduction theorem of the McKay conjecture (Isaacs-Malle-Navarro 2007).

Proving the reduction theorem of Dade's Conjecture

- Induction on $|G : Z(G)|$
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Let $\mathrm{Irr}_{p'}(G) := \{\chi \in \mathrm{Irr}(G) \mid p \nmid \chi(1)\}$.

McKay Conjecture (1972)

Let $P \in \mathrm{Syl}_p(G)$. Then:

$$|\,\mathrm{Irr}_{p'}(G)| = |\,\mathrm{Irr}_{p'}(\mathrm{N}_G(P))|.$$

Known for:

- all p-solvable groups (Okuyama-Wajima 1979)
- all quasi-simple groups (Michler-Olsson, Green-Lehrer-Lusztig, Malle-S., Wilson 1976 - 2010)

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McKay Conjecture - a relative version

Let $\nu \in \mathrm{Irr}(\mathrm{Z}(G))$ and $\mathrm{Irr}_{p'}(G|\nu) := \{\chi \in \mathrm{Irr}_{p'}(G) \mid \chi \text{ is a constituent of } \nu^G\}$.

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Let $P \in \mathrm{Syl}_p(G)$. Then:

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Proposition

Let G be a finite group. Assume that the relative McKay conjecture holds for all groups H with $|H : \mathrm{Z}(H)| < |G : \mathrm{Z}(G)|$.

Then one of the following holds

1. *the McKay conjecture holds for G, or*
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(iMcK) – the inductive McKay condition I

Definition (Equivalence relation on character triples)

Let $G \lhd X$, $\theta \in \mathrm{Irr}(G)$ and $\theta' \in \mathrm{Irr}(\mathrm{N}_G(P))$. Then we write

$$(X_\theta, G, \theta) \sim_G (\mathrm{N}_X(P)_{\theta'}, \mathrm{N}_G(P), \theta'),$$

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S a non-abelian simple group

The inductive McKay condition for S

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Step 3: Specific case $K = S^r$ and $G = S^r \rtimes \mathrm{Aut}(S^r)$

We assume:

- $K \lhd G$ with $K \cong S^r$ for some simple group S
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Recall: G is a minimal counterexample, i.e. $\exists\, K \lhd G$: $K/\mathrm{Z}(K) \cong S \times \cdots \times S$ for some simple group S and $K\,\mathrm{N}_G(\widetilde{P}) = G$ for $\widetilde{P} \in \mathrm{Syl}_p(G)$.

Assume that S is a non-abelian simple group with trivial Schur multiplier.

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