Quantised tilting modules and cellular algebras

.

(jt. H. H. Andersen and D. Tubbenhauer)

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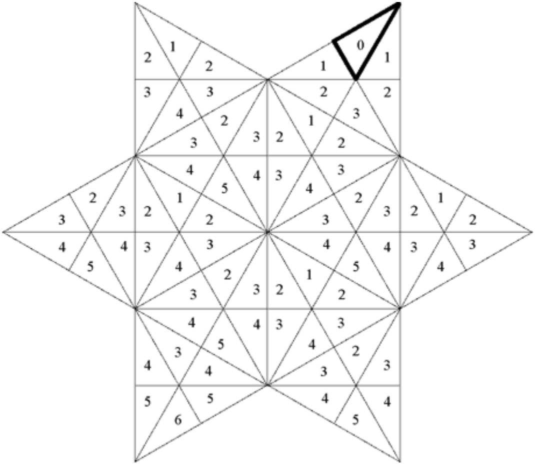
Some hidden stars that should be studied...



Endomorphism algebras

Theorem [ATS]

If T is a *tilting module for* $U_q = U_q(\mathfrak{g})$ *then* $\mathrm{End}_{U_q}(T)$ *is a cellular algebra.*

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- Works in principal over any field

Toy Example: Schur-Weyl duality

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Example

$G = GL(V)$ over any field $\mathbb{K}$. Then

$$\mathrm{End}_G(V^{\otimes d}) \cong \mathbb{K}[S_d]$$

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If $q \neq 1$ then we get $\mathrm{End}_{U_q}(\mathfrak{gl}_n)(V^{\otimes d}) \cong \mathcal{H}_q(S_d)$, the (finite) Iwahori-Hecke algebra and its cellularity.

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$$\rightsquigarrow \quad \text{Hopf algebra } U_q \text{ over } \mathbb{K}$$

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Definition

$T \in \mathcal{C}_q$ is tilting if T has a Δ_q- and a ∇_q-flag

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Assume M has a Δ_q-filtration and N has a ∇_q-filtration then

Corollary

$\dim \mathrm{Hom}_{\mathcal{C}_q}(M, N) = \sum_\lambda (M : \Delta_q(\lambda))(N : \nabla_q(\lambda))$

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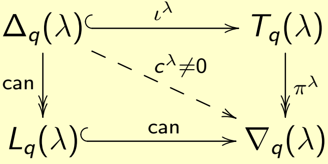

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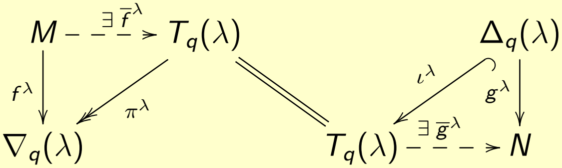
Assume we have homomorphisms f^λ and g^λ as indicated. Then



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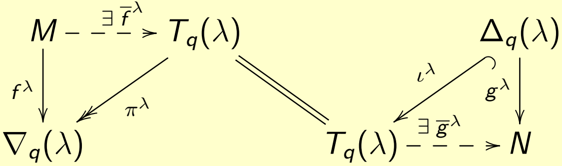
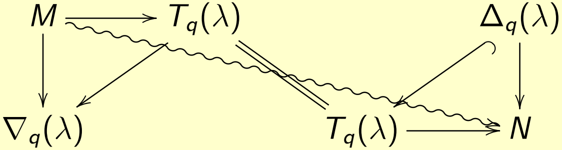


Hence get the composition map $c^\lambda = \overline{g}^\lambda \circ \overline{f}^\lambda$:



Bases of Homs

Assume still that M has a Δ_q-filtration and N has a ∇_q-filtration.
Let

- $\{f_j^\lambda\}$ be a basis of $\mathrm{Hom}_{\mathcal{C}_q}(M, \nabla_q(\lambda))$ and
- $\{g_k^\lambda\}$ be a basis of $\mathrm{Hom}_{\mathcal{C}_q}(\Delta_q(\lambda), N)$

Proposition

Then $\{c_{j,k}^\lambda = \overline{g}_k^\lambda \circ \overline{f}_j\}$ is a basis of $\mathrm{Hom}_{\mathcal{C}_q}(M, N)$.

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3. $ac^\lambda_{j,k} = \sum_i \underbrace{r_{i,j}(a)}_{\text{indep. of } k} c^\lambda_{i,k}$ mod A^λ for any $a \in A$.

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- set $c_{i,j}^\lambda = \overline{g}_i^\lambda \circ \overline{f}_j^\lambda$.

Many cellular bases of $\mathrm{End}_{\mathcal{C}_q}(T)$

Now take $M = N = T$ tilting and set $A = \mathrm{End}_{\mathcal{C}_q}(T)$

- $\Lambda = \Lambda_T = \{\lambda \in X^+ \mid (T : \Delta_q(\lambda)) \neq 0\}$,
- $I^\lambda = \{1, 2, \ldots, m_\lambda\}$, where $m_\lambda = \dim \mathrm{Hom}_{\mathcal{C}_q}(\Delta_q(\lambda), T)$,
- $\{g_k^\lambda\}$ any basis of $\mathrm{Hom}_{\mathcal{C}_q}(\Delta_q(\lambda), T)$,
- choose lifts $\overline{g}_k^\lambda : T_q(\lambda) \to T$ of g_k^λ,
- set $f_i^\lambda = D(g_i^\lambda)$ and $\overline{f}_i^\lambda = D(\overline{g}_i^\lambda)$, where D is duality
- set $c_{i,j}^\lambda = \overline{g}_i^\lambda \circ \overline{f}_j^\lambda$.

This is a cell datum for $A = \mathrm{End}_{\mathcal{C}_q}(T)$.

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$\Rightarrow$ (New) method to deduce (old and new) semisimplicity criteria

Easy weight combinatorics and Jantzen sum formula gives:

Example

Let $p > 2$. Then the Hecke algebra $\mathcal{H}_q(S_d)$ is semisimple if and only if

- q not a root of unity and $p > d$ or
- q is a root of unity of order $\ell > d$ if ℓ is odd and of order $\ell > 2d$ if ℓ is even.

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Let $\delta = q + q^{-1}$. Then the Temperley-Lieb algebra $\mathcal{TL}_d(\delta)$ is semi-simple iff $[i] \neq 0$ for all $i = 1, ..., d$ iff q is not a root of unity with $d < \ell = \mathrm{ord}(q^2)$.

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- Spider algebras in the sense of Kuperberg

- Wreath products:

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  Quantised: Ariki-Koike algebras (Ariki, Sakamoto-Shoji)
- Includes Hecke algebras of type B, and blob algebras, and Quantised Rook monoids, and Solomon algebras, and Mirabolic Hecke algebras, ...

Brauer algebras

- $\mathrm{End}_{\mathfrak{g}}(V^{\otimes d})$ for $\mathfrak{so}(V)$ or $\mathfrak{sp}(V)$ gives Brauer algebras.
- $\mathrm{End}_{\mathfrak{gl}(V)}(V^{\otimes r} \otimes V^{*\otimes s})$ gives walled Brauer algebras $B_{r,s}(n)$.

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Methods apply to tilting modules in category $\mathcal{O}$

Theorem

If T is a *tilting module in* $\mathcal{O}$ *and* E *a finite dimensional module then* $\mathrm{End}_{\mathcal{O}}(T \otimes E)$ *is a cellular algebra.*

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This includes

- cyclotomic degenerate affine Hecke algebras (Brundan-Kleshchev, Brundan-S.),
- cyclotomic affine BMW-algebras, (Benkart-Ram-Leduc)
- cyclotomic affine VW-algebras (Ehrig-S.)
- cyclotomic affine Hecke algebras at roots of unity (Vasserot-Varagnolo-Shan)

Graded cellular algebras

All the algebras from the last theorem are graded (KLR-algebras, Koszul grading on category $\mathcal{O}$).

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Conjecture

1. If T is a *tilting module for* $U_q = U_q(\mathfrak{g})$ *then* $\mathrm{End}_{U_q}(T)$ *can be equipped with a graded cellular algebra structure.*
2. *In case* $\mathbb{K} = \mathbb{C}$ *the graded decomposition numbers are given by affine Kazhdan-Lusztig polynomials (refining Kazdhan-Lusztig, Soergel, LTT).*

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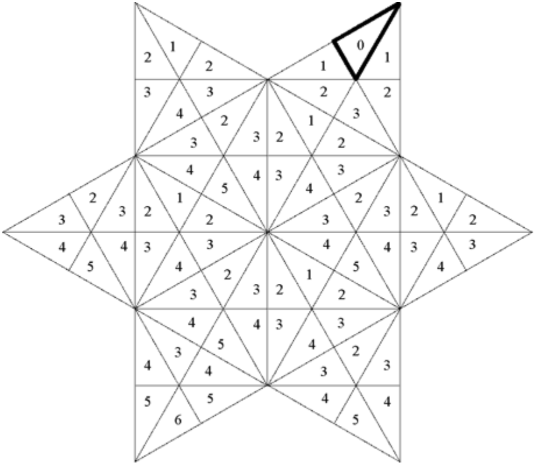
- true in type A
- very explicit for Temperley-Lieb algebra

Follow the stars . . .



but: where are the stars in positive characteristics ???