RESTRICTED RATIONAL CHEREDNIK ALGEBRAS

Ulrich Thiel

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University of Stuttgart
http://www.mathematik.uni-stuttgart.de/~thiel

SINGULARITIES

$$\left.\begin{array}{ll} V & \text{a finite-dimensional complex vector space} \\ G & \text{a finite subgroup of } \mathrm{GL}(V) \text{ generated by reflections} \end{array}\right\} \Gamma$$

Classical fact

The quotient $V/G = \mathrm{Spec}(\mathbb{C}[V]^G)$ is smooth. $\leftarrow$ boring

Consider instead G with the natural action on $V \oplus V^*$. Call this $\mathbb{D}\Gamma$.

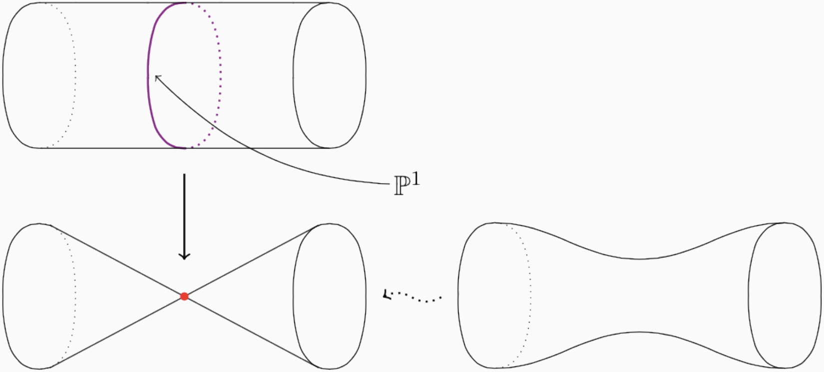
This is a symplectic reflection group and

$$(V \oplus V^*)/G = \mathrm{Spec}(\mathbb{C}[V \oplus V^*]^G)$$

is a symplectic singularity (Beauville).

Example (Type A_1 Kleinian singularity)

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \subseteq \mathrm{SL}_2(\mathbb{C})$$



Aim

Find deformations of $(V \oplus V^*)/G = \operatorname{Spec}(\mathbb{C}[V \oplus V^*]^G)$.

Idea

Instead of deforming $\mathbb{C}[V \oplus V^*]^G$ directly, we first deform

$$\mathbb{C}[\mathbb{D}\Gamma] := \mathbb{C}[V \oplus V^*] \rtimes G\ .$$

Reasons

- $Z(\mathbb{C}[\mathbb{D}\Gamma]) = \mathbb{C}[V \oplus V^*]^G$
- $\mathbb{C}[\mathbb{D}\Gamma]$ has an easy presentation: it is the quotient of

$$\mathbb{C}\langle\mathbb{D}\Gamma\rangle := \mathbb{C}\langle V \oplus V^*\rangle \rtimes G$$

  by $[y, y'] = [x, x'] = [y, x] = 0$ for all $y, y' \in V$ and $x, x' \in V^*$.

Etingof and Ginzburg defined in 2002 the rational Cherednik algebras.

Let $t \in \mathbb{C}$ and let $c : \mathrm{Ref}(\Gamma)/G \to \mathbb{C}$ be a map. Then $\mathsf{H}_{t,c}$ is the quotient of $\mathbb{C}\langle \mathbb{D}\Gamma \rangle = \mathbb{C}\langle V \oplus V^* \rangle \rtimes G$ by the relations

$$[x, x'] = 0 = [y, y'] \quad \forall x, x' \in V^*, y, y' \in V$$

$$[y, x] = t\langle y, x \rangle + \sum_{s \in \mathrm{Ref}(\Gamma)} c(s) \frac{\langle \alpha_s^\vee, x \rangle \langle y, \alpha_s \rangle}{\langle \alpha_s^\vee, \alpha_s \rangle} s$$

where $\alpha_s \in V^*$ with $\mathrm{Ker}(\alpha_s) = \mathrm{Ker}(\mathrm{id}_V - s)$ and $\langle \alpha_s^\vee \rangle \subseteq V$ is s-stable.

The $\mathsf{H}_{t,c}$ are a flat family of filtered deformations of $\mathbb{C}[V \oplus V^*] \rtimes G$, i.e., $\mathrm{gr}\, \mathsf{H}_{t,c} \cong \mathbb{C}[V \oplus V^*] \rtimes G$. In particular, $\mathsf{H}_{t,c} \underset{\mathbb{C}}{=} \mathbb{C}[V \oplus V^*] \otimes_{\mathbb{C}} \mathbb{C}G$.

We have $\mathsf{H}_{0,0} = \mathbb{C}[V \oplus V^*] \rtimes G$ and $\mathsf{H}_{1,0} = \mathcal{D}(V) \rtimes G$.

This indicates already that $t = 0$ and $t \neq 0$ will behave differently.

Theorem (Etingof–Ginzburg)

- The center $\mathsf{Z}_{t,c}$ of $\mathsf{H}_{t,c}$ is non-trivial if and only if $t = 0$.
- $X_c := \mathrm{Spec}(\mathsf{Z}_{0,c})$ is an irreducible variety.
  This is the Calogero–Moser space in c.
- $(X_c)_c$ is a flat family of filtered deformations of $(V \oplus V^*)/G$.

$t \neq 0$

A lot of structure and theory for $t \neq 0$:

- Highest weight category $\mathcal{O}_c \subseteq \mathrm{H}_{1,c}$-mod with $\mathrm{Irr}(\mathcal{O}_c) \cong \mathrm{Irr}(G)$. This is a highest weight cover of a cyclotomic Hecke algebra $\mathcal{H}_{q_c}$, i.e., there is a fully faithful functor

$$\mathrm{KZ} : \mathcal{O}_c\text{-proj} \to \mathcal{H}_{q_c}\text{-mod} .$$

  (Ginzburg–Guay–Opdam–Rouquier).
- Parabolic restriction and induction functors for category $\mathcal{O}_c$ (Bezrukavnikov–Etingof).
- Categorification of an $U_q(\widehat{\mathfrak{sl}}_e)$ Fock space representation (Shan).
- ...

We do not have anything like this for $t = 0$ (so far?)

RATIONAL CHEREDNIK ALGEBRAS AT T=0

$$\mathsf{H}_c := \mathsf{H}_{0,c}$$

and

$$\mathsf{Z}_c := \mathsf{Z}_{0,c}$$

Theorem (Etingof–Ginzburg)

$$\mathcal{Z} := \mathbb{C}[V]^G \otimes_{\mathbb{C}} \mathbb{C}[V^*]^G = \mathbb{C}[V \oplus V^*]^{G\times G}$$

is a central subalgebra of H_c and H_c is a finite $\mathcal{Z}$-module.

So, "H_c is almost commutative". This implies:

- $\dim L \leq \text{PI-deg}\, \mathsf{H}_c$. In fact, $\text{PI-deg}\, \mathsf{H}_c = |G|$.
- $\operatorname{Irr} \mathsf{H}_c \xrightarrow{\sim} \operatorname{Max} \mathsf{H}_c$ via $L \mapsto \operatorname{Ann}_{\mathsf{H}_c} L$.
- The contractions

$$\begin{aligned} \operatorname{Max} \mathsf{H}_c &\to \operatorname{Max} \mathsf{Z}_c \\ \Upsilon_c : \operatorname{Max} \mathsf{Z}_c &\to \operatorname{Max} \mathcal{Z} \\ \operatorname{Max} \mathsf{H}_c &\to \operatorname{Max} \mathcal{Z} \end{aligned}$$

  are surjective.

Theorem (Etingof–Ginzburg, Brown)

The following are equivalent for $L \in \mathrm{Irr}(\mathsf{H}_c)$ with $\mathfrak{m} := \mathrm{Ann}_{Z_c} L$:

- $\mathfrak{m}$ is a smooth point of X_c.
- $\dim L = |G|$.
- L is the only simple H_c-module lying over $\mathfrak{m}$.

Theorem (Etingof–Ginzburg, Gordon, Gordon–Martino, Martino, Bellamy; 2002–2009)

Classification of those G for which X_c is smooth for some c.

This gives a classification of those G for which $(V \oplus V^*)/G$ admits a symplectic resolution.

For $\mathfrak{m} \in \operatorname{Max} \mathcal{Z}$ or $\mathfrak{m} \in \operatorname{Max} Z_c$ let

$$H_c^{\mathfrak{m}} := H_c / \mathfrak{m} H_c \ .$$

This is a finite-dimensional $\mathbb{C}$-algebra. Call this a restriction of H_c.

Its simple modules are those simple H_c-modules annihilated by $\mathfrak{m}$.

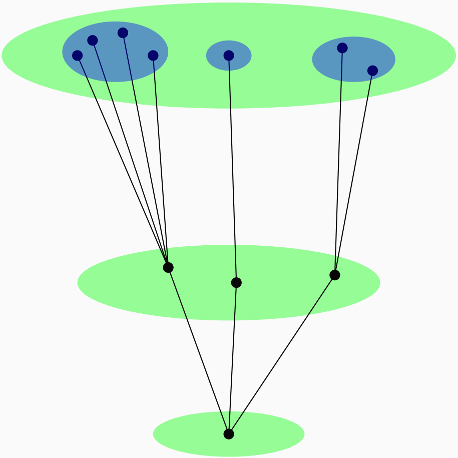
Theorem (Müller)

If $\mathfrak{m} \in \operatorname{Max} \mathcal{Z}$, then

$$H_c^{\mathfrak{m}} = \bigoplus_{\mathfrak{b} \in \Upsilon_c^{-1}(\mathfrak{m})} H_c^{\mathfrak{b}}$$

is the block decomposition of $H_c^{\mathfrak{m}}$.

PICTURE



$$\operatorname{Max} \mathsf{H}_c$$

$$\Big\downarrow \quad \mathfrak{M} \mapsto \mathfrak{M} \cap \mathsf{Z}_c$$

$$\operatorname{Max} \mathsf{Z}_c$$

$$\Upsilon_c \Big\downarrow \quad \mathfrak{b} \mapsto \mathfrak{b} \cap \mathcal{Z}$$

$$\operatorname{Max} \mathcal{Z} = V/G \times V^*/G$$

H_c is naturally $\mathbb{Z}$-graded by putting

$$\left.\begin{array}{lll} V & \text{in degree} & -1 \\ G & \text{in degree} & 0 \\ V^* & \text{in degree} & +1 \end{array}\right\} \mathsf{H}_{0,c} \underset{\mathbb{C}}{=} \mathbb{C}[V^*] \otimes_{\mathbb{C}} \mathbb{C}[G] \otimes_{\mathbb{C}} \mathbb{C}[V]$$

Both Z_c and $\mathcal{Z}$ are graded subalgebras. The grading induces a $\mathbb{C}^*$-action on X_c and $\operatorname{Max} \mathcal{Z}$.

The only $\mathbb{C}^*$-fixed point of $\operatorname{Max} \mathcal{Z} = V/G \times V^*/G$ is the origin $\mathfrak{o} = (0,0)$. The restriction

$$\overline{\mathsf{H}}_c := \mathsf{H}_c^{\mathfrak{o}} = \mathsf{H}_c / \mathfrak{o}\, \mathsf{H}_c \underset{\mathbb{C}}{=} \mathbb{C}[V^*]_G \otimes_{\mathbb{C}} \mathbb{C}G \otimes_{\mathbb{C}} \mathbb{C}[V]_G$$

is thus $\mathbb{Z}$-graded. This is the restricted rational Cherednik algebra.

Note: Blocks of $\overline{\mathsf{H}}_c$ = $\Upsilon_c^{-1}(\mathfrak{o})$ = $X_c^{\mathbb{C}^*}$.

Theorem (Bellamy–Martino)

X_c is smooth if and only if $\Upsilon_c^{-1}(\mathfrak{o})$ consists of smooth points.

RESTRICTED RATIONAL CHEREDNIK ALGEBRAS

Let $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ be a representation of G.

This lifts to a representation $\hat{\rho} : \mathbb{C}\langle V \oplus V^* \rangle \rtimes G \to \mathrm{Mat}_n(\mathbb{C})$ with

$$\hat{\rho}(y) = 0 = \hat{\rho}(x) \text{ for } y \in V, x \in V^* \text{ and } \hat{\rho}(g) = \rho(g) \text{ for } g \in G\,.$$

When does $\hat{\rho}$ descend to a representation of H_c? Precisely when

$$0 = \hat{\rho}([y,x]) = \sum_{s \in \mathrm{Ref}(\Gamma)} c(s) \frac{\langle y, \alpha_s \rangle \langle \alpha_s^\vee, x \rangle}{\langle \alpha_s^\vee, \alpha_s \rangle} \rho(s)$$

for all $y \in V$ and $x \in V^*$. In this case $\hat{\rho}$ is annihilated by $\mathfrak{o}$, so it is a representation of $\overline{\mathrm{H}}_c$. If ρ is simple, so is $\hat{\rho}$.

Call $\hat{\rho}$ (or ρ) a c-rigid representation of H_c.

Lemma (T.)

Let G be an odd dihedral group. Then for any $c \neq 0$ all but one of the two-dimensional simple representations of G are c-rigid.

The exception is the chosen reflection representation.

Similar result for even dihedral groups.

We have a triangular decomposition

$$\overline{\mathrm{H}}_c \underset{\mathbb{C}}{=} \underbrace{\mathbb{C}[V^*]_G \otimes_{\mathbb{C}} \overbrace{\mathbb{C}G}^{}}_{\mathbb{C}[V^*]_G \rtimes G} \hspace{-2em}\overbrace{\phantom{\mathbb{C}G} \otimes_{\mathbb{C}} \mathbb{C}[V]_G}^{\mathbb{C}[V]_G \rtimes G}$$

And thus a functor $\Delta_c : \mathbb{C}G\text{-mod} \to \overline{\mathrm{H}}_c\text{-mod}$ defined by

$$\Delta_c(\lambda) = \mathrm{Ind}_{\mathbb{C}[V^*]_G \rtimes G}^{\overline{\mathrm{H}}_c} \circ \mathrm{Inf}_{\mathbb{C}G}^{\mathbb{C}[V^*]_G \rtimes G} \lambda \, .$$

Theorem (Holmes–Nakano, Gordon)

If $\lambda \in \mathrm{Irr}(G)$, then $\Delta_c(\lambda)$ has simple head $\mathrm{L}_c(\lambda)$ and $\mathrm{Irr}(G) \xrightarrow{\sim} \mathrm{Irr}(\overline{\mathrm{H}}_c)$.

Since $\mathrm{Irr}(G) \xrightarrow{\sim} \mathrm{Irr}(\overline{\mathsf{H}}_c)$, the block partition of $\overline{\mathsf{H}}_c$ partitions $\mathrm{Irr}(G)$.

Call the families Calogero–Moser c-families.

Conjecture (Gordon–Martino, Bonnafé–Rouquier)

Suppose that G is a Coxeter group and that $c : \mathrm{Ref}(\Gamma)/G \to \mathbb{R}_{>0}$. Then the Calogero–Moser c-families are equal to Lusztig's c-families.

Theorem (Lusztig, Gordon–Martino, Bellamy)

The conjecture holds for classical Weyl groups and dihedral groups.

Theorem (T.)

The conjecture holds for H_3.

There is an extension of this conjecture to complex reflection groups and Rouquier families (Martino's conjecture).

As for $\mathcal{O}_c$ the simples of $\overline{\mathrm{H}}_c$-mod are naturally parametrized by $\mathrm{Irr}(G)$.

Is $\overline{\mathrm{H}}_c$-mod the correct analogue of $\mathcal{O}_c$ in $t = 0$? Probably not...

Lemma (T.)

$\overline{\mathrm{H}}_c$ is semisimple if and only if G is cyclic and X_c is smooth.

Theorem (Brown–Gordon–Stroppel)

H_c is a symmetric $\mathcal{Z}$-algebra. Hence, $\overline{\mathrm{H}}_c$ is a symmetric $\mathbb{C}$-algebra.

Corollary

$\overline{\mathrm{H}}_c$-mod is not a highest weight category unless G is cyclic and X_c is smooth.

The module $\Delta_c(\lambda)$ is naturally $\mathbb{Z}$-graded, and so is its head $L_c(\lambda)$.

Hence, $\mathrm{Irr}(\overline{H}_c\text{-grmod}) = \mathrm{Irr}(G) \times \mathbb{Z}$. Define a partial order $\preceq$ on this set by $(\lambda, m) \preceq (\mu, n)$ if and only if $m \geq n$.

Theorem (Bellamy–T.)

With respect to $\preceq$ the category $\overline{H}_c$-grmod is a highest weight category with standard and costandard objects.

(General result for $\mathbb{Z}$-graded algebras with triangular decomposition and semisimple middle part)

Is $\overline{H}_c$-grmod the correct analogue of category $\mathcal{O}_c$ in $t = 0$? Work in progress...

POISSON GEOMETRY

The commutator on $\mathsf{H}_{t,c}$ for $t \neq 0$ induces a Poisson structure on Z_c.

Let $\mathbf{t}$ be an indeterminate. Note that $\mathsf{H}_{\mathbf{t},c} / \mathbf{t}\, \mathsf{H}_{\mathbf{t},c} = \mathsf{H}_c$.

Lift $z_1, z_2 \in \mathsf{Z}_c$ to $\hat{z}_1, \hat{z}_2 \in \mathsf{H}_{\mathbf{t},c}$ and define

$$\{z, z'\} := \left(\frac{1}{\mathbf{t}} [\hat{z}_1, \hat{z}_2] \right) \mod \mathbf{t}\, \mathsf{H}_{\mathbf{t},c} \ .$$

Hence, X_c is a Poisson deformation of $(V \oplus V^*)/G$.

Theorem (Brown–Gordon)

There is a certain (finite) stratification of X_c into locally closed subsets called symplectic leaves.

A zero-dimensional leaf is called cuspidal.

Theorem (Bellamy)

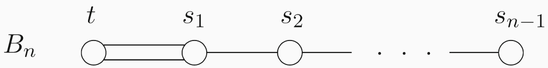
The cuspidal leaves of X_c are contained in $\Upsilon_c^{-1}(\mathfrak{o})$.

Hence, cuspidal leaves are blocks of $\overline{\mathrm{H}}_c$.

TYPE B

BASICS

Consider the Weyl group of type B_n.



We have two parameters: $a := c(s_i)$ and $b := c(t)$.

The simples of B_n are labeled by bipartitions (λ^1, λ^2) of n.

If $a = 0$, then $\mathrm{H}_c = \mathrm{H}_b(\mathbb{Z}_2)^{\otimes n} \rtimes \mathrm{S}_n$. $\leftarrow$ can be understood

Assume $a \neq 0$ from now on.

Theorem (Martino)

X_c is singular if and only if $b = \pm ma$ with $0 \leq m \leq n-1$.

Assume $b = \pm ma$ with $0 \leq m \leq n - 1$ from now on.

Theorem (Martino)

The symplectic leaves of X_c are parametrized by

$$\{\mathcal{L}_k \mid k \in \mathbb{N} \text{ with } km + k^2 \leq n\}$$

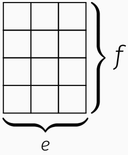
and $\dim \mathcal{L}_k = 2(n - km - k^2)$.

Corollary

There is a cuspidal leaf if and only if $n = k(k + m)$.

A RIGID SIMPLE MODULE

Let $\lambda = (e^f)$ be a box partition of n with $f \geq e$, e.g.,



Lemma (Bellamy–T.)

$(e^f, \emptyset)$ is c-rigid if and only if $b = a(f - e)$.

Hence, in this case $\mathsf{L}_c(e^f, \emptyset) = (e^f, \emptyset)$.

Theorem (Bellamy–T.)

Suppose that we are in the cuspidal case $n = k(k + m)$.

Then the c-rigid simple $\overline{\mathsf{H}}_c$-module $\mathsf{L}_c(k^{(k+m)}, \emptyset)$ is cuspidal.

Assume $b = a$, so $m = 1$ (equal parameter case).

There is a (unique) cuspidal leaf if and only if $n = k(k + m) = k^2 + k$.

There is a (unique) cuspidal Lusztig family if and only if $n = k^2 + k$.

Theorem (Bellamy–T.)

The (unique) cuspidal Calogero–Moser family is equal to the (unique) cuspidal Lusztig family.

This is a Poisson geometric interpretation of Lusztig's cuspidality!

(We expect the same for unequal parameters—work in progress)

CHAMP

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CHAMP (CHerednik Algebra Magma Package)
Version v1.5-71-g8d818cf
Copyright (C) 2013-2015 Ulrich Thiel
http://thielul.github.io/CHAMP/

> G  := ShephardTodd(2,1,2);
> H  := RationalCherednikAlgebra(G,0);
> eu := EulerElement(H); eu;

[1 0]                          [0 1]                    [1  0]
[0 1]*(y1*x1 + y2*x2) + [1 0]*(1/2*c1) + [0 -1]*(1/2*c2)
+
[-1  0]                    [0  -1]
[ 0  1]*(1/2)*(c2) + [-1  0]*(1/2*c1)

> IsCentral(eu);
true
```

CHAMP

```
> eu^2;
[ 0  -1]              [ 1   0]                    [ 0   1]
[ 1   0]*(c1*c2) + [ 0  -1]*(c2*y1*x1) + [-1   0]*(c1*c2)
+
[1  0]
[0  1]*(y1^2*x1^2 + 2*y1*y2*x1*x2 + y2^2*x2^2 + 1/2*c1^2 + 1/2*c2^2)
+
[ 0  -1]
[-1   0]*(1/2*c1*y1*x1 - 1/2*c1*y1*x2 - 1/2*c1*y2*x1 + 1/2*c1*y2*x2)
+
[-1   0]                                  [-1  0]
[ 0  -1]*(1/2*c1^2 + 1/2*c2^2) + [0   1]*(c2*y2*x2)
+
[0  1]
[1  0]*(1/2*c1*y1*x1 + 1/2*c1*y1*x2 + 1/2*c1*y2*x1 + 1/2*c1*y2*x2)

> PoissonBracket(x1,eu);
[1  0]
[0  1]*(x1)
```

What CHAMP can already do

- Compute in (restricted) rational Cherednik algebras — for any t and c (also generic), in any characteristic.
- Representation theory of RRCAs (simple modules and their graded W-character, decomposition matrices of Verma modules, Calogero–Moser families).

Future work (joint with C. Bonnafé)

- Calogero–Moser families for many more cases as above.
- Symplectic leaves and cuspidal families.
- Explicit presentations of Calogero–Moser spaces.
- Calogero–Moser cellular characters.
- Calogero–Moser cells.