Specifying smooth vectors for semibounded representations by single elements and applications

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Contents

Preliminaries

Let G be a (loc. convex) Lie group with exp-map $\exp : \mathfrak{g} \to G$.
Let $\pi : G \to \mathrm{U}(\mathcal{H})$ be a unitary representation.

- π is called **ray-continuous** if $t \mapsto \pi(\exp(tx)) = e^{tB_x}$ is strongly continuous $\forall x \in \mathfrak{g}$.
- **support function**: $s_\pi(x) := \sup \mathrm{Spec}(iB_x), x \in \mathfrak{g}$.
- $W_\pi := \{x_0 \in \mathfrak{g} : s_\pi(x)$ is bounded on a neighborhood of $x_0\}$.
- π is called **smooth** if $\mathcal{H}^\infty := \{v \in \mathcal{H} : g \mapsto \pi(g)v \text{ smooth}\}$ is dense in $\mathcal{H}$.
- If π smooth, **derived representation**:

$$\mathrm{d}\pi : \mathfrak{g} \to \mathrm{End}(\mathcal{H}^\infty), \quad \mathrm{d}\pi(x)v = \frac{d}{dt}\Big|_{t=0} \pi(exp(tx))v$$

  Then: $\overline{\mathrm{d}\pi(x)} = B_x$.
- π is called **semibounded** if π is smooth and $W_\pi \neq \emptyset$.

Nelson's Commutator Theorem

Let $N \geq 1$ self-adjoint operator on Hilbert space $\mathcal{H}$.

- $\mathcal{H}_k :=$ completion of $\mathcal{D}(N^{k/2})$ w.r.t. $\|v\|_k := \langle N^k v, v\rangle^{1/2} = \|N^{k/2}v\|, k \in \mathbb{Z}$.
  Then: $\mathcal{H}_k = \mathcal{D}(N^{k/2})$ for $k \geq 0$ and $\mathcal{H}_{-k} \cong \mathcal{H}_k^*$.
- Scale of subspaces:

$$\cdots \subset \mathcal{H}_2 \subset \mathcal{H}_1 \subset \mathcal{H} \subset \mathcal{H}_{-1} \subset \mathcal{H}_{-2} \subset \dots$$

- For $A \in B(\mathcal{H}_k, \mathcal{H}_{-k})$ define

$$[N, A] \in B(\mathcal{H}_{k+2}, \mathcal{H}_{-k-2}), \quad [N, A]v := NAv - ANv.$$

- We write $[N, A] \in B(\mathcal{H}_k, \mathcal{H}_{-k})$ if $[N, A](\mathcal{H}_{k+2}) \subset \mathcal{H}_{-k}$ and $\exists c > 0$ s.t. $\|[N, A]v\|_{-k} \leq c\|v\|_k, v \in \mathcal{H}_{k+2}$.

Nelson's Commutator Theorem

$N \geq 1$ self-adjoint operator on Hilbert space $\mathcal{H}$.

- A dense subspace $D \subset \mathcal{D}(N)$ is called a **core** for N if $\overline{N|_D} = N$.
- For $A \in B(\mathcal{H}_m, \mathcal{H}_n)$ let $\|A\|_{m,n}$ operator norm w.r.t $\mathcal{H}_m, \mathcal{H}_n$.

Theorem (Nelson)

Let $A \in B(\mathcal{H}_1, \mathcal{H}_{-1})$ and $[N, A] \in B(\mathcal{H}_1, \mathcal{H}_{-1})$. Then:

- *$A \in B(\mathcal{H}_2, \mathcal{H})$ and*

$$\|A\|_{2,0} \leq \|A\|_{1,-1}^{1/2}(\|A\|_{1,-1} + \|[N, A]\|_{1,-1})^{1/2} =: c,$$

  i.e., $\|Av\| \leq c\|Nv\|$ for all $v \in \mathcal{D}(N) = \mathcal{H}_2$.
- *If $\hat{A} := A|_{\mathcal{D}(N)}$ is a symmetric operator on $\mathcal{H}$ then $\hat{A}$ is essentially self-adjoint on any core for N.*

Example

- $p := \frac{1}{i}\frac{d}{dx}$, $q := x$ self-adj. on $\mathcal{H} = L^2(\mathbb{R}, \mathbb{C})$.
- Harmonic oscillator:
  $N_S := p^2 + q^2 = -\Delta + x^2$ on $S(\mathbb{R})$, $\quad N := \overline{N_S} \geq 1$
- Fact: N is ess. self-adjoint on $C_0^\infty(\mathbb{R})$.
- $A := p^2 - q^2$ on $S(\mathbb{R})$. Then $\forall v, w \in \mathcal{S}(\mathbb{R})$:

$$|\langle Av, w\rangle| \leq \|pv\|\|pw\| + \|qv\|\|qw\| \leq 2\|N^{1/2}v\|\|N^{1/2}w\|.$$

$$\Rightarrow \|N^{-1/2}Av\| = \sup_{\|w\|=1} |\langle Av, N^{-1/2}w\rangle| \leq 2\|N^{1/2}v\|.$$

$$[A, N] = 2[p^2, q^2] = 4(pq + qp) \leq 4(p^2 + q^2) = 4N \text{ on } \mathcal{S}(\mathbb{R}).$$

- $\Rightarrow$ A extends to $\hat{A} \in B(\mathcal{H}_1, \mathcal{H}_{-1})$ with $[N, \hat{A}] \in B(\mathcal{H}_1, \mathcal{H}_{-1})$.
- $\Rightarrow$ $p^2 - q^2 = -\Delta - x^2$ is ess. self-adjoint on $C_0^\infty(\mathbb{R})$ by Nelson's Commutator Theorem.

Specifying $\mathcal{H}^\infty$ for π semibounded by a single element

Theorem

Let G be a loc. conv. Lie group with exp-map $\exp : \mathfrak{g} \to G$. Let $\pi : G \to \mathrm{U}(\mathcal{H})$ be a semibounded representation and $x_0 \in W_\pi$. Then

$$\mathcal{H}^\infty = \mathcal{H}^\infty(\overline{\mathrm{d}\pi}(x_0)),$$

where $\mathcal{H}^\infty(\overline{\mathrm{d}\pi}(x_0)) = \{v \in \mathcal{H} : \mathbb{R} \ni t \mapsto \pi(\exp(tx_0))v \text{ is smooth}\}$.

Proof (sketch).

Choose cont. seminorm p on $\mathfrak{g}$ such that $\forall x \in \mathfrak{g}, p(x) \leq 1$:

$$\sup_{v \in \mathcal{H}^\infty, \|v\|=1} \langle i\mathrm{d}\pi(x_0 + x)v, v\rangle = s_\pi(x_0 + x) \leq s_\pi(x_0) + 1.$$

Set $N := \frac{1}{i}\mathrm{d}\pi(x_0) + (s_\pi(x_0) + 1)\mathbf{1} \ \geq \ \mathbf{1}$. Then

$$|\langle \mathrm{d}\pi(x)v, v\rangle| \leq \langle Nv, v\rangle, \quad p(x) \leq 1, v \in \mathcal{H}^\infty.$$

Proof (continuation).

Set $A_x := \frac{1}{i}\mathrm{d}\pi(x)$. Then:

$$|\langle NN^{-1}A_x v, v\rangle| = |\langle A_x v, v\rangle| \leq p(x)\langle Nv, v\rangle, \quad x \in \mathfrak{g}, v \in \mathcal{H}^\infty.$$

$\Rightarrow A_x$ extends to $\tilde{A}_x \in B(\mathcal{H}_1, \mathcal{H}_{-1})$. ($\mathcal{H}_1$ HS with SKP $\langle Nx, y\rangle$.)
Moreover: $[N, \tilde{A}_x] \in B(\mathcal{H}_1, \mathcal{H}_{-1})$ (implied by $[N, A_x] = iA_{[x,x_0]}$).
Moreover: $\|\tilde{A}_x\|_{1,-1} \leq p(x)$ and $\|[N, \tilde{A}_x]\|_{1,-1} \leq p([x, x_0])$.
Commutator Theorem $\Rightarrow \tilde{A}_x \in B(\mathcal{H}_2, \mathcal{H})$ and

$$\|\mathrm{d}\pi(x)v\| \leq (p(x) + \tfrac{1}{2}p([x, x_0]))\|Nv\|, \quad x \in \mathfrak{g}, v \in \mathcal{H}^\infty.$$

Thus $\beta : \mathfrak{g} \times \mathcal{H}^\infty(\overline{\mathrm{d}\pi}(x_0)) \to \mathcal{H}, \beta(x, v) = \overline{\mathrm{d}\pi}(x)v$ continuous.
Moreover it can be shown that

$$\beta : \mathfrak{g} \times \mathcal{H}^\infty(\overline{\mathrm{d}\pi}(x_0)) \to \mathcal{H}^\infty(\overline{\mathrm{d}\pi}(x_0))$$

is continuous. This implies $\mathcal{H}^\infty(\overline{\mathrm{d}\pi}(x_0)) \subset \mathcal{H}^\infty$. □

Remark

Let $\pi : G \to \mathrm{U}(\mathcal{H})$ *be semibounded and* $x_0 \in W_\pi$.

(a) *For* $c > s_\pi(x_0)$ *consider the Minkowski functional*

$$p(x) = \inf\{t > 0 : s_\pi(x_0 \pm \tfrac{x}{t}) < c, x_0 \pm \tfrac{x}{t} \in W_\pi\}. \quad (1)$$

Then for all $x \in \mathfrak{g}, v \in \mathcal{H}^\infty$:

$$\|\mathrm{d}\pi(x)v\| \leq (p(x) + \tfrac{1}{2}p([x, x_0]))(\|\mathrm{d}\pi(x_0)v\| + (c+1)\|v\|).$$

(b) *Let* τ *be the locally convex topology on* $\mathfrak{g}$ *generated by the the seminorm* (1). *Then for all* $v \in \mathcal{H}^\infty, k \in \mathbb{N}$ *the map*

$$\mathfrak{g}^k \to \mathcal{H}, (x_1, \ldots, x_k) \mapsto \mathrm{d}\pi(x_1) \cdots \mathrm{d}\pi(x_k)v$$

is continuous when $\mathfrak{g}$ *is equipped with the locally convex topology generated by* $f_k : \mathfrak{g} \to (\mathfrak{g}, \tau), x \mapsto \mathrm{ad}(x_0)^k x, k \in \mathbb{N}_0$.

Example for $\mathcal{H}^\infty = \mathcal{H}^\infty(\mathrm{d}\pi(x_0))$

Example

- $p := \frac{1}{i}\frac{d}{dx}$, $q := x$ self-adj. on $\mathcal{H} = L^2(\mathbb{R}, \mathbb{C})$.
- Hamiltonian: $N := \frac{p^2+q^2-1}{2}$.
- Heisenberg algebra: $\mathfrak{heis}_3 = \mathbb{R}\text{-span}\{i\mathbf{1}, ip, iq\}$, $[q, p] = i\mathbf{1}$.
- Oscillator algebra: $\mathfrak{g} := \mathfrak{heis}_3 \oplus \mathbb{R}iN$.
- Oscillator group G with Lie algebra $\mathfrak{g}$.
- $\pi : G \to \mathrm{U}(L^2(\mathbb{R}, \mathbb{C}))$ Schrödinger representation with generators $\mathbf{1}, p, q, N$.
- Then: $x_0 := (\mathbf{0}, 1) \in W_\pi$ and $\mathrm{d}\pi(x_0) = iN$.
- Thus: $\mathcal{H}^\infty(\pi) = \mathcal{H}^\infty(N) = \bigcap_{k\in\mathbb{N}} \mathcal{D}(N^k) = S(\mathbb{R})$.

Smoothness of action on $\mathcal{H}^\infty$

Proposition

Let $\pi : G \to \mathrm{U}(\mathcal{H})$ be a semibounded representation and $x_0 \in W_\pi$. Then the action $G \times \mathcal{H}^\infty \to \mathcal{H}^\infty, (g, v) \mapsto \pi(g)v$ is smooth when $\mathcal{H}^\infty = \mathcal{H}^\infty(\overline{\mathrm{d}\pi}(x_0))$ is equipped with the C^∞-topology, generated by the seminorms $v \mapsto \|\mathrm{d}\pi(x_0)^k v\|, k \in \mathbb{N}_0$.

Remark (Neeb)

(a) *If G is a Banach Lie group and $(\pi, \mathcal{H})$ a smooth representation of G then $\mathcal{H}^\infty$ carries a natural Fréchet topology such that $\pi : G \times \mathcal{H}^\infty \to \mathcal{H}^\infty$ is smooth.*

(b) *Consider $\pi : (\mathbb{R}^{\mathbb{N}}, +) \to \mathrm{U}(\ell^2(\mathbb{N}, \mathbb{C})), (\pi(g)x)_n = e^{ig_n} x_n$. Then $\mathcal{H}^\infty = \mathbb{C}^{(\mathbb{N})}$ and there is no locally convex topology on $\mathcal{H}^\infty$ such that*

$$\mathfrak{g} \times \mathcal{H}^\infty \to \mathcal{H}, \quad (x, v) \mapsto \mathrm{d}\pi(x)v$$

is continuous.

Smoothing operators

Let G be a Lie group such that $\mathfrak{g}$ is metrizable (e.g. G Fréchet). Let $\pi : G \to \mathrm{U}(\mathcal{H})$ be a smooth representation.

Definition

$A \in B(\mathcal{H})$ is called a **smoothing operator** for π if $A(\mathcal{H}) \subset \mathcal{H}^\infty$.

Theorem (Neeb/Salmasian)

A is smoothing operator $\Leftrightarrow$ $G \to B(\mathcal{H}), g \mapsto \pi(g)A$ is smooth.

Remark

- *If $\mathcal{H}^\infty = \mathcal{H}^\infty(\overline{\mathrm{d}\pi}(x))$ then $\pi_x(f) := \int_{\mathbb{R}} f(t)\pi(\exp(tx))dt$ is a smoothing operator $\forall f \in S(\mathbb{R})$.*
- *If π semibounded then $e^{i\mathrm{d}\pi(x)}$ is smoothing operator $\forall x \in W_\pi$.*

- For a C^*-algebra $\mathcal{A} \subset B(\mathcal{H})$ consider the **multiplier algebra** $M(\mathcal{A}) := \{x \in B(\mathcal{H}) : x\mathcal{A} + \mathcal{A}x \subset \mathcal{A}\}$.
- Fact: Every non-deg. representation ρ of $\mathcal{A}$ extends canonically to a representation $\tilde{\rho}$ of $M(\mathcal{A})$.

Proposition

Let $\mathcal{B} \subset B(\mathcal{H})$ be a $*$-invariant subset of smoothing operators for π. Consider the C^*-algebra

$$\mathcal{A} := C^*(\pi(G)\mathcal{B}\pi(G)) = \overline{\mathrm{span}}\left(\cup_{n\geq 1}\pi(G)(\mathcal{B}\pi(G))^n\right).$$

Then: $\forall\rho$ non-deg. repr. of $\mathcal{A}$, $\rho_G := \tilde{\rho} \circ \pi$ is a smooth repr.

Proposition

Assume π is semibounded, $\mathcal{A} := C^*(\pi(G)e^{i\mathrm{d}\pi(W_\pi)}\pi(G))$. Then

- $\mathcal{A} = C^*(\pi(G)e^{i\mathrm{d}\pi(x_0)}\pi(G))\ \forall x_0 \in W_\pi$.
- *$\forall\rho$ non-deg. repr. of $\mathcal{A}$, $\rho_G = \tilde{\rho} \circ \pi$ is semibounded, $W_{\rho_G} \supset W_\pi$ and $\rho_G(G)' = \rho(\mathcal{A})'$.*

Direct integral decomposition

Let G be a Lie group with $\mathfrak{g}$ metrizable.

Theorem

Assume G separable (e.g. G connected and $\mathfrak{g}$ separable). Let $\pi : G \to U(\mathcal{H})$ be semibounded and $\mathcal{H}$ separable. Then π is equivalent to a direct integral of irreducible semibounded representations of G.

Proof (idea).

For $x_0 \in W_\pi$ consider $\mathcal{A} = C^*(\pi(G)e^{i\mathrm{d}\pi(x_0)}\pi(G))$. Then $\mathcal{A}$ is separable. Let $\rho : \mathcal{A} \subset B(\mathcal{H})$ identical representation. Then ρ is non-degenerate and $\rho_G = \pi$. Disintegration of ρ into irreducible representations and preceding proposition yield the assertion. □

Oscillator groups

Let (V, ω) be a locally convex symplectic vector space and $\gamma : \mathbb{R} \to \mathrm{Sp}(V)$ defining a smooth action $\mathbb{R} \times V \to V$.

Definition

Oscillator group: $G = G(V, \omega, \gamma) := \mathrm{Heis}(V, \omega) \rtimes_\gamma \mathbb{R}$.

Let H be a complex Hilbert space, $\gamma : \mathbb{R} \to \mathrm{U}(H)$ a strongly continuous unitary one-parameter group, $\gamma(t) = e^{iAt}$ such that $A \geq 0$ and $\ker A = 0$. We define:

- $\mathcal{D}^\infty(A) :=$ space of γ-smooth vectors in H, equipped with C^∞-topology (given by norms $x \mapsto \|A^k x\|, k \in \mathbb{N}_0$).
- $\omega_A(x, y) := \mathrm{Im}\langle Ax, y\rangle$.

Definition

$G_A := \mathrm{Heis}(\mathcal{D}^\infty(A), \omega_A) \rtimes_\gamma \mathbb{R}$ is called a **standard oscillator group**.

Applications to oscillator groups

Let $G = G(V, \omega, \gamma)$ be an oscillator group. Further assume $\exists$ dense embedding $\iota : V \hookrightarrow C^\infty(A)$ such that

$$\iota : G(V, \omega, \gamma) \hookrightarrow G_A, (t, v, s) \mapsto (t, \iota(v), s)$$

is a morphism of Lie groups.

Proposition

Assume $DV \subset V$ is dense where $D = \gamma'(0)$. Then every semibounded repr. π of G extends to a (unique) semibounded repr. $\hat{\pi}$ of G_A.

Proposition

Assume A is diagonalizable. Let $\pi : G_A \to \mathrm{U}(\mathcal{H})$ be a continuous positive energy representation, i.e., $-i\mathrm{d}\pi(0, 0, 1) \geq 0$. Then π is semibounded.

Application to double extensions of loop groups

Let K be a simply-connected compact simple Lie group with Lie algebra $\mathfrak{k}$ and κ (normalized) invariant inner product on $\mathfrak{k}$.

- Loop group: $\mathcal{L}(K) := C^\infty(S^1, K)$,
  Loop algebra: $\mathcal{L}(\mathfrak{k}) = C^\infty(S^1, \mathfrak{k})$.
- γ rotation action of $\mathbb{R}$ on $\mathcal{L}(\mathfrak{k}), (\gamma(t)f)(x) := f(x+t)$.
  $Df := f'$ generator of γ.
- $\omega(f, g) := \frac{1}{2\pi}\int_{S^1} \kappa(Df(x), g(x))dx, \quad f, g \in \mathcal{L}(\mathfrak{k})$.
- double extension: $\hat{\mathcal{L}}(\mathfrak{k}) = \mathbb{R} \oplus_\omega \mathcal{L}(\mathfrak{k}) \rtimes_D \mathbb{R}$.
- integrates to: 2-connected Fréchet-Lie group $\hat{\mathcal{L}}(K)$ with Lie algebra $\hat{\mathcal{L}}(\mathfrak{k})$.

Proposition

Let $\pi : \hat{\mathcal{L}}(K) \to \mathrm{U}(\mathcal{H})$ be a continuous positive energy representation, i.e., $-i\mathtt{d}\pi(0,0,1) \geq 0$. Then π is semibounded.

Open problems and perspectives

Let G be a Lie group.

- **Problem:** Let $\pi : G \to U(\mathcal{H})$ continuous representation with $W_\pi \neq \emptyset$. Does it follow that π is semibounded?
- Let $\pi : G \to U(\mathcal{H})$ semibounded, $x_0 \in W_\pi$. Consider:

$$\mathcal{H}^\omega := \left\{ v \in \mathcal{H}^\infty : \sum_{n=0}^\infty \frac{\|\mathrm{d}\pi(x)^n v\|}{n!} < \infty \text{ for } x \text{ in a neighbh. of } 0 \right\},$$

$$\mathcal{H}^\omega(\overline{\mathrm{d}\pi}(x_0)) = \left\{ v \in \mathcal{H}^\infty : \exists t > 0 \text{ s.t. } \sum_{n=0}^\infty \frac{t^n \|\mathrm{d}\pi(x_0)^n v\|}{n!} < \infty \right\}.$$

  Is true that $\mathcal{H}^\omega = \mathcal{H}^\omega(\overline{\mathrm{d}\pi}(x_0))$?
- Consider situations where $\mathcal{H}^\infty = \mathcal{H}^\infty(\pi|_T)$ for $T \subset G$ a finite-dim. subgroup and π not necessarily semibounded. In this case we also obtain smoothing operators and C^*-algebras.

  Recall: G finite-dim., π cont. repr. of G, $(e_k)_k$ basis of $\mathfrak{g}$

$$\Rightarrow \mathcal{H}^\infty = \mathcal{H}^\infty(\Delta), \ \Delta = \overline{\sum_k \mathrm{d}\pi(e_k)^2}. \text{ (Nelson)}$$