1)

Branching laws for Verma modules

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joint with

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Verma modules versus flag manifolds

$\Updownarrow$ algebra $\longleftrightarrow$ S. Lie $\longleftrightarrow$ analysis / geometry $\Updownarrow$

Goal Understand branching laws for Verma modules and the relation to exceptional differential operators on flag manifolds.

Plan
① Motivating examples
② Branching of Verma modules
③ Parabolically induced representations
④ Constructions and examples.

2)

(a) Rankin - Cohen brackets : $F = F(x,y)$

$$D_n^{k_1 k_2} F = \sum_{m=0}^{n} (-1)^m \binom{k_1+n-1}{m} \binom{k_2+n-1}{n-m} \partial_x^m \partial_y^{n-m} F \Big|_{x=y}$$

with $n, k_1, k_2 \in \mathbb{N}$, satisfying for

$$[f_1, f_2]_n = D_n^{k_1 k_2}(f_1 \otimes f_2)$$

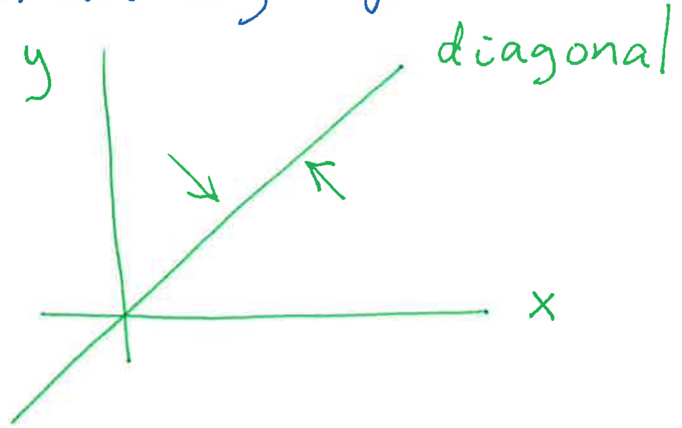
$$\pi_{k_j}(g) f_j(x) = (cx+d)^{-k_j} f_j\left(\frac{ax+b}{cx+d}\right)$$

$$g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$$

$$\boxed{[\pi_{k_1}(g) f_1, \pi_{k_2}(g) f_2]_n = \pi_{k_1+k_2+2n}(g)[f_1, f_2]_n}$$

i.e. $D_n^{k_1 k_2} : \pi_{k_1} \otimes \pi_{k_2} \longrightarrow \pi_{k_1+k_2+2n}$

is an intertwining operator



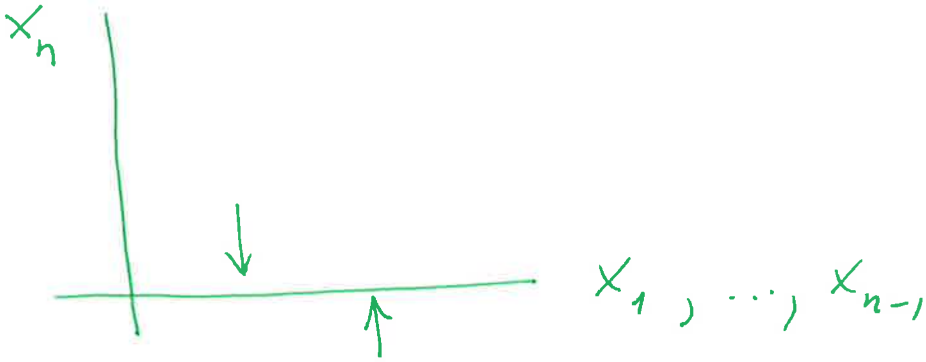
(b) A. Juhl's operators in conformal geometry : restricting functions from the sphere S^n to S^{n-1} :

3)

$$D_{2N}(\lambda) = \sum_{j=0}^{N} a_j(\lambda)\, \Delta_{n-1}^{j} \left(\frac{\partial}{\partial x_n}\right)^{2N-2j}$$

$$a_0(\lambda)\, x^{2N} - a_1(\lambda)\, x^{2N-2} + \cdots + (-1)^N x^0 = C_{2N}^{-\lambda-\frac{n-1}{2}}(x)$$

Gegenbauer polynomial



is an intertwining operator for the conformal group of $\mathbb{R}^{n-1} \hookrightarrow S^{n-1}$

$$\boxed{\pi_{\lambda-2N}(g) \circ D_{2N}(\lambda) = D_{2N}(\lambda) \circ \pi_\lambda(g)}$$

viewing $D_{2N}(\lambda) : C^\infty(S^n) \longrightarrow C^\infty(S^{n-1})$ and $\lambda \in \mathbb{C}$ a parameter for the spherical principal series on S^n resp. S^{n-1} via

$$G^n \curvearrowright S^n = G^n/P^n = K^n/M^n \supset S^{n-1}$$

$$G^n = SO(n+1, 1) \supset G^{n-1}$$

$$\boxed{\pi_\lambda(g)\varphi(k) = a(g^{-1}k)^\lambda \varphi(\kappa(g^{-1}k))}$$

via $G = KAN$ Iwasawa.

4)

Verma modules

$\mathfrak{g}$ = semisimple Lie algebra over $\mathbb{C}$
$\cup$
$\mathfrak{h}$ = Cartan $\subset$ Borel $= \mathfrak{h} \oplus \mathfrak{n} = \mathfrak{b}$

BGG category $\mathcal{O}$: $\mathfrak{g}$-modules, $\mathfrak{h}$ semisimple, locally $\mathfrak{n}$-finite, f.g.

parabolic $= \mathfrak{p} \supset \mathfrak{b}$, $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}_+(\mathfrak{l})$

subcategory $\mathcal{O}^{\mathfrak{p}}$: locally $\mathfrak{n}_-(\mathfrak{l})$-finite

$\Lambda^+(\mathfrak{l}) = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{N} \ (\alpha \in \Delta^+(\mathfrak{l}))\}$

Define $M^{\mathfrak{g}}_{\mathfrak{p}}(\lambda) = M^{\mathfrak{g}}_{\mathfrak{p}}(F_\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} F_\lambda$

– with infinitesimal char. $= \lambda + \rho$

Now branching to reductive $\mathfrak{g}' \subset \mathfrak{g}$

of $X \in \mathcal{O}^{\mathfrak{p}}$:

$$\boxed{X|_{\mathfrak{g}'} = ?}$$

Note difficult, e.g. could be no simple submodule – and even then complicated composition series, but still interesting

5)

Define $X|_{\mathfrak{g}'}$ discretely decomposable if there is an increasing sequence of finite length $\mathfrak{g}'$-modules with $X = \bigcup_{j=0}^{\infty} X_j$

Theorem [Kobayashi] If $G'P$ is closed in $G = \mathrm{Int}(\mathfrak{g})$, then $X|_{\mathfrak{g}'}$ discr. dec. and vice versa for $G' \subset G$ symmetric.

Note ok in compatible situations: $P' = P \cap G'$ parabolic in G'. In such a situation

K(OP)
$$\boxed{M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)\Big|_{\mathfrak{g}'} \cong \sum_{\mu \in \Delta^+(\mathfrak{l}')}^{\oplus} m(\lambda,\mu) M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mu)}$$

with
$$m(\lambda,\mu) = \dim \mathrm{Hom}_{\mathfrak{l}'}(F'_\mu, F_\lambda|_{\mathfrak{l}'} \otimes S(\mathfrak{n}_-/\mathfrak{n}_- \cap \mathfrak{g}'))$$

Parabolic induction (real groups)

G = connected real reductive Lie group

hyperbolic $x \in \mathfrak{g} = \mathrm{Lie}(G)$

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{l} \oplus \mathfrak{n}_+ \quad \text{rel. } \mathrm{ad}(x)$$

$$\text{parabolic } \mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}_+$$

$$P = \{ g \in G \mid \mathrm{Ad}(g)(\mathfrak{p}) = \mathfrak{p} \}$$

6)

$$N_\pm = \exp \mathfrak{n}_\pm$$

Define flag manifold G/P with big Schubert cell

$$\phi : \mathfrak{n}_- \to G/P, \quad \phi(X) = \exp(X).o$$

Now V = irr. finite-dim. P-module

π = induced repr. $\mathrm{Ind}_P^G(V) \cong C^\infty(G,V)^P$

i.e. smooth sections of $G \times_P V$.

Define $\mathcal{D}'_{[o]}(G/P)$ = distributions with support at base pt. o

Observation There is a natural G-invar. bilinear pairing

$$\boxed{C^\infty(G,V)^P \times \left(\mathcal{D}'_{[o]}(G/P) \times V^\vee\right) \to \mathbb{C}}$$

where $V^\vee$ = dual to V. Also, the generalized Verma module

$$M_\mathfrak{p}^\mathfrak{g}(V^\vee) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} V^\vee \cong \mathcal{U}(\mathfrak{n}_-) \times V^\vee$$

and we have an isomorphism

$$\varphi : \mathcal{U}(\mathfrak{n}_-) \xrightarrow{\simeq} \mathrm{Diff}_{N_-}(\mathfrak{n}_-)$$

7) by extending for $Y \in \mathfrak{n}_-$, $x \in N_-$

$$(D_Y f)(x.o) = \frac{d}{dt} f(x \exp tY.o)\Big|_{t=0}$$

and our pairing is now

$$\boxed{\begin{aligned} &C^\infty(\mathfrak{n}_-, V) \times (\mathcal{U}(\mathfrak{n}_-) \times V^\vee) \to \mathbb{C} \\ &\langle Y, f \rangle = (Y\delta_0)(f) \end{aligned}}$$

PICTURE:

$$\mathrm{Ind}_P^G(V) \xleftrightarrow{\text{duality}} M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)$$

G-maps here

g-maps here by transpose

$$\begin{array}{ccc} \mathrm{Ind}_P^G(V) & & M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee) \\ \big\downarrow & & \big\uparrow \\ \mathrm{Ind}_{P'}^{G'}(W) & & M_{\mathfrak{p}'}^{\mathfrak{g}'}(W^\vee) \end{array}$$

G'-map

g'-map

WANT Find such maps, i.e. differential operators, resp. imbeddings of gen. Verma modules, i.e. look for g'-singular vectors $\boxed{M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)^{\mathfrak{n}'_+}}$

8)

Recall the representation $d\pi$ of $\mathfrak{g}$ on $C^\infty(\mathfrak{n}_-, V)$ and consider the contragredient

$$\langle d\pi^\vee(X)T, f\rangle = -\langle T, d\pi(X)f\rangle$$
$$(T \in \mathrm{Diff}_{N_-}(\mathfrak{n}_-) \otimes V^\vee,\ f \in C^\infty(\mathfrak{n}_-, V))$$

Assume $\mathfrak{n}_-$ Abelian, and consider the Fourier transform giving

$$\mathrm{Diff}_{N_-}(\mathfrak{n}_-) \otimes V^\vee \cong \mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee$$

and action by (higher order) differentia operators $d\tilde{\pi}(X)$ $(X \in \mathfrak{g})$

Definition

$$\mathrm{Sol} = \{ f \in \mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee \mid d\tilde{\pi}(Z) f = 0 \ (Z \in \mathfrak{n}_+') \}$$

– reducing our problem to one of solving these differential equations.

Note that the targets $W^\vee$ will be the $\mathfrak{l}'$-submodules in $M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)^{\mathfrak{n}_+'}$

9)

Examples (a) $G = SO(p,q)$ preserving

$$Q = 2x_0 x_{n+1} + \sum_{i=1}^{n} \varepsilon_i x_i^2, \quad n = p+q-2,$$

$\varepsilon_i = 1 \quad (i = 1, \ldots, p-1), \quad \varepsilon_i = -1 \ (i = p, \ldots, n)$

$$P = \left\{ \begin{pmatrix} a & * & * \\ 0 & m & * \\ 0 & 0 & a^{-1} \end{pmatrix} \middle| \ a > 0, \ m \in SO(p-1, q-1) \right\}$$

G/P = projective null cone

induce from the character a^λ

Lemma For the standard $\mathfrak{n}_+$ - basis

(i) $d\pi_\lambda(E_j) = -\frac{1}{2} \varepsilon_j x^2 \partial_j + x_j (\lambda + E)$

(ii) $d\tilde{\pi}_\lambda(E_j) = i\left(\frac{1}{2} \varepsilon_j \xi_j \Box + (\lambda - E)\partial_{\xi_j}\right)$

with E the Euler $\sum_k x_k \partial_k$ resp. $\sum_k \xi_k \partial_{\xi_k}$.

Now we are in the situation to find the key space Sol, say for the case $G' = SO(p, q-1)$

$$\mathfrak{n}'_- \cong \mathbb{R}^{n-1} \subset \mathbb{R}^n \cong \mathfrak{n}_-$$

Proposition (i) For each $\lambda \in \mathbb{C}$, Sol contains

$$F_M(\xi) = \xi_n^M P_M(t), \quad t = \sum_{i=1}^{n-1} \varepsilon_n \varepsilon_i \xi_i^2 / \xi_n^2$$

with $P_M(t)$ an explicit Gegenbauer with paramer $\alpha = -\lambda - \frac{n-1}{2}$

(ii) For $\lambda = k \in \mathbb{N}$ Sol also contains

$$\sum_{j=0}^{k+1}{}^{\oplus} \mathcal{H}^j(\mathbb{R}^{p-1,q-2}) \qquad L'\text{-modules}$$

Note This list is exhaustive, (ii) gives rise to a direct sum of $\mathfrak{g}'$-Verma submodules, and (i) recovers the Juhl operators.

Theorem For $\lambda \in \mathbb{C} \setminus \{\frac{1}{2}(k-n) \mid k = 2, 3, \dots\}$

$$\boxed{M_p^{\mathfrak{g}}(\lambda)|_{\mathfrak{g}'} \cong \sum_{N \in \mathbb{N}}^{\oplus} M_{p'}^{\mathfrak{g}'}(\lambda - N)}$$

Note In the non-generic case $\lambda_k = \frac{1}{2}(-n+k+1)$ we have to work with extensions

$$0 \to M_{p'}^{\mathfrak{g}'}(\lambda_{2k} - j) \to M_{j,2k-j} \to M_{p'}^{\mathfrak{g}'}(\lambda_{2k} - (2k-j)) \quad \&$$

$$M_p^{\mathfrak{g}}(\lambda_{2k}) \cong \sum_{j=0}^{k-1}{}^{\oplus} M_{j,2k-j} \oplus M_{p'}^{\mathfrak{g}'}(\lambda_{2k} - k) \oplus \sum_{N=2k+1}^{\infty}{}^{\oplus} M_{p'}^{\mathfrak{g}'}(\lambda_{2k} - N)$$

where $M_{j,2k-j} \subset M_p^{\mathfrak{g}}(\lambda_{2k})$.

11) (b) Spinors and Dirac operators

(double coverings of) $G = SO(p,q)$ etc.

$F \in C^\infty(G, \mathbb{S})^P$, $\mathbb{S}$ = spin reprn.

$F(gman) = a^{-\lambda} m^{-1} \cdot F(g)$

$$d\pi_\lambda(E_j) = \tfrac{1}{2}\varepsilon_j x^2 \partial_j - x_j(\lambda + E + \tfrac{1}{2}) - \tfrac{1}{2}\varepsilon_j e_j \underline{x}$$

$$d\tilde{\pi}_\lambda(E_j) = -i\left(\tfrac{1}{2}\varepsilon_j \xi_j \Box + (\lambda - E + \tfrac{1}{2})\partial_{\xi_j} - \tfrac{1}{2}\varepsilon_j e_j \underline{D}\right.$$

where $\underline{D} = \sum_{k=1}^{n} e_k \partial_{\xi_k}$ is the Dirac operator

Again the task is to find polynomials annihilated by $d\tilde{\pi}_\lambda(E_j)$, $j = 1, \dots, n-1$ corresponding to $G' = SO(p, q-1)$.

Proposition With coefficients from Gegenbauer polynomials we get

$$D_{2N}(\lambda) : C^\infty(\mathbb{R}^{\bar{p},\bar{q}}, \mathbb{S}) \longrightarrow C^\infty(\mathbb{R}^{\bar{p},\bar{q}-1}, \mathbb{S})$$

$$D_{2N}(\lambda) = \sum_{j=0}^{N} a_j(\lambda)\, \Delta_{n-1}^{j}\, \partial_n^{2N-2j} + \sum_{j=0}^{N-1} b_j(\lambda)\, \Delta_{n-1}^{j}\, \partial_n^{2N-2j}\, \underline{D}\, \underline{\partial}_n$$

$\underline{D} = \sum_{i=1}^{n-1} e_i \partial_i$, $\underline{\partial}_n = e_n \partial_n$, $\bar{p} = p-1$, $\bar{q} = q-1$

— and similarly $D_{2N+1}(\lambda)$.

12) (c) Grassmannian geometry

$G = GL(n,\mathbb{R}) \supset P$ maximal parabolic

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad p = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

$N_+ : C = 0,\ A = I_{p\times p},\ D = I_{q\times q}$ & N_-

induce from $(\det A)^\lambda (\det D)^\mu$

$$d\tilde{\pi}_{\lambda,\mu}(E_{ij}) = -i\left(-(\lambda+\mu)\partial_{ji} + \sum_{a,b} \xi_{ab}\,\partial_{ai}\,\partial_{jb}\right)$$

$L = GL(p,\mathbb{R}) \times GL(q,\mathbb{R})$

The case $G' = G$

Proposition The space Sol contains a one-dimensional L-invariant subspace iff $p = q = n/2$. For this case $\lambda + \mu = \ell - \frac{n}{2}$, $\ell \in \mathbb{N}$ and uniquely

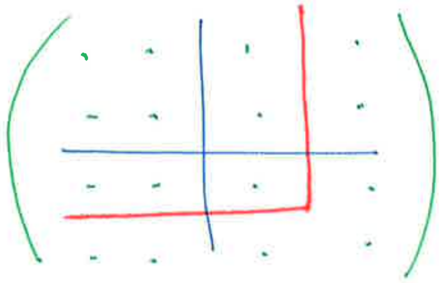
$$p_\ell(\xi_{ij}) = (\det \xi_{ij})^\ell .$$

Corollary n even, $E_{\lambda,\mu} = G \times_P \mathbb{C}_{\lambda,\mu}$ admits the invariant differential operators

$$D_\ell = (\det \partial_{ij})^\ell : E_{\lambda,\ell-\frac{n}{2}-\lambda} \longrightarrow E_{\lambda+\ell,\lambda-\frac{n}{2}}$$

(on smooth sections)

13) (d) $G = GL(4,\mathbb{R}) \supset \underline{G'} = GL(3,\mathbb{R})$



The polynomials we look for are

$$F(\xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}) = F(\xi_{21}, \xi_{22})$$

$\mathcal{P}_\ell$ = those homogeneous of degree $\ell = 0, 1, \dots$

Theorem For each character λ, the branching of $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ to $\mathfrak{g}'$ is

$$\boxed{M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda) \cong \sum_{\ell=0}^{\infty}{}^{\oplus} M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathcal{P}_\ell)}$$

(e) Rankin-Cohen brackets

$SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \supset \operatorname{diag} SL(2,\mathbb{R})$

spherical principal series induced from character $\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \to \alpha^\lambda$ (two such)

$$d\pi(E) = +\lambda x + x^2\partial_x + \mu y + y^2\partial_y$$

$$d\tilde{\pi}(E) = i(-\lambda\partial_\xi + \xi\partial_\xi^2 - \mu\partial_\eta + \eta\partial_\eta^2)$$

and again look for polynomial solutions:

14) Let $t = \xi/\eta$ and write a homogeneous polynomial as $\eta^N Q(t)$, $\deg Q = N$, then the condition becomes

$$\left[t(t+1)\frac{d^2}{dt^2} + \left(t(\mu - 2(N-1)) - \lambda\right)\frac{d}{dt} + N(N-1-\mu)\right]Q = 0$$

which via $t = \frac{-1+z}{2}$ becomes Jacobi's equation :

Theorem With

$$\tilde{P}_n^{-\lambda-1,\mu+\lambda-2n+1}(\xi,\eta) = \eta^n P_n^{-\lambda-1,\mu+\lambda-2n+1}\left(2\frac{\xi}{\eta}+1\right)$$

we have

$$Sol = \sum_{n=0}^{\infty} \langle \tilde{P}_n^{-\lambda-1,\mu+\lambda-2n+1} \rangle$$

recovering the usual R.-C. brackets.

Theorem If $\lambda+\mu \notin \mathbb{N}$ we have

$$M_b^{sl_2}(\lambda) \otimes M_b^{sl_2}(\mu) \simeq \sum_{n\in\mathbb{N}} M_b^{sl_2}(\lambda+\mu-2n)$$

If $\lambda+\mu \in \mathbb{N}$, $\lambda,\mu \notin \mathbb{N}$ we have instead

$$\simeq \sum_{\nu\in\Lambda_r(\lambda+\mu)}^{\oplus} M_b^{sl_2}(\nu) \oplus \sum_{\nu\in\Lambda_s(\lambda+\mu)}^{\oplus} P_b^{sl_2}(\nu)$$

for certain projective objects ($\nu \in \mathbb{N}$)

$$0 \to M(-\nu-2) \longrightarrow P(\nu) \longrightarrow M(\nu) \longrightarrow 0.$$

SO WE HAVE SEEN

decomposing Verma modules / branching

$\updownarrow$

intertwining differential operators between flag manifolds

$\updownarrow$

finding singular vectors by solving differential equations

with motivation from number theory, conformal geometry, and representation theory. Want: systematic study

FURTHER PERSPECTIVE In parabolic geometry a manifold M is modeled on a flag manifold, and one would like in this setting to construct natural

16) differential operators, extending those found in the model situation

$$D : \mathrm{Ind}_P^G(V) \to \mathrm{Ind}_P^G(W)$$

or $$D : \mathrm{Ind}_P^G(V) \longrightarrow \mathrm{Ind}_{P'}^{G'}(W).$$

Define semi-holonomic

$$\bar{\mathcal{U}}(\mathfrak{g},\mathfrak{p}) = T(\mathfrak{g}) \Big/ \left\langle X\otimes Y - Y\otimes X - [X,Y],\ \begin{matrix} X\in\mathfrak{p} \\ Y\in\mathfrak{g}\end{matrix} \right\rangle$$

$$\bar{M}_{\mathfrak{p}}^{\mathfrak{g}}(V) = \bar{\mathcal{U}}(\mathfrak{g},\mathfrak{p})\otimes V \Big/ \left\langle X\otimes v - 1\otimes Xv,\ \begin{matrix} X\in\mathfrak{p} \\ v\in V\end{matrix} \right\rangle$$

with projections $\left.\begin{matrix} \bar{M}(V) \to M(V) \\ \bar{\mathcal{U}}(\mathfrak{g},\mathfrak{p}) \to \mathcal{U}(\mathfrak{g}) \end{matrix}\right\}$

FACT A semi-holonomic Verma morphism induces an invariant differential operator on the corresponding parabolic geometry.

ref [Eastwood - Slovak]

Theorem There is a semi-holonomic lift $\bar{D}_M$ of all the Juhl operators D_M for each $\lambda \neq -n+3, -n+4, \dots, -n+N+2$, $M = 2N$ or $M = 2N+1$.