# Semibounded representations of oscillator groups

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- **Coadjoint orbits**: $\mathcal{O}_\lambda := \mathrm{Ad}^*(G)\lambda$ for $\lambda \in \mathfrak{g}'$.
- $E \subset \mathfrak{g}'$ is called **semi-equicontinuous** if the map $s_E : \mathfrak{g} \to \mathbb{R} \cup \{\infty\}, x \mapsto -\inf\langle E, x\rangle$ is bounded for $x$ in a non-empty open subset of $\mathfrak{g}$.

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## Remark

*Let $\pi : G \to \mathrm{U}(\mathcal{H})$ be a semibounded representation. If all semi-equicont. coadjoint orbits in $\mathfrak{g}'$ are trivial then $\pi|_{\mathbb{R} \times D(V) \times \{0\}}$ is trivial. Thus, if in addition $D(V) \subset V$ dense $\Rightarrow$ repr. of $\mathbb{R}$.*

# Standard oscillator groups

Let $H$ be a complex Hilbert space, $\gamma : \mathbb{R} \to \mathrm{U}(H)$ a strongly continuous unitary one-parameter group, $\gamma(t) = e^{iAt}$ such that

$$A \geq 0 \text{ and } \ker A = 0.$$

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We define:

- $C^\infty(A) :=$ space of $\gamma$-smooth vectors in $H$, equipped with $C^\infty$-topology (given by norms $x \mapsto \|A^k x\|, k \in \mathbb{N}_0$).
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$C^\infty(A)$ is a Fréchet space and $(C^\infty(A), \omega_A)$ is a symplectic space.

**Definition**

$G_A := \mathrm{Heis}(C^\infty(A), \omega_A) \rtimes_\gamma \mathbb{R}$ is called a *standard oscillator group*.

# Standard oscillator groups

**Theorem (Extension to standard oscillator groups)**

*Suppose $G = G(V, \omega, \gamma)$ has non-trivial semi-equicont. coadjoint orbits. Then, after possibly replacing $\gamma$ by $\gamma^{-1}$:*

(a) *$\exists$ a standard oscillator group $G_A$ and a dense embedding $\iota : V \hookrightarrow C^\infty(A)$ such that $\iota : G \to G_A, (t, v, s) \mapsto (t, \iota(v), s)$ is a morphism of Lie groups.*

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(b) *Assume $DV \subset V$ dense and $\gamma(\mathbb{R}) \subset \mathrm{End}(V)$ equicontinuous. Then every semibounded repr. $\pi$ of $G$ extends to a (unique) semibounded repr. $\hat{\pi}$ of $G_A$, where $G_A$ is as in (a).*

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Let $\pi : G_A \to \mathrm{U}(\mathcal{H})$ be a semibounded representation.

**Proposition (Central disintegration)**

*Suppose $C^\infty(A)$ and $\mathcal{H}$ are both separable. Then $\pi \cong \int_{\mathbb{R}} \pi_\lambda d\mu(\lambda)$, where $\pi_\lambda : G_A \to \mathrm{U}(\mathcal{H})$ is semibounded with $\pi_\lambda(t, 0, 0) = e^{i\lambda t}\mathbf{1}$.*

# Open invariant cones in $\mathfrak{g}_A$

$\mathfrak{g}_A = \mathbb{R} \oplus_{\omega_A} C^\infty(A) \rtimes_D \mathbb{R}$ is the Lie algebra of $G_A$, $D = iA|_{C^\infty(A)}$.
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W.l.o.g. $W \subset W_\infty$. Define

$$W_d := \left\{ (t + \tfrac{1}{2s}\|x\|^2, x, s) \in \mathfrak{g}_A : s > 0, t + ds > 0 \right\} \quad \text{for } d \in \mathbb{R}.$$

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**Proposition**

*The non-empty open invariant cones $W$ in $\mathfrak{g}_A$ with $W \subset W_\infty$ are given by $W_d$ for $d \in \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$.*

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- open complex subsemigroups: $S_d := G_A^{\mathcal{O}} \exp_{G_{\mathbb{C}}}(iW_d^{\mathcal{O}}) \subset G_{\mathbb{C}}$ for $d \in \overline{\mathbb{R}}$.

# Holomorphic extension of a semibounded representation

**Proposition (Polar decomposition)**

$\psi : G_A^{\mathcal{O}} \times W_d^{\mathcal{O}} \to S_d, (g, w) \mapsto g \exp(iw)$ *is an analytic diffeomorphism for every* $d \in \overline{\mathbb{R}}$.

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Then: $id\pi(w)$ is bounded above for $w \in W_d$. Define:

$$\hat{\pi}(g \exp(iw)) := \pi(g) e^{id\pi(w)} \in \mathcal{B}(\mathcal{H}) \text{ for } (g, w) \in G_A^{\mathcal{O}} \times W_d^{\mathcal{O}}.$$

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**Theorem**

*For every $d \in \overline{\mathbb{R}}$ the assignment $\pi \mapsto \hat{\pi}$ yields a bijection*

$$\left\{ \begin{array}{c} \textit{semibounded reps. } \pi \textit{ of} \\ G_A \textit{ with } B_\pi \supset W_d \end{array} \right\} \to \left\{ \begin{array}{c} \textit{non-deg. holomorphic reps.} \\ \hat{\pi} : (S_d, *) \to (B(\mathcal{H}), *) \end{array} \right\}$$

*which preserves commutants, i.e. $\pi(G_A)' = \hat{\pi}(S_d)'$.*

# Associated $C^*$-algebras

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**Corollary**

*Let $C^\infty(A), \mathcal{H}$ be separable. Every semibd. rep. $\pi : G_A \to B(\mathcal{H})$ is a direct integral of semibd. factor representations.*

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**Fock representation**: $\pi_F : G_A \to \mathrm{U}(\mathcal{F}(C^\infty(A)))$ defined by the pos. def. function $\varphi(t, x, s) = e^{it - \langle Ax, x \rangle}$ (with GNS-construction).
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(c) *Let* $\pi(t,0,0) = e^{it}\mathbf{1}$. *Then*

$$(\pi, \mathcal{H}) \cong (\pi_F \otimes \nu, \mathcal{F}(C^\infty(A)) \otimes \mathcal{K}),$$

*where* $\nu(t,x,s) = \nu(s)$ *is a semibd. repr. of* $\mathbb{R}$ *on* $\mathcal{K}$. *If* $\pi$ *is irreducible then* $\dim \mathcal{K} = 1$, *hence* $\nu(t,x,s) = e^{i\beta s}\mathbf{1}, \beta \in \mathbb{R}$.

# Extending representations of the CCRs

$H$ Hilbert space with ONB $\{e_n\}_{n\in\mathbb{N}}$. Set $\omega(x,y) := \mathrm{Im}\langle x,y\rangle$.
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**Definition**

A representation $\pi : \mathrm{Heis}(V,\omega) \to \mathrm{U}(\mathcal{H})$ is called **regular** if $\pi(t,0,0) = e^{it}1$ and $\mathbb{R} \ni t \mapsto \pi(tv)$ is strongly cont. for all $v \in V$.

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## Theorem

*For every regular repr. $\pi : \mathrm{Heis}(V,\omega) \to \mathrm{U}(\mathcal{H})$ there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$, $a_n \to 0$ such that $\pi$ extends to a semibounded repr. $\hat{\pi} : G_A \to \mathrm{U}(\mathcal{H})$ with $\hat{\pi}(G_A)' = \pi(\mathrm{Heis}(V,\omega))'$.*