Category $\mathcal{O}$ and 1-quasi-hereditary algebras

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Overview

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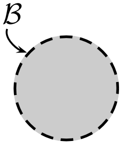

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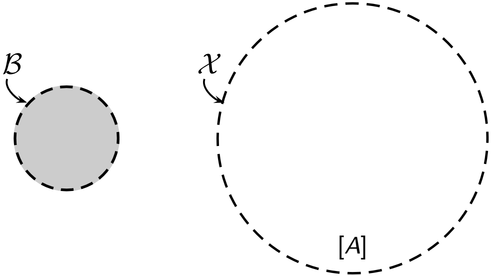

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$\mathcal{X} :=$ {basic K-algebras of dominant dimension ≥ 2}

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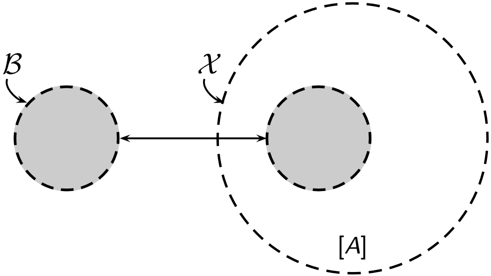

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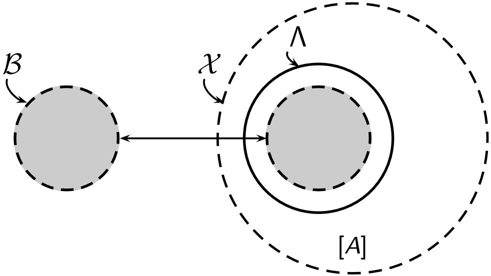

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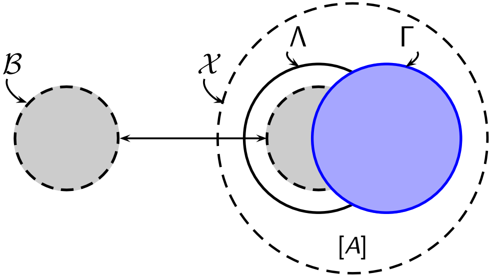

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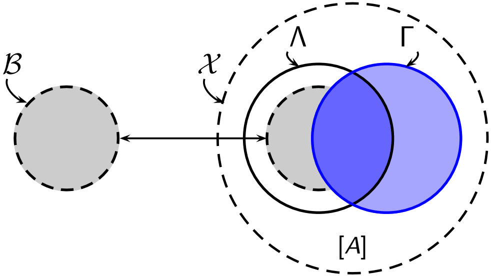

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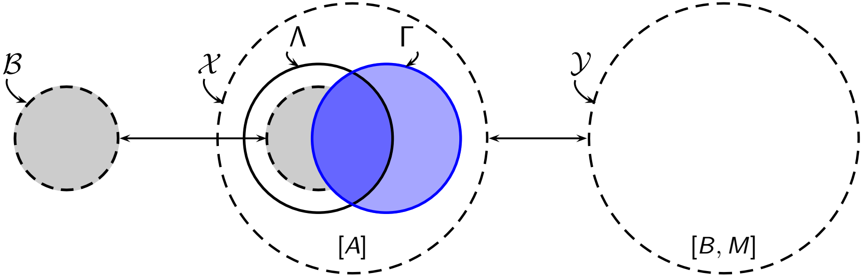

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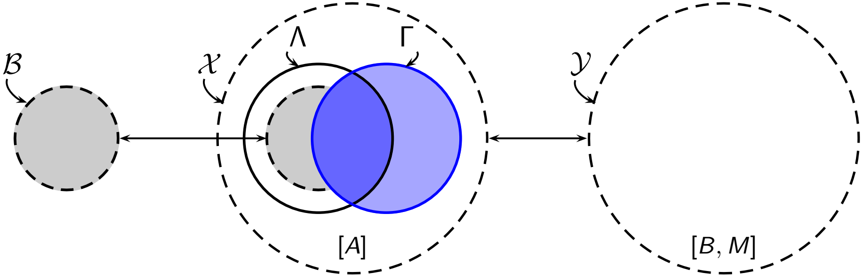

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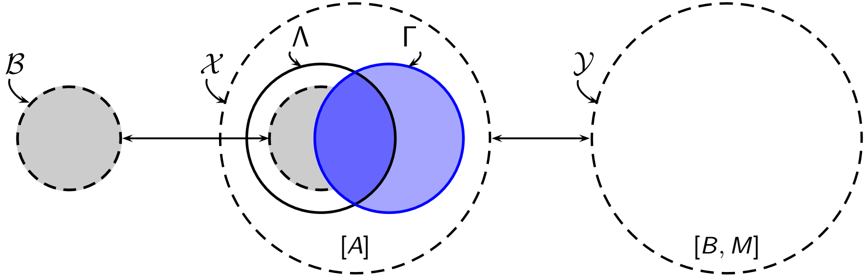

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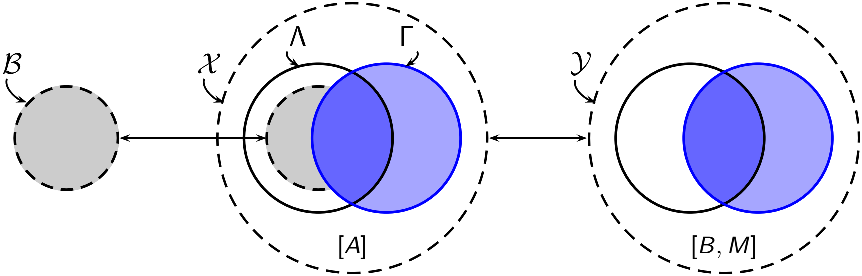

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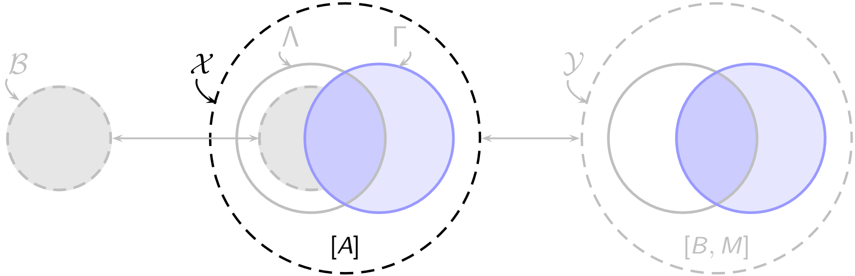

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Basic K-algebra over $K = \bar{K}$ with dom.dim (A)≥ 2

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Theorem (Gabriel)

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The dominant dimension of A is ≥ 2 if there exists an exact sequence $0 \to {}_AA \longrightarrow I_1 \longrightarrow I_2$

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The dominant dimension of A is ≥ 2 if there exists an exact sequence $0 \rightarrow {}_A A \longrightarrow I_1 \longrightarrow I_2$ where I_1, I_2 are projective - injective modules

$KQ/\mathcal{I}$-modules corresponding to $j \in Q_0$

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Q_0

j

$KQ/\mathcal{I}$-modules corresponding to $j \in Q_0$

$$\begin{tikzcd}
 & \begin{matrix}\{\text{simple}\} \\ S(j)\end{matrix} \\
\begin{matrix} Q_0 \\ j \end{matrix} \arrow[ur] &
\end{tikzcd}$$

$KQ/\mathcal{I}$-modules corresponding to $j \in Q_0$

$$\begin{array}{ccccc} & & \{\text{simple}\} & \overset{1:1}{\longleftrightarrow} & \{\text{projective indec.}\} \\ & \nearrow & S(j) & & P(j) \\ Q_0 & & & & \\ j & & & & \end{array}$$

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\begin{tikzcd}
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$[M : S(j)]$ is the multiplicity of $S(j)$ in a composition series of M

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$$\begin{array}{c} (Q_0, \leqslant) \\ j \end{array} \nearrow$$

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\begin{matrix}(Q_0, \leqslant)\\ j\end{matrix} \arrow[ur] \arrow[r] & \{\text{standard}\} \quad \Delta(j) & &
\end{tikzcd}
$$

$KQ/\mathcal{I}$-modules corresponding to $j \in Q_0$

$$\begin{array}{ccccccc} & & \{\text{simple}\} & \xleftrightarrow{1:1} & \{\text{projective indec.}\} & \xleftrightarrow{1:1} & \{\text{injective indec.}\} \\ & \nearrow & S(j) & & P(j) & & I(j) \\ (Q_0, \leqslant) & \longrightarrow & \{\text{standard}\} & & & & \\ j & & & & & & \end{array}$$

$\Delta(j)$ is the largest factor module of $P(j)$ such that

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$S(j)$ $\quad$ $P(j)$ $\quad$ $I(j)$

j

$\Delta(j)$ is the largest factor module of $P(j)$ such that $[\Delta(j) : S(i)] = 0$ for all $i \nleqslant j$

$KQ/\mathcal{I}$-modules corresponding to $j \in Q_0$

$$\begin{tikzcd}[row sep=small]
 & \{\text{simple}\} \ni S(j) \arrow[r, leftrightarrow, "1:1"] & \{\text{projective indec.}\} \ni P(j) \arrow[r, leftrightarrow, "1:1"] & \{\text{injective indec.}\} \ni I(j) \\
(Q_0, \leqslant) \ni j \arrow[ur] \arrow[r] \arrow[dr] & \{\text{standard}\} & & \\
 & \{\text{costandard}\}\ \nabla(j) & &
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$\{$standard$\}$ $\Delta(j)$ is the largest factor module of $P(j)$ such that $[\Delta(j) : S(i)] = 0$ for all $i \not\leqslant j$

$\{$costandard$\}$ $\nabla(j)$ (dual)

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Example.

$KQ/\mathcal{I}$-modules corresponding to $j \in Q_0$

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$\{\text{costandard}\}$ $\nabla(j)$ (dual)

Example. The K-algebra $\quad A = K\left(\overset{1}{\bullet} \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \overset{2}{\bullet}\right)$

$KQ/\mathcal{I}$-modules corresponding to $j \in Q_0$

$$\begin{tikzcd} & \{\text{simple}\} \; S(j) \arrow[r, leftrightarrow, "1:1"] & \{\text{projective indec.}\} \; P(j) \arrow[r, leftrightarrow, "1:1"] & \{\text{injective indec.}\} \; I(j) \\ (Q_0, \leqslant) \; j \arrow[ur] \arrow[r] \arrow[dr] & \{\text{standard}\} & & \\ & \{\text{costandard}\} & & \end{tikzcd}$$

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$1 \rightsquigarrow \{e_1, \alpha, \beta\alpha, \alpha\beta\alpha, \dots\}$

$KQ/\mathcal{I}$-modules corresponding to $j \in Q_0$

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 & S(j) & P(j) & I(j) \\
(Q_0, \leqslant) \arrow[uur] \arrow[r] \arrow[ddr] & \{\text{standard}\} & \multicolumn{2}{l}{} \\
j & & & \\
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$$2 \rightsquigarrow \{e_2, \beta, \alpha\beta, \beta\alpha\beta, \ldots\}$$

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\nearrow & & & & \\
(Q_0, \leqslant) \longrightarrow & \{\text{standard}\} & & \multicolumn{2}{l}{\Delta(j) \text{ is the largest factor module of } P(j) \text{ such that } [\Delta(j) : S(i)] = 0 \text{ for all } i \nleqslant j} \\
j \searrow & & & & \\
 & \{\text{costandard}\} \; \nabla(j) \; (\text{dual}) & & &
\end{array}
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Example. The K-algebra

$$A = K\left(\overset{1}{\bullet} \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \overset{2}{\bullet}\right) / \langle \alpha\beta \rangle$$

$$1 \rightsquigarrow \{e_1, \alpha, \beta\alpha, \alpha\beta\alpha, \ldots\}$$

$$2 \rightsquigarrow \{e_2, \beta, \alpha\beta, \beta\alpha\beta, \ldots\}$$

$KQ/\mathcal{I}$-modules corresponding to $j \in Q_0$

$$\begin{tikzcd}[column sep=small]
 & \{\text{simple}\} \; S(j) \arrow[r, leftrightarrow, "1:1"] & \{\text{projective indec.}\} \; P(j) \arrow[r, leftrightarrow, "1:1"] & \{\text{injective indec.}\} \; I(j) \\
(Q_0, \leqslant) \ni j \arrow[ur] \arrow[r] \arrow[dr] & \{\text{standard}\} & & \\
 & \{\text{costandard}\} & &
\end{tikzcd}$$

$\{\text{standard}\}$: $\Delta(j)$ is the largest factor module of $P(j)$ such that $[\Delta(j) : S(i)] = 0$ for all $i \nleqslant j$

$\{\text{costandard}\}$: $\nabla(j)$ (dual)

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(Q_0, \leqslant) \; j \arrow[urr] \arrow[rr] \arrow[drr] & & \{\text{standard}\} & & \\
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$KQ/\mathcal{I}$-modules corresponding to $j \in Q_0$

$$\begin{array}{ccccc} & & \{\text{simple}\} \xleftrightarrow{1:1} \{\text{projective indec.}\} \xleftrightarrow{1:1} \{\text{injective indec.}\} \\ & \nearrow & S(j) \qquad\qquad P(j) \qquad\qquad I(j) \\ (Q_0, \leqslant) & \longrightarrow & \{\text{standard}\} \\ j & \searrow & \\ & & \{\text{costandard}\}\end{array}$$

$\Delta(j)$ is the largest factor module of $P(j)$ such that $[\Delta(j) : S(i)] = 0$ for all $i \nleqslant j$

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Example. The K-algebra $\dim_K[A = K\left(\overset{1}{\bullet} \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \overset{2}{\bullet}\right) / \langle \alpha\beta \rangle] = 5$,

$$1 \rightsquigarrow \{e_1, \alpha, \beta\alpha, \alpha\beta\alpha, \ldots\} = \{e_1, \alpha, \beta\alpha\}$$

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$KQ/\mathcal{I}$-modules corresponding to $j \in Q_0$

$$\begin{array}{ccccc} & \{\text{simple}\} \; S(j) & \overset{1:1}{\longleftrightarrow} & \{\text{projective indec.}\} \; P(j) & \overset{1:1}{\longleftrightarrow} \; \{\text{injective indec.}\} \; I(j) \end{array}$$

$$\begin{tikzcd} & \{\text{simple}\} \\ \underset{j}{(Q_0, \leqslant)} \arrow[ur] \arrow[r] \arrow[dr] & \{\text{standard}\} \\ & \{\text{costandard}\} \end{tikzcd}$$

$\{$standard$\}$ $\Delta(j)$ is the largest factor module of $P(j)$ such that $[\Delta(j) : S(i)] = 0$ for all $i \not\leqslant j$

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$$\{\text{simple}\} \quad \overset{1:1}{\longleftrightarrow} \quad \{\text{projective indec.}\} \quad \overset{1:1}{\longleftrightarrow} \quad \{\text{injective indec.}\}$$

$$S(j) \qquad\qquad P(j) \qquad\qquad I(j)$$

$(Q_0, \leqslant) \longrightarrow \{\text{standard}\}$ $\quad \Delta(j)$ is the largest factor module of $P(j)$ such that $[\Delta(j) : S(i)] = 0$ for all $i \not\leqslant j$

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$$P(2) \leftrightsquigarrow \{e_2, \beta\}$$

$$\begin{array}{c} \langle e_1 \rangle = P(1) \\ | \\ \langle \alpha \rangle \\ | \\ \langle \beta\alpha \rangle \\ | \\ 0 \end{array}$$

$KQ/\mathcal{I}$-modules corresponding to $j \in Q_0$

$$\{\text{simple}\} \overset{1:1}{\longleftrightarrow} \{\text{projective indec.}\} \overset{1:1}{\longleftrightarrow} \{\text{injective indec.}\}$$
$$S(j) \qquad\qquad P(j) \qquad\qquad I(j)$$

$(Q_0, \leqslant) \nearrow$ {simple}

$\underset{j}{(Q_0, \leqslant)} \longrightarrow \{\text{standard}\}$ $\quad \Delta(j)$ is the largest factor module of $P(j)$ such that $[\Delta(j) : S(i)] = 0$ for all $i \not\leqslant j$

$(Q_0, \leqslant) \searrow$ $\{\text{costandard}\}$ $\nabla(j)$ (dual)

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$$P(1) \leftrightsquigarrow \{e_1, \alpha, \beta\alpha\}$$
$$P(2) \leftrightsquigarrow \{e_2, \beta\}$$

$$\begin{tikzcd}[row sep=large]
\langle e_1 \rangle = P(1) \arrow[d, dash] & \\
\langle \alpha \rangle \arrow[d, dash] & \langle e_2 \rangle = P(2) \arrow[d, dash] \\
\langle \beta\alpha \rangle \arrow[d, dash] & \langle \beta \rangle \arrow[d, dash] \\
0 & 0
\end{tikzcd}$$

$KQ/\mathcal{I}$-modules corresponding to $j \in Q_0$

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 & \{\text{simple}\} \; S(j) \arrow[r, leftrightarrow, "1:1"] & \{\text{projective indec.}\} \; P(j) \arrow[r, leftrightarrow, "1:1"] & \{\text{injective indec.}\} \; I(j) \\
(Q_0, \leqslant) \; j \arrow[ur] \arrow[r] \arrow[dr] & \{\text{standard}\} & & \\
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$$
\begin{tikzcd}[row sep=large]
\langle e_1 \rangle = P(1) = I(1) \arrow[d, dash] & \\
\langle \alpha \rangle \arrow[d, dash] & \langle e_2 \rangle = P(2) \arrow[d, dash] \\
\langle \beta\alpha \rangle \arrow[d, dash] & \langle \beta \rangle \arrow[d, dash] \\
0 & 0
\end{tikzcd}
$$

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$$\begin{array}{c} \langle e_1 \rangle = P(1) = I(1) \\ | \\ \langle \alpha \rangle \\ | \\ \langle \beta\alpha \rangle \\ | \\ 0 \end{array} \qquad \begin{array}{c} \langle e_2 \rangle = P(2) \\ | \\ \langle \beta \rangle \\ | \\ 0 \end{array} \qquad \begin{array}{c} \langle \bar{e_1} \rangle = I(2) \\ | \\ \langle \bar{\alpha} \rangle \\ | \\ \langle \bar{\beta}\alpha \rangle \end{array}$$

$KQ/\mathcal{I}$-modules corresponding to $j \in Q_0$

$$\begin{array}{lll}
 & \nearrow & \{\text{simple}\} \ S(j) \xleftrightarrow{1:1} \{\text{projective indec.}\} \ P(j) \xleftrightarrow{1:1} \{\text{injective indec.}\} \ I(j) \\
\underset{j}{(Q_0, \leqslant)} & \longrightarrow & \{\text{standard}\} \quad \Delta(j) \text{ is the largest factor module of } P(j) \text{ such that } [\Delta(j) : S(i)] = 0 \text{ for all } i \nleqslant j \\
 & \searrow & \{\text{costandard}\} \ \nabla(j) \ (\text{dual})
\end{array}$$

Example. The K-algebra $\dim_K [A = K\left(\overset{1}{\bullet} \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \overset{2}{\bullet} \right) / \langle \alpha\beta \rangle] = 5,$

$$\begin{tikzcd}[row sep=small]
\langle e_1 \rangle = P(1) = I(1) \arrow[d, dash] & \langle e_2 \rangle = P(2) \arrow[d, dash] & \langle \bar{e_1} \rangle = I(2) \arrow[d, dash] \\
\langle \alpha \rangle \arrow[d, dash] & \langle \beta \rangle \arrow[d, dash] & \langle \bar{\alpha} \rangle \arrow[d, dash] \\
\langle \beta\alpha \rangle \arrow[d, dash] & 0 & \langle \bar{\beta\alpha} \rangle \\
0 & &
\end{tikzcd}$$

$KQ/\mathcal{I}$-modules corresponding to $j \in Q_0$

$$\begin{array}{lllll} & \{\text{simple}\} \; S(j) & \overset{1:1}{\longleftrightarrow} & \{\text{projective indec.}\} \; P(j) & \overset{1:1}{\longleftrightarrow} \; \{\text{injective indec.}\} \; I(j) \\ \nearrow & & & & \\ (Q_0, \leqslant) \; j \longrightarrow & \{\text{standard}\} & & \Delta(j) \text{ is the largest factor module of } P(j) \text{ such that } [\Delta(j) : S(i)] = 0 \text{ for all } i \not\leqslant j & \\ \searrow & & & & \\ & \{\text{costandard}\} & \nabla(j) \text{ (dual)} & & \end{array}$$

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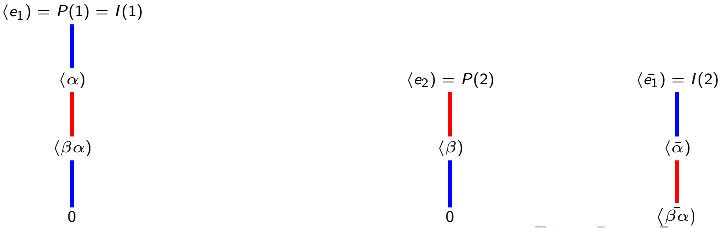

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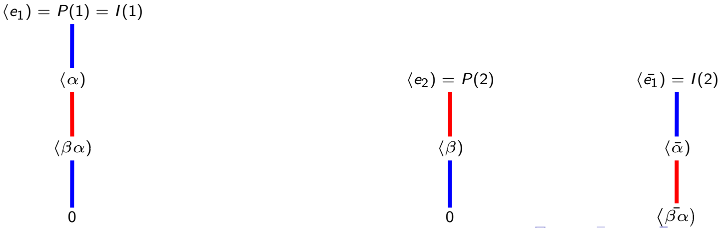
$(Q_0, \leqslant) \longrightarrow$ {standard} $\quad \Delta(j)$ is the largest factor module of $P(j)$ such that $[\Delta(j) : S(i)] = 0$ for all $i \nleqslant j$

j

{costandard} $\nabla(j)$ (dual)

Example. The K-algebra $\dim_K [A = K\left(\overset{1}{\bullet} \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \overset{2}{\bullet}\right) / \langle \alpha\beta \rangle] = 5, \quad 1 < 2$



$KQ/\mathcal{I}$-modules corresponding to $j \in Q_0$

$$
\begin{array}{lllll}
 & \{\text{simple}\} & \overset{1:1}{\longleftrightarrow} & \{\text{projective indec.}\} \overset{1:1}{\longleftrightarrow} & \{\text{injective indec.}\} \\
 & S(j) & & P(j) & I(j) \\
\underset{j}{(Q_0, \leqslant)} \longrightarrow & \{\text{standard}\} & & \multicolumn{2}{l}{\Delta(j) \text{ is the largest factor module of } P(j) \text{ such that}} \\
 & & & \multicolumn{2}{l}{[\Delta(j) : S(i)] = 0 \text{ for all } i \not\leqslant j} \\
 & \{\text{costandard}\} & \nabla(j) \text{ (dual)} & &
\end{array}
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$\Delta(1) = \{$ $\langle e_1 \rangle = P(1) = I(1)$

$\langle \alpha \rangle$

$\langle \beta\alpha \rangle$

0

$\langle e_2 \rangle = P(2)$

$\langle \beta \rangle$

0

$\langle \bar{e_1} \rangle = I(2)$

$\langle \bar{\alpha} \rangle$

$\langle \bar{\beta}\alpha \rangle$

$KQ/\mathcal{I}$-modules corresponding to $j \in Q_0$

$$\begin{tikzcd}[column sep=small]
 & \{\text{simple}\} \arrow[r, leftrightarrow, "1:1"] & \{\text{projective indec.}\} \arrow[r, leftrightarrow, "1:1"] & \{\text{injective indec.}\} \\
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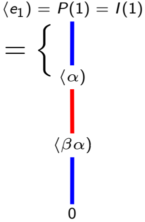
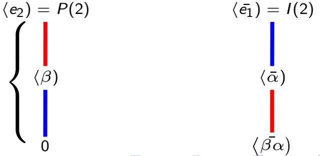


$\Delta(2) = \{$



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$S(j)$ $\qquad$ $P(j)$ $\qquad$ $I(j)$

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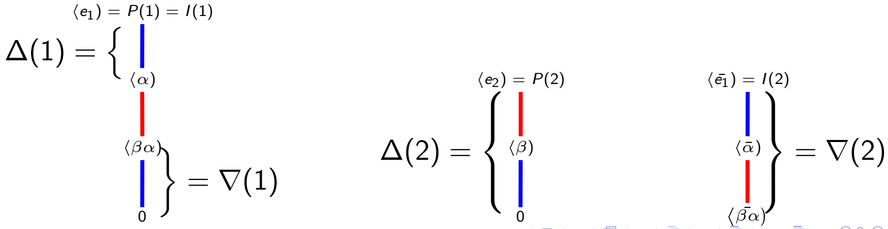

Quasi-hereditary algebra

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Definition (Cline, Parshall, Scott)

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An algebra $A = KQ/\mathcal{I}$ is quasi-hereditary with respect to $(Q_0, \leqslant)$ if for all $i, j \in Q_0$ we have

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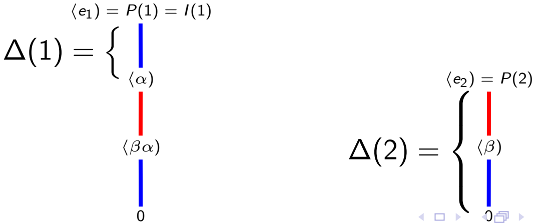

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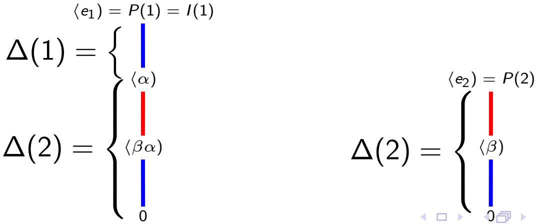
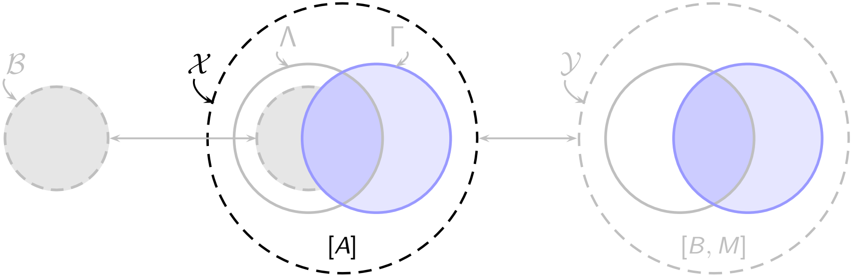


Overview



$\mathcal{B} := \{\text{blocks of } \mathcal{O}(\mathfrak{g}) \mid \mathfrak{g} \text{ is a semisimple Lie-algebra } / \ \mathbb{C}\ \}$

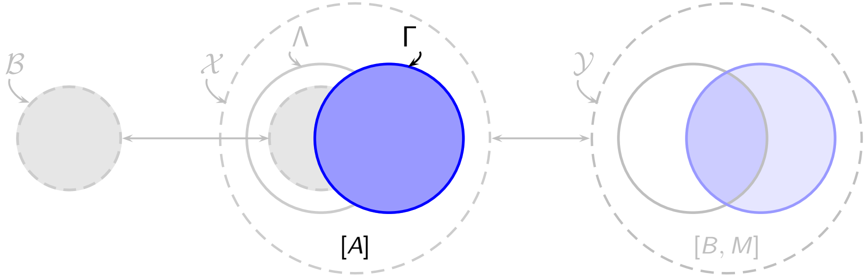
$\mathcal{X} := \{\text{basic } K\text{-algebras of dominant dimension} \geq 2\}$

$\Lambda := \{\text{factor algebras of type } A(i) \text{ for } A \in \mathcal{B} \subset \mathcal{X}\}$

$\Gamma := \{\text{1-quasi-hereditary algebras}\}$

$$\mathcal{Y} := \left\{ [B, M] \;\middle|\; \begin{array}{l} B \text{ is a basic } K\text{-algebra} \\ M \text{ is a mult.-free generator-cogenerator of mod-}B \end{array} \right\}$$

Overview



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1-quasi-hereditary algebras

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1. $1 \leqslant i \leqslant n$,
2. $(P(j) : \Delta(i)) = [\Delta(i) : S(j)] = 1$ for $j \leqslant i$,
3. $\operatorname{soc} P(i) \cong \operatorname{top} I(i) \cong S(1)$,

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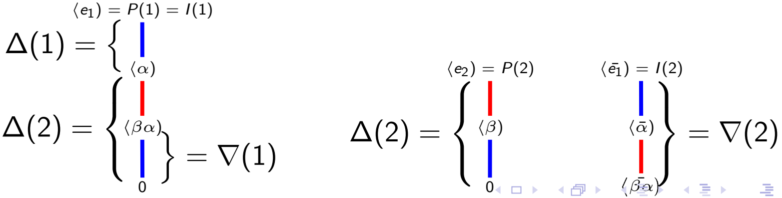

The quiver and relations of a 1-quasi-hereditary algebra

The quiver and relations of a 1-quasi-hereditary algebra

Theorem

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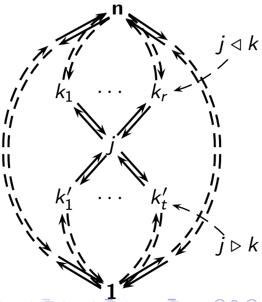

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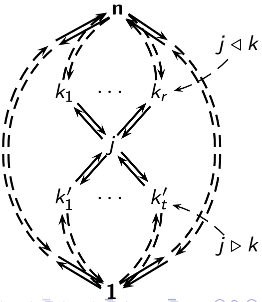

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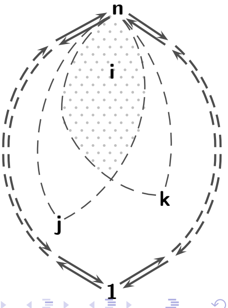

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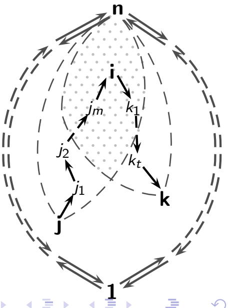

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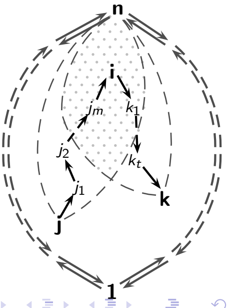

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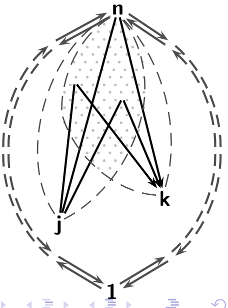

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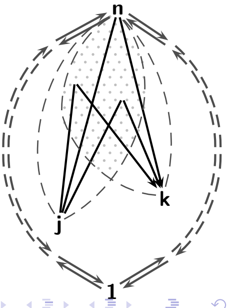

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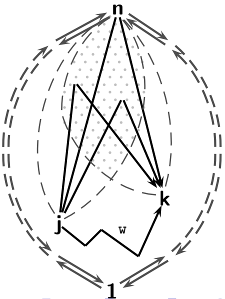

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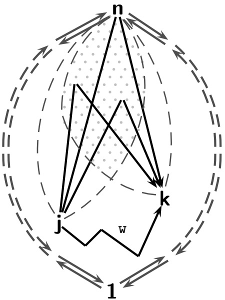

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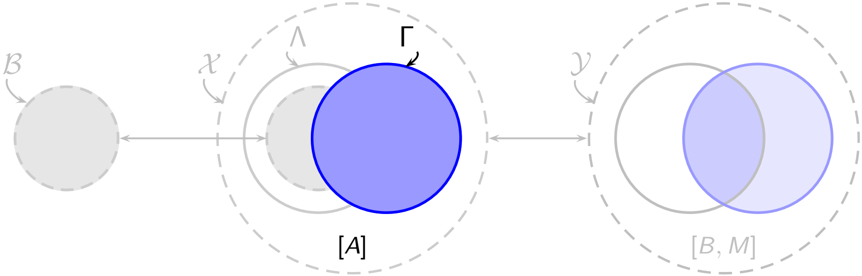

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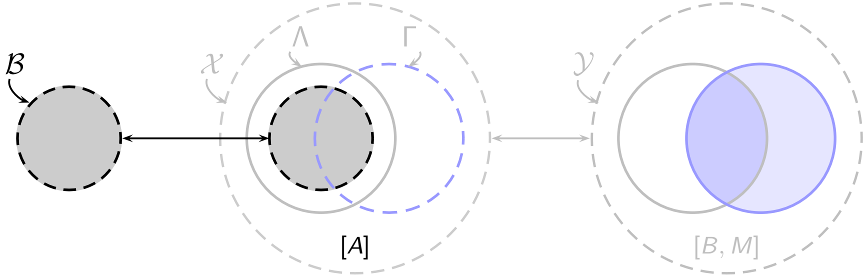

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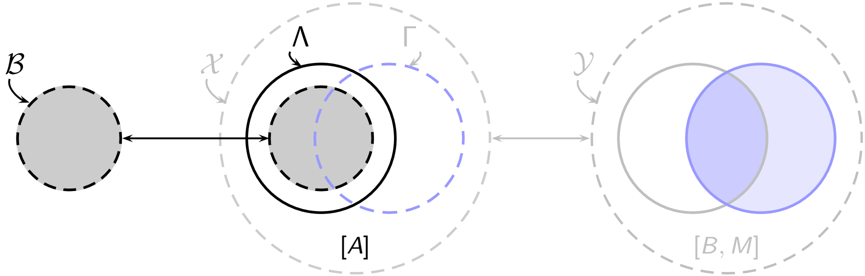

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- $\mathcal{O}_\lambda(\mathfrak{g}) \leftrightsquigarrow W_\lambda \cdot \lambda \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{projective indec.} \\ P(w\cdot\lambda) \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Verma} \\ V(w\cdot\lambda) \end{array} \right\}$
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- The algebra $A_\lambda(\mathfrak{g}) \cong \mathbb{C}Q/\mathcal{I}$
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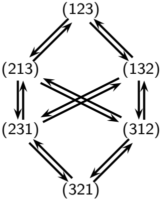
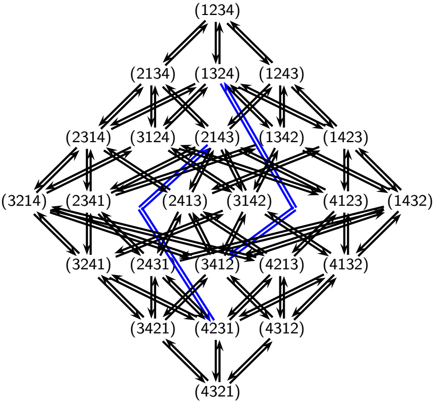
$Q(3) =$ 

$Q(4) =$ 

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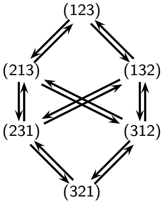

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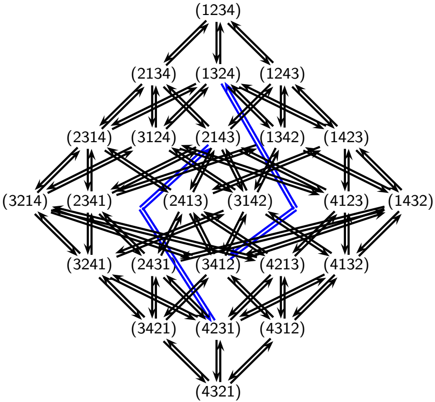

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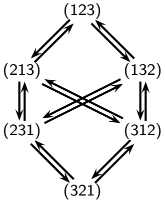

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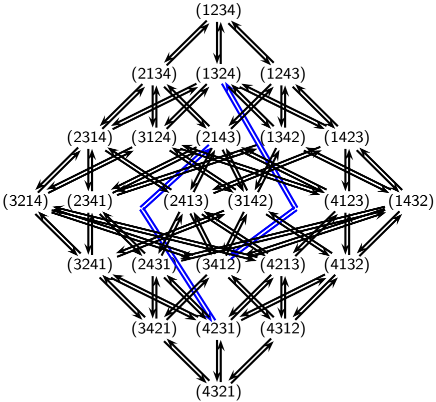

Let $(\mathcal{A}, \leqslant)$ be a quasi-hereditary algebra and $i \in Q_0(\mathcal{A})$

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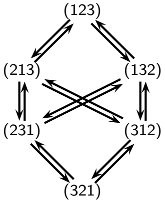

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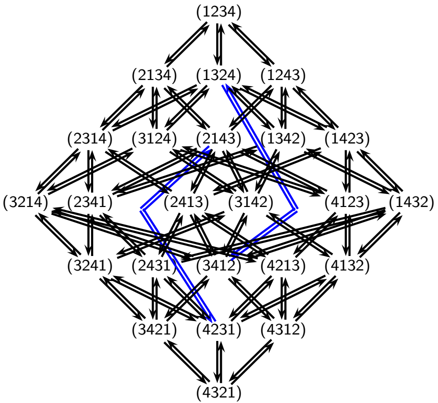

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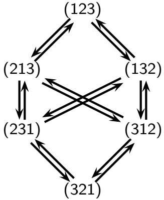
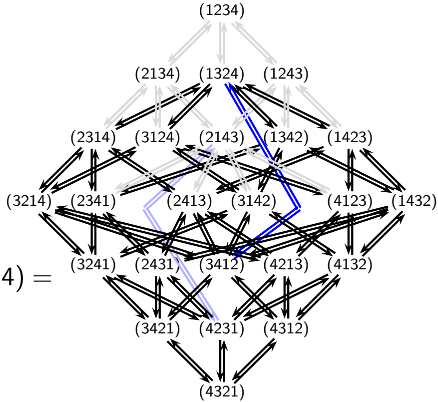


$Q(1324) =$



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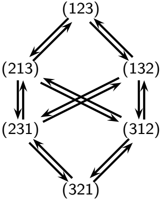

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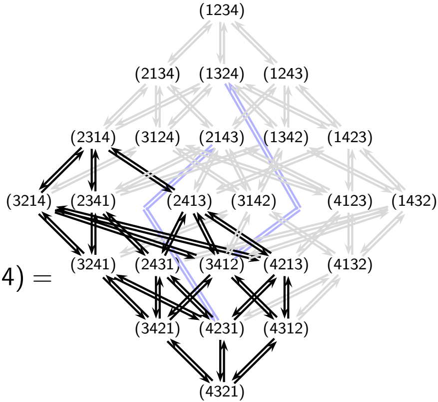

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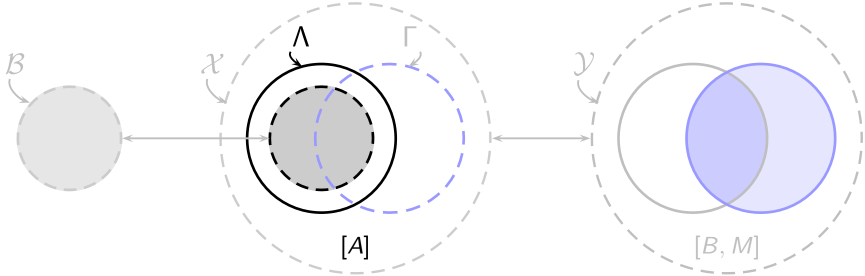
Overview



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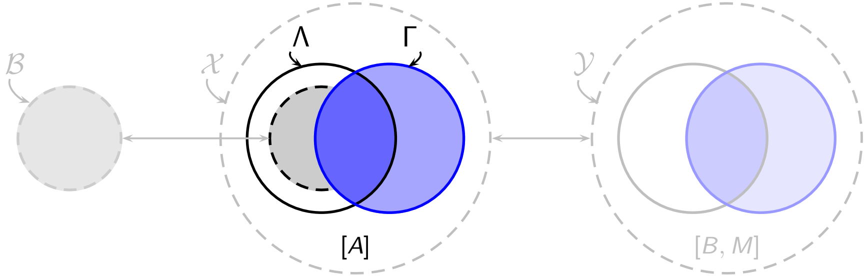

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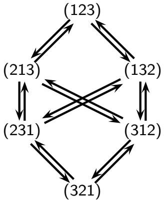
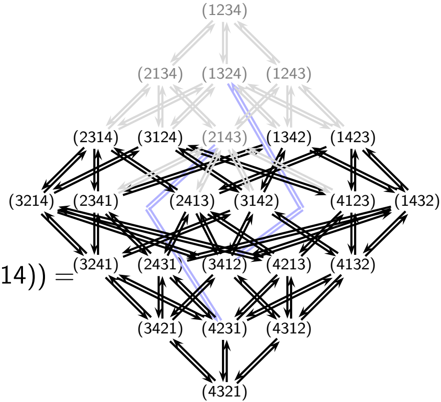
$Q(3) =$ 

$Q(A_{(4)}(2314)) =$ 

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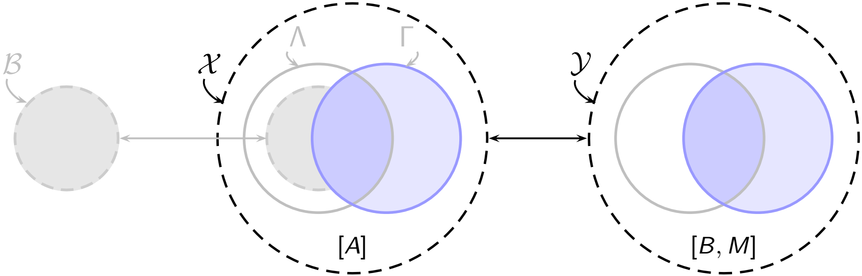

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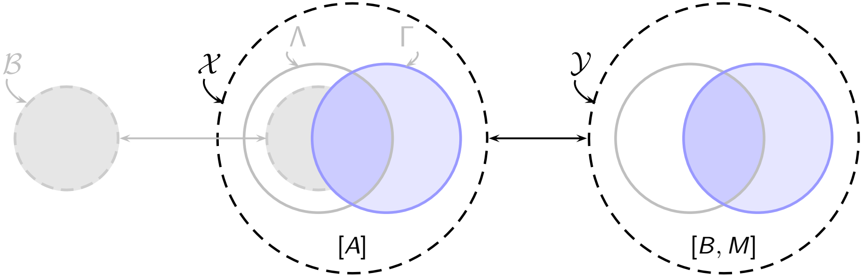

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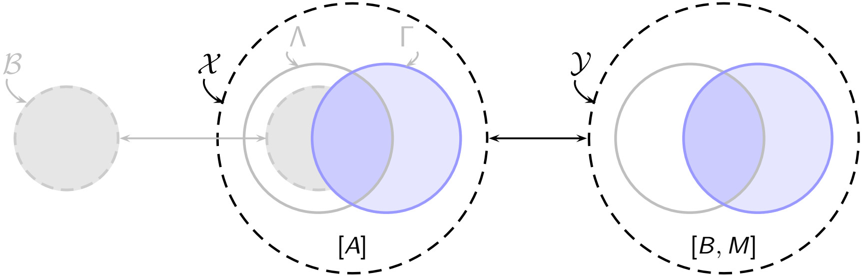

There exist a bijection between $\mathcal{X}$ and $\mathcal{Y}$

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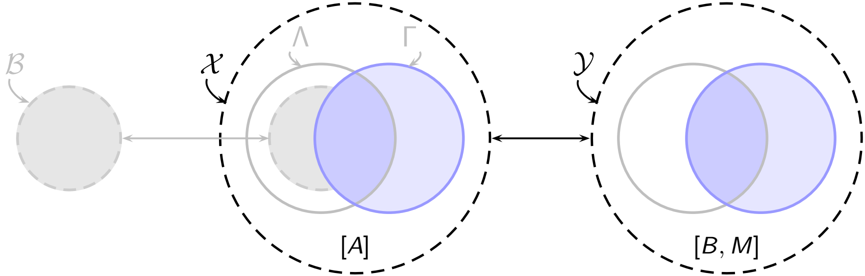

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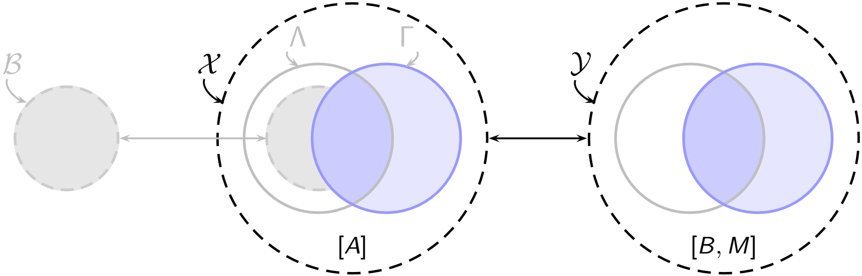

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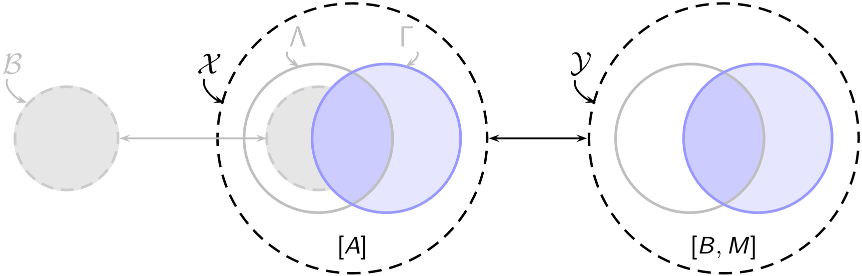

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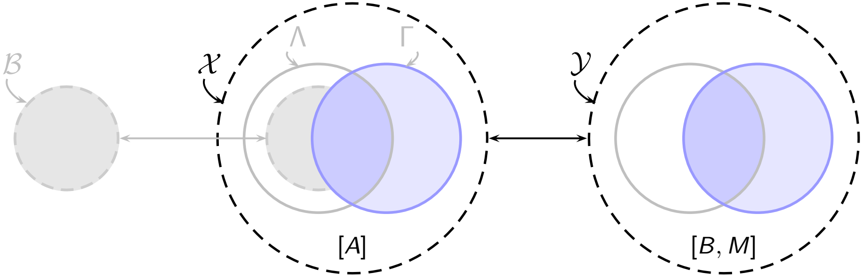

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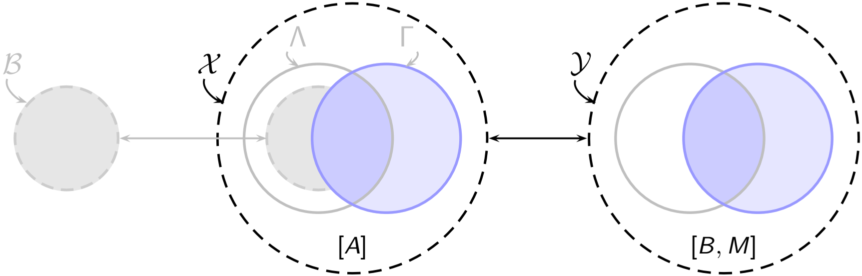

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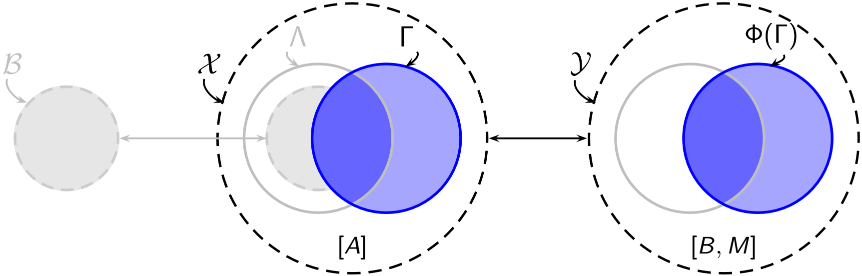

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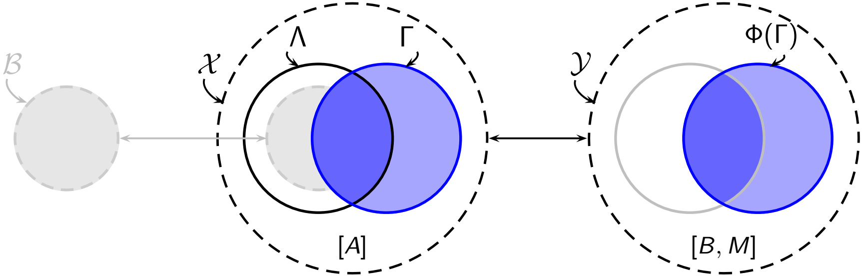

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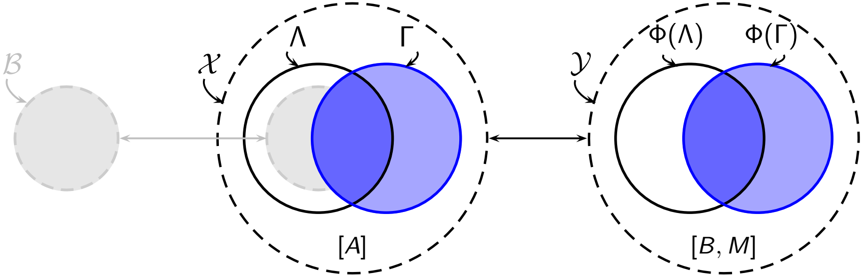

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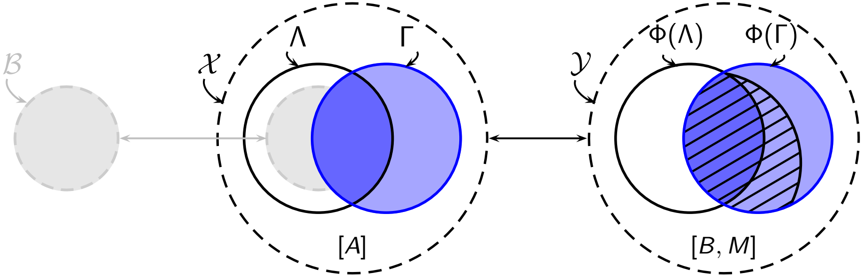

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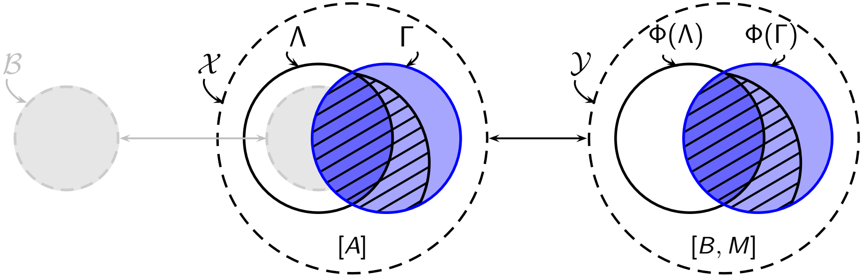

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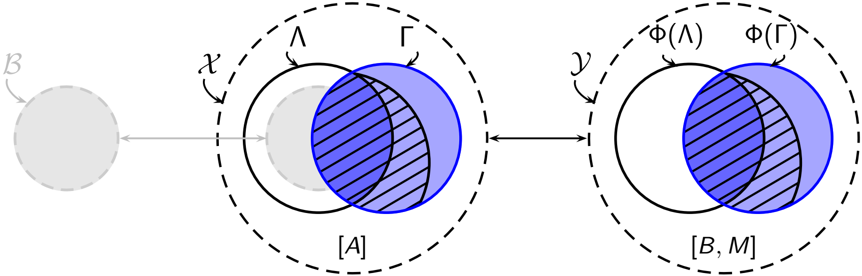

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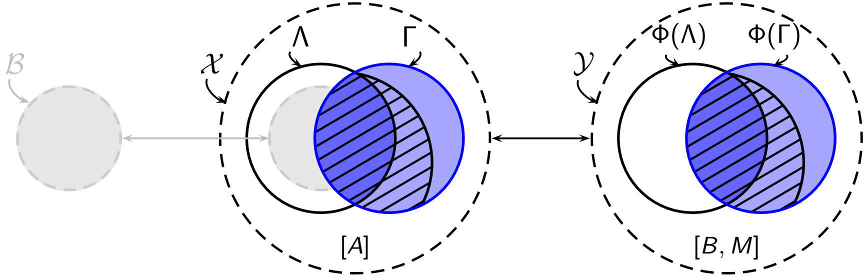

Theorem

Let $[B, M] \in \Phi(\Gamma)$ and $A \cong \mathrm{End}_B(M)^{op} \cong KQ/\mathcal{I}$ the correspon-ding 1-quasi-hereditary algebra.

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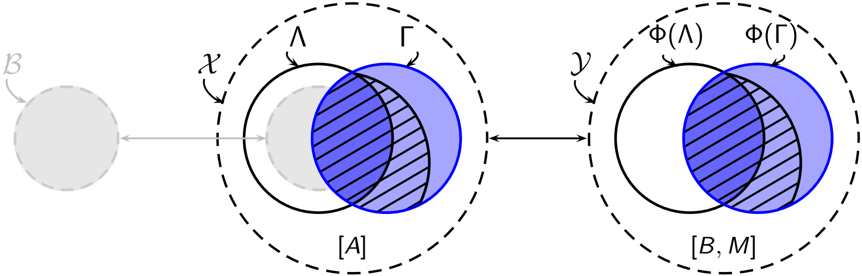

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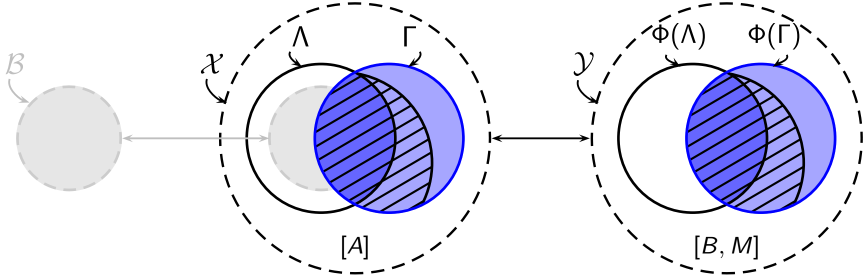

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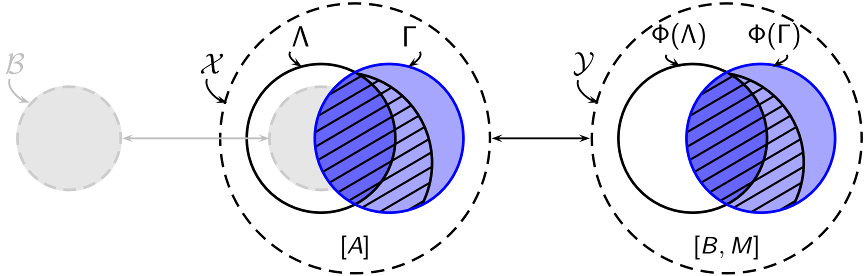

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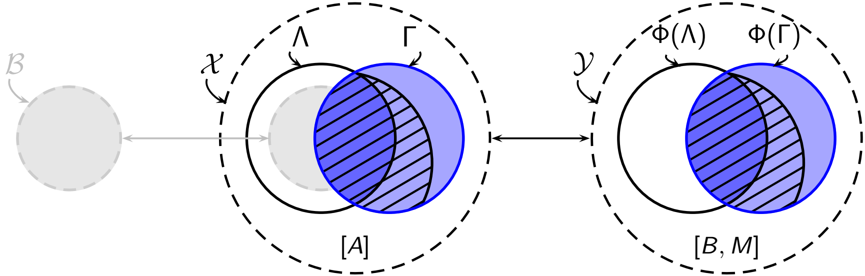

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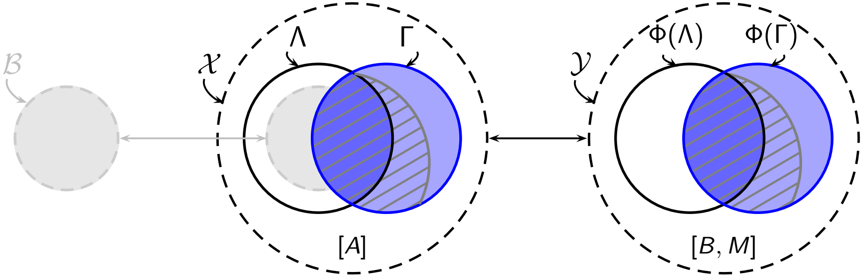

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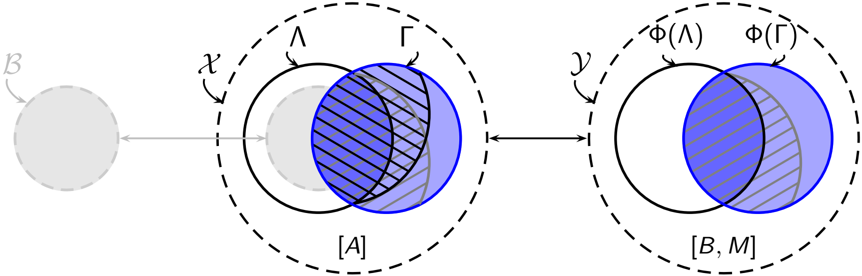

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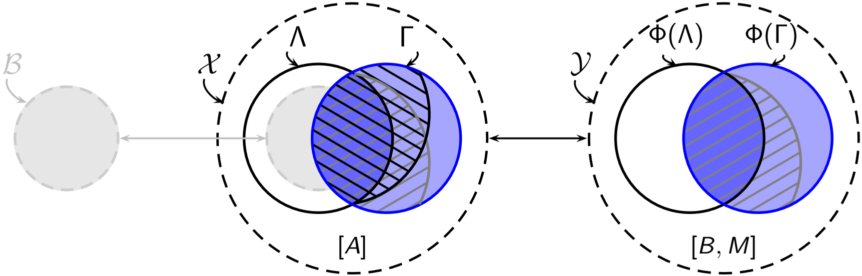

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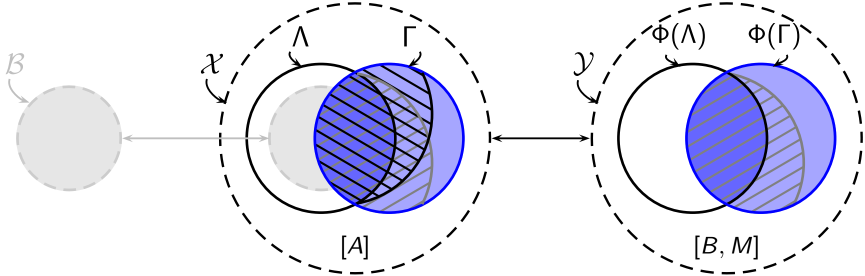

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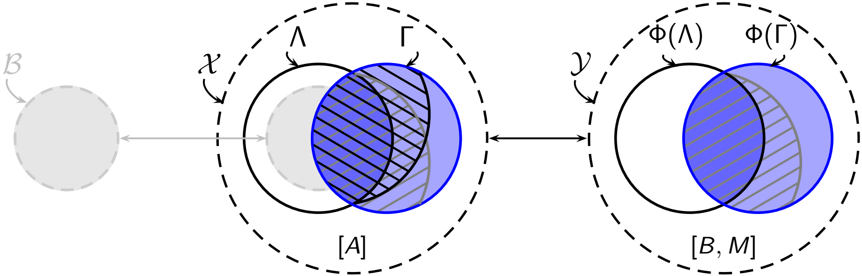

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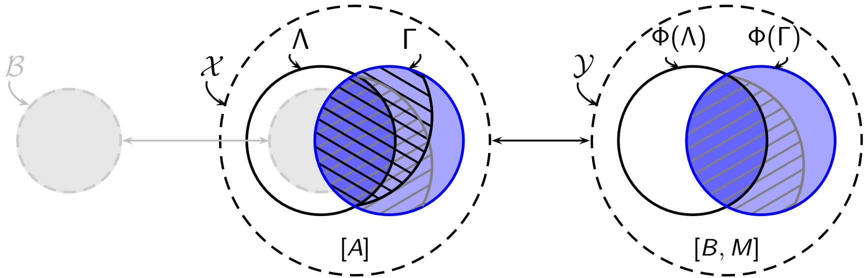

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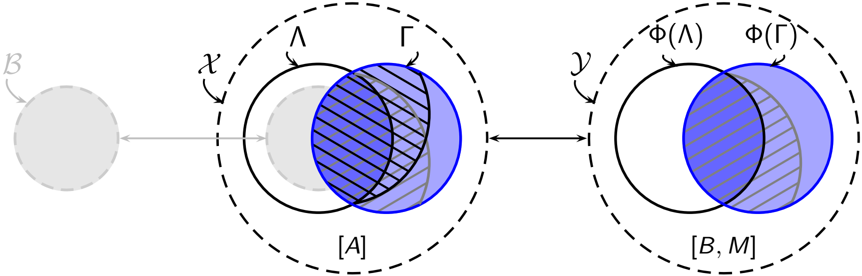

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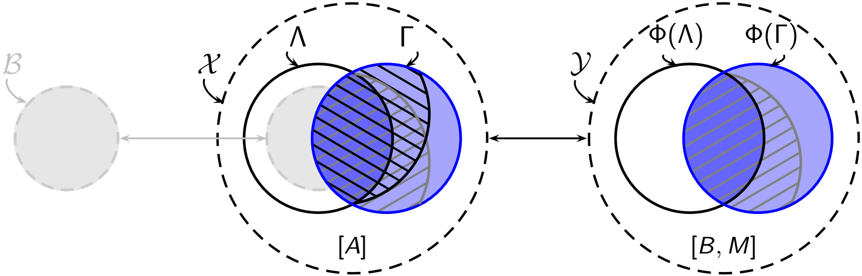

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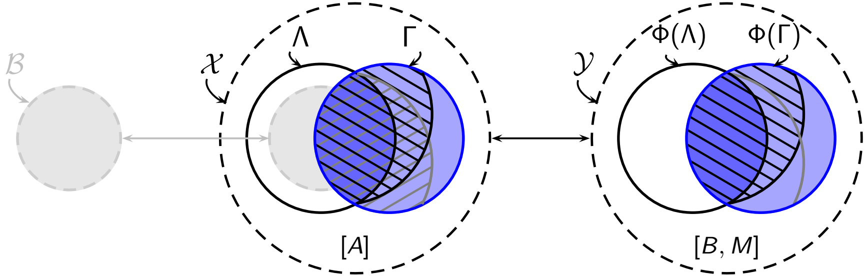

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