Lattices and Categories

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Cone surface singularities

S affine, isolated surface singularity with good $\mathbb{C}^*$-action
$\rightsquigarrow S = \mathrm{Spec}(R)$ with $R = \bigoplus_{i \geq 0} R_i, \quad R_0 = \mathbb{C}$.

S cone over the smooth, projective curve $\mathrm{Proj}(R)$

Simplest case: hypersurface $S = \{f = 0\}$
$R = \mathbb{C}[x, y, z]/(f)$ with weighted-homogenous polynomial f

Kleinian singularities

$(S, 0)$ *Kleinian singularity* if

- canonical surface singularities
- quotient plane singularities $\mathbb{C}^2/G$ with $G \subset \mathrm{SU}(2)$
- hypersurfaces from ADE polynomials

A_n	$x^{n+1} + yz$	cyclic group
D_n	$x^{n-1} + xy^2 + z^2$	binary dihedral group
E_6	$x^4 + y^3 + z^2$	binary tetrahedral group
E_7	$x^3y + y^3 + z^2$	binary octahedral group
E_8	$x^5 + y^3 + z^2$	binary icosahedral group

Topologically: resolution = deformation.

Fuchsian singularities

$$\begin{array}{lll} \overline{S} := \mathrm{Proj}(R[t]) & \text{canonical compactification} \\ \bigcup & \\ \overline{S}_\infty := \mathrm{Proj}(R) & \text{boundary divisor} \end{array}$$

$(S, 0)$ Fuchsian singularity if

- $\overline{S}$ Gorenstein
- $S = (T_{\mathbb{H}} \backslash \mathbb{H})/\Gamma$ with $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ Fuchsian group

Analogy between Kleinian and Fuchsian types:

$$\begin{array}{ll} S_{Klein} = \mathbb{C}^2/G & G \subset \mathrm{SU}(2) \\ R_{Klein} = \bigoplus_{k \geq 0} H^0(\mathbb{P}^1, T_{\mathbb{P}^1}^{\otimes k})^\Gamma & \Gamma \subset \mathrm{Aut}(\mathbb{P}^1) = \mathrm{PSU}(2) \\ R_{Fuchs} = \bigoplus_{k \geq 0} H^0(\mathbb{H}, T_{\mathbb{H}}^{-k})^\Gamma & \Gamma \subset \mathrm{Aut}(\mathbb{H}) = \mathrm{PSU}(1,1) \end{array}$$

Genus of $(S, 0)$ is $g := g(\mathbb{H}/\Gamma) = g(\overline{S}_\infty)$.

Milnor lattice

S hypersurface singularity, $f\colon \mathbb{C}^3 \to \mathbb{C}$

$S_t := f^{-1}(t) \cap B_\epsilon \to D_\delta \backslash \{0\}$ Milnor fibre

$H^2(\mathcal{X}_t, \mathbb{Z}) \cong \mathbb{Z}^\mu$ lattice with cup pairing (Milnor lattice)

Note: differential topological (even symplectic) construction
$\rightsquigarrow$ A side of mirror symmetry

Milnor lattices are root lattices (basis of -2-classes)
$\rightsquigarrow$ Coxeter element (monodromy operator)

Quest for nice bases (Brieskorn, Knörrer, Ebeling)

Milnor number and modality

$S = \{f = 0\}$ isolated surface singularity in 0

Milnor number $\mu(f)$ is

- rank of Milnor lattice
- dimension of the algebra $Q_f := \mathbb{C}[[x,y,z]]/(\partial f/\partial x, \partial f/\partial y, \partial f/\partial z)$
- number of parameters for minimal versal unfolding

$$F\colon \mathbb{C}^3 \times \mathbb{C}^\mu \to \mathbb{C}, \quad (p,u) \mapsto f(p) + \sum_{i=1}^{\mu} g_i(p) u_i$$

Modality $m(f)$ is the dimension of

$$\{u \in \mathbb{C}^\mu \mid F_u(0) = 0 \text{ singularity with Milnor number } \mu\}$$

Arnol'd classification for modalities 0,1,2.

Bimodal singularities

Arnol'd: 8 series and 14 exceptional classes.
20 bimodal singularities given by a weighted-homogeneous polynomial, and have an invertible representative.

Want to describe the Milnor lattice (A side) using coherent sheaves (B side).
$\Rightarrow$ Work with mirror partner of the singularity — Berglund-Hübsch dual for invertible polynomials.

Example: $x^6y + y^3 + z^2$ with Berglund-Hübsch dual $x^6 + xy^3 + z^2$.

Connections to representation theory

- Lattices can come up in representation theory (Fuchsian singularities $\leftrightarrow$ canonical algebras: Lenzing, de la Peña).
- Homological algebra of the coordinate rings R: MCM modules, matrix factorisations.

Categorification

Goal: Find a triangulated category whose K-group is the Milnor lattice.

- Gives connection between singularity, geometry (and algebra)
- Allows geometric explanation of known effects
- Higher level structure (e.g. for mirror symmetry)

This approach allows to treat lattice elements, roots, Coxeter elements geometrically (sheaves, curves, spherical twist functors).

From singularities to K3 surfaces to categories

S singularity $\rightsquigarrow$ X K3 surface $\rightsquigarrow$ $\mathcal{D}$ triangulated category

$S \rightsquigarrow X$ using geometry:

- resolution
- compactification
- deformation

Define $\mathcal{D}$ as a subcategory of $D^b(\mathrm{coh} X)$ such that

- $\mathcal{D}$ generated by spherical objects
- $\mathcal{D}$ 2-Calabi-Yau
- $\mathcal{D}$ lifts Milnor lattice: $K(\mathcal{D}) \cong H^2(S_t, \mathbb{Z})$

Kleinian singularities

$S = \mathbb{C}^2/G \cong \{f_{ADE} = 0\}$
$X \to S$ minimal resolution
$\mathcal{E} \subset X$ exceptional divisor (ADE configuration of -2-curves)

$D^b_{\mathcal{E}}(X)$ Hom-finite, triangulated 2-CY category
Slightly too big (Gorenstein parameter -1), reduce to

$$\mathcal{D} := D^b_{\mathcal{E}}(X) \cap \mathcal{O}_X^{\perp} = \langle \mathcal{O}_{E_1}, \dots, \mathcal{O}_{E_n} \rangle$$

Fuchsian singularities

$$\begin{array}{lll} \pi\colon & Y \to \overline{S} & \text{minimal resolution} \\ & \bigcup & \\ & \pi^{-1}(\overline{S}_\infty) =: \mathcal{E} & \text{total transform at infinity} \end{array}$$

$\mathcal{E} = Y_\infty \cup \bigcup E_i$ with $E_i^2 = -2$ and $Y_\infty^2 = 2g-2$.

Assumption: Y has negative smoothing X
(e.g. hypersurface).

Then X is a K3 surface containing $\mathcal{E}$ (Pinkham).

$\mathcal{D} := \langle D^b_{\mathcal{E}}(Y), \mathcal{O}_Y \rangle$

Extend the subcategory supported on $\mathcal{E}$ by one more spherical generator (Gorenstein parameter 1).

Bimodal singularities

$S = \{f = 0\}$ one of the 20 bimodal singularities from an invertible polynomial

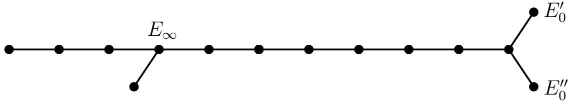
Dualisation Use Berglund-Hübsch dual f^T of f.

Compactification and deformation Compactify $S^T = \{f^T(x, y, z) = 0\}$ as a subset of a weighted projective space $\mathbb{P}(q_0, q_1, q_2, q_3)$ by adding a monomial containing w.

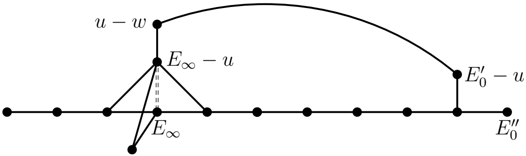
Quotient Divide by isotropy group G of weighted projective space. (G is trivial or $G = \mathbb{Z}/2$ or $G = \mathbb{Z}/3$.)

Resolution X is the minimal resolution of the quotient.

Example: $x^6y + y^3 + z^2$
Dualise: $x^6 + xy^3 + z^2$
Compactify: $x^6 + xy^3 + z^2 + w^{18}$ gives $Z \subset \mathbb{P}(1,3,5,9)$
Quotient: $Y := Z/\pm 1$ with action $(1,-1,-1,1)$
Resolution has the following configuration of -2-curves:



$$\mathcal{D} := \langle \mathcal{O}_{E_1^1}(-1), \mathcal{O}_{E_2^1}(-1), \mathcal{O}_{E_3^1}(-1), \mathcal{O}_{E_1^2}(-1), \mathsf{T}_{\mathcal{O}_{E_1^3}(-1)}(\mathcal{O}_{E_2^3}(-1)), \mathcal{O}_{E_3^3}(-1), \ldots, \mathcal{O}_{E_7^3}(-1), \mathcal{O}_{E_\infty}(-1), \mathcal{O}_{E_\infty}, \mathcal{O}_X, \mathcal{O}_{E_0} \rangle$$



Singularities:
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The singularities
Kleinian
Fuchsian
Some bimodals
Categorification
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Applications

McKay correspondence for Fuchsian singularities, $g = 0$

$$p_R(t) := \sum_{k\geq 0} \dim(R_t) t^k \text{ Poincaré series of} R = \bigoplus_{k\geq 0} R_k$$

Relation between groups (invariant rings) and geometry (Coxeter-Dynkin diagrams):

$$p_{R_{Klein}} = \frac{\Delta_-}{\Delta_0}$$

$$p_{R_{Fuchs}} = \frac{\Delta_+}{\Delta_0}$$

Δ_* is the characteristic polynomial of the Coxeter functor on

$$\mathcal{D}_- = D^b_{\mathcal{E}}(X) \cap \mathcal{O}_X^\perp$$

$$\mathcal{D}_- = D^b_{\mathcal{E}}(X)$$

$$\mathcal{D}_- = \langle D^b_{\mathcal{E}}(X), \mathcal{O}_X \rangle$$

Generalised Coxeter

S Fuchsian singularity with $g > 0$

Milnor lattice without root basis
$\Rightarrow$ Ebeling defined new Coxeter element using Eichler-Siegel transformation

Categorical: $\mathsf{T}_{\mathcal{O}_C(-1)}\mathsf{T}_{\mathcal{O}_C} = (\cdot) \otimes \mathcal{O}(C)$ line bundle twist if $C^2 = -2$.

Justification: new Coxeter element satisfies same relation for Poincaré series

Future research

Stability conditions for the category $D^b_{\mathcal{E}}(Y)$

Accessible via Bridgeland's work on K3 surfaces;
expect relation to semi-universal deformations of a Fuchsian singularity.