Branching laws for tensor modules of $\mathrm{sl}(\infty)$

Elitza Hristova

Jacobs University Bremen

Schwerpunkttagung Darstellungstheorie
March 19-22, 2012

Outline

- Introduction
- Branching laws for $\mathrm{gl}(n)$
- The Lie algebras $\mathrm{gl}(\infty)$ and $\mathrm{sl}(\infty)$
- Characterizing the embeddings
- Specifying the irreducible representations
- Statements of the branching laws

Introduction

- Finite dimensional classical Lie algebras
  - All finite dimensional representations are completely reducible
  - Huge amount of possible embeddings
- Lie algebras $\mathrm{gl}(\infty)$ and $\mathrm{sl}(\infty)$
  - Representations are not completely reducible
  - Characterization of the possible embeddings

Finite dimensional case

Gelfand-Tsetlin branching law

- Let us have the embedding $\mathrm{gl}(n-1) \to \mathrm{gl}(n)$ given by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ for any $A \in \mathrm{gl}(n-1)$.
- Let V_λ^n denote the irreducible representaiton of $\mathrm{gl}(n)$ with highest weight $\lambda = (\lambda_1, \dots \lambda_n)$, where $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0}$ for $i = 1, \dots, n-1$.

Then

$$V^n_{\lambda\,|\mathrm{gl}(n-1)} \cong \bigoplus_\sigma V_\sigma^{n-1}$$

where $\sigma = (\sigma_1, \dots, \sigma_{n-1})$ and
$\lambda_i - \sigma_i \in \mathbb{Z}_{\geq 0}$ and $\sigma_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0}$.

Finite dimensional case

Generalization of Gelfand-Tsetlin branching law

Let the embedding be $\mathrm{gl}(n) \to \mathrm{gl}(n+s)$ given by $A \mapsto \begin{pmatrix} A & \\ & 0_{s\times s} \end{pmatrix}$ for any $A \in \mathrm{gl}(n)$. Then

$$V_\lambda^{n+s}{}_{|\mathrm{gl}(n)} \cong \bigoplus_\sigma m_{\lambda,\sigma}^s V_\sigma^n$$

where $m_{\lambda,\sigma}^s$ are Gelfand-Tsetlin multiplicities.
$m_{\lambda,\sigma}^s$ is the number of sequences of weights
$\sigma^1 = (\sigma_1^1, \ldots, \sigma_{n+s-1}^1), \ldots, \sigma^{s-1} = (\sigma_1^{s-1}, \ldots, \sigma_{n+1}^{s-1})$ such that

$$\begin{array}{c} \lambda_1 \lambda_2 \ldots \lambda_{n+s-1} \lambda_{n+s} \\ \sigma_1^1 \sigma_2^1 \ldots \sigma_{n+s-1}^1 \\ \sigma_1^2 \ldots \sigma_{n+s-2}^2 \\ \ldots\ldots \\ \sigma_1 \ldots \sigma_n \end{array}$$

Finite dimensional case

Diagonal embedding $\mathrm{gl}(n) \to \mathrm{gl}(2n)$, $A \mapsto \begin{pmatrix} A & \\ & A \end{pmatrix}$

- Let λ and μ be two non-negative integer partitions with p and q parts, where $p + q \leq m$.
- Let $V^m_{\lambda,\mu}$ denote the irreducible $\mathrm{gl}(m)$-module with highest weight $(\lambda, \mu) = (\lambda_1, \dots \lambda_p, 0, \dots 0, -\mu_q, \dots, -\mu_1)$.

Then for $m = 2n$

$$V^{2n}_{\lambda,\mu\,|\mathrm{gl}(n)} \cong \bigoplus_{\substack{\alpha^+,\beta^+,\alpha^-,\beta^- \\ \lambda',\mu'}} c^{(\lambda,\mu)}_{(\alpha^+,\alpha^-),(\beta^+,\beta^-)} d^{(\lambda',\mu')}_{(\alpha^+,\alpha^-),(\beta^+,\beta^-)} V^n_{\lambda',\mu'}$$

$$c^{(\lambda,\mu)}_{(\alpha^+,\alpha^-),(\beta^+,\beta^-)} = \sum_{\gamma^+,\gamma^-,\delta} c^{\lambda}_{\gamma^+\delta} c^{\mu}_{\gamma^-\delta} c^{\gamma^+}_{\alpha^+\beta^+} c^{\gamma^-}_{\alpha^-\beta^-}$$

$$d^{(\lambda',\mu')}_{(\alpha^+,\alpha^-),(\beta^+,\beta^-)} = \sum_{\substack{\alpha_1,\alpha_2,\beta_1,\beta_2 \\ \gamma_1,\gamma_2}} c^{\alpha^+}_{\alpha_1\gamma_1} c^{\beta^-}_{\gamma_1\beta_2} c^{\alpha^-}_{\beta_1\gamma_2} c^{\beta^+}_{\gamma_2\alpha_2} c^{\lambda'}_{\alpha_2\alpha_1} c^{\mu'}_{\beta_2\beta_1}$$

Finite dimensional case

Diagonal embedding
For the module $V^{2n}_{\lambda,0}$ with highest weight $(\lambda_1, \dots, \lambda_p, 0, \dots 0)$ the decomposition is

$$V^{2n}_{\lambda,0}{}_{|\mathrm{gl}(n)} \cong \bigoplus_{\lambda',\alpha,\beta} c^{\lambda}_{\alpha,\beta} c^{\lambda'}_{\alpha,\beta} V^{n}_{\lambda',0}$$

Generalization of the diagonal embedding $\mathrm{gl}(n) \to \mathrm{gl}(kn)$.
Then

$$V^{kn}_{\lambda,0}{}_{|\mathrm{gl}(n)} \cong \bigoplus_{\lambda',\alpha_1,\dots\alpha_k} c^{\lambda}_{\alpha_1,\dots\alpha_k} c^{\lambda'}_{\alpha_1,\dots,\alpha_k} V^{n}_{\lambda',0}$$

$c^{\lambda}_{\alpha_1,\dots,\alpha_k}$ are called generalized Littlewood-Richardson coefficients.

The Lie algebras $\mathrm{gl}(\infty)$ and $\mathrm{sl}(\infty)$

Definition

- *Let V and V_* be countable dimensional vector spaces with a nondegenerate bilinear form $<,>$ between them. Then $V \otimes V_*$ is an associative algebra with respect to*

$$(u \otimes u^*)(v \otimes v^*) = < u^*, v > u \otimes v^*$$

for all $u, v \in V$, $u^*, v^* \in V_*$. Then we define

$$\mathrm{gl}(\infty) = V \otimes V_* \quad \mathrm{sl}(\infty) = [\mathrm{gl}(\infty), \mathrm{gl}(\infty)]$$

V and V_* are called the natural and the conatural representations of $gl(\infty)$ and $\mathrm{sl}(\infty)$.

- *Direct limit Lie algebras with respect to the upper left corner inclusions*

$$\mathrm{gl}(\infty) = \lim_{\rightarrow} \mathrm{gl}(n) \quad \mathrm{sl}(\infty) = \lim_{\rightarrow} \mathrm{sl}(n)$$

Classification of embeddings

Theorem (Dimitrov, Penkov)

Let $\mathfrak{g}$ be one of $\mathrm{gl}(\infty)$ and $\mathrm{sl}(\infty)$ and $\mathfrak{g}'$ be an infinite dimensional semisimple subalgebra of $\mathfrak{g}$. Let V and V_* be respectively the natural and conatural represenations of $\mathfrak{g}$, and V', and V'_* of $\mathfrak{g}'$. Then

$$\mathrm{soc}\, V \cong kV' \oplus lV'_* \oplus N_a \qquad V/\mathrm{soc}\, V \cong N_b$$
$$\mathrm{soc}\, V_* \cong lV' \oplus kV'_* \oplus N_c \qquad V_*/\mathrm{soc}\, V_* \cong N_d$$

where $k, l \in \mathbb{Z}_{>0}$ and N_a, N_b, N_c, and N_d are finite or countable dimensional trivial $\mathfrak{g}'$-modules.

A family of $\mathrm{sl}(\infty)$ modules

Definition

Let $T = T(V \oplus V_*)$, be the tensor algebra of $\mathrm{sl}(\infty)$, i.e. $T = \bigoplus_{p \geq 0, q \geq 0} V^{\otimes(p,q)}$, where $V^{\otimes(p,q)} = V^{\otimes p} \otimes V_*^{\otimes q}$. A **tensor module** *is any subquotient of $\bigoplus_{p+q \leq r} V^{\otimes(p,q)}$.*

Theorem (Penkov, Styrkas)

Any simple tensor module of $\mathrm{sl}(\infty)$ is a submodule of $V^{\otimes(p,q)}$ for some p and q. Moreover, it is a highest weight $\mathrm{sl}(\infty)$-module with highest weight $(\lambda, \mu) = (\lambda_1, \dots, \lambda_p, 0, \dots 0, -\mu_q, \dots, -\mu_1)$ where (λ, μ) is a pair of nonnegative integer partitions. We denote this module by $V_{\lambda,\mu}$.

Branching problem for $\mathrm{sl}(\infty)$

- Given an embedding $\mathfrak{g}' \subset \mathfrak{g}$ where $\mathfrak{g}' \cong \mathrm{sl}(\infty)$ and $\mathfrak{g} \cong \mathrm{sl}(\infty)$ and a simple tensor $\mathfrak{g}$-module $V_{\lambda,\mu}$, find the socle filtration of $V_{\lambda,\mu}$ over $\mathfrak{g}'$.

Branching problem for $\mathrm{sl}(\infty)$

- Given an embedding $\mathfrak{g}' \subset \mathfrak{g}$ where $\mathfrak{g}' \cong \mathrm{sl}(\infty)$ and $\mathfrak{g} \cong \mathrm{sl}(\infty)$ and a simple tensor $\mathfrak{g}$-module $V_{\lambda,\mu}$, find the socle filtration of $V_{\lambda,\mu}$ over $\mathfrak{g}'$.

Let the embedding $\mathfrak{g}' \subset \mathfrak{g}$ be of the general type. Then we construct subalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ of $\mathfrak{g}$ such that $\mathfrak{g}' \subset \mathfrak{g}_2 \subset \mathfrak{g}_1 \subset \mathfrak{g}$ and the following hold:

(1) The embedding $\mathfrak{g}_1 \subset \mathfrak{g}$ satisfies the properties:

$$\mathrm{soc}\, V \cong V_1 \oplus N_{a_1} \qquad V/\mathrm{soc}\, V \cong N_b$$
$$\mathrm{soc}\, V_* \cong V_{1*} \oplus N_{c_1} \qquad V_*/\mathrm{soc}\, V_* \cong N_d$$

where N_{a_1} and N_{c_1} are the largest submodules of N_a and N_c which pair trivially.

(2) The embedding $\mathfrak{g}_2 \subset \mathfrak{g}_1$ satisfies the properties:

$$V_1 \cong V_2 \oplus N_{a_2} \qquad V_{1*} \cong V_{2*} \oplus N_{c_2}$$

where N_{a_2} and N_{c_2} are such that $N_a = N_{a_1} \oplus N_{a_2}$ and $N_c = N_{c_1} \oplus N_{c_2}$.

Branching problem for $\mathrm{sl}(\infty)$

(3) The embedding $\mathfrak{g}' \subset \mathfrak{g}_2$ satisfies the properties:

$$V_2 \cong kV' \oplus lV'_* \qquad V_{2*} \cong lV' \oplus kV'_*$$

Branching problem for $\mathrm{sl}(\infty)$

(3) The embedding $\mathfrak{g}' \subset \mathfrak{g}_2$ satisfies the properties:

$$V_2 \cong kV' \oplus lV'_* \qquad\qquad V_{2*} \cong lV' \oplus kV'_*$$

Lemma

Let the embedding $\mathfrak{g}'' \subset \mathfrak{g}' \subset \mathfrak{g}$ be of any of the described above types. Then the socle filtration over $\mathfrak{g}''$ of any simple $\mathfrak{g}$-module $V_{\lambda,\mu}$ can be determined using the following formula

$$\overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}''} V_{\lambda,\mu} \cong \bigoplus_{m+n=r} \overline{\mathrm{soc}}^{(m+1)}_{\mathfrak{g}''} (\overline{\mathrm{soc}}^{(n+1)}_{\mathfrak{g}'} V_{\lambda,\mu})$$

Here $\overline{\mathrm{soc}}^{(r+1)} V_{\lambda,\mu} := \mathrm{soc}^{(r+1)} V_{\lambda,\mu} / \mathrm{soc}^{(r)} V_{\lambda,\mu}$.

Branching problem for $\mathrm{sl}(\infty)$

Theorem

Let the embedding $\mathfrak{g}_1 \subset \mathfrak{g}$ satisfy the following properties:

$$\mathrm{soc}\, V \cong V_1 \oplus N_{a_1} \qquad V/\mathrm{soc}\, V \cong N_b$$

$$\mathrm{soc}\, V_* \cong V_{1*} \oplus N_{c_1} \qquad V_*/\mathrm{soc}\, V_* \cong N_d$$

where N_{a_1} and N_{c_1} pair trivially. Denote $\dim N_{a_1} = a_1$, $\dim N_b = b$, $\dim N_{c_1} = c_1$, $\dim N_d = d$. *Then for any $V_{\lambda,\mu} \subset V^{\otimes(p,q)}$*

$$\overline{\mathrm{soc}}^{(r+1)} V_{\lambda,\mu} \cong \bigoplus_{m+n=r} \bigoplus_{\lambda'',\mu''} \bigoplus_{\substack{|\lambda_1|=|\lambda''|-m \\ |\mu_1|=|\mu''|-n}} m^{a_1}_{\lambda,\lambda''} m^{b}_{\lambda'',\lambda_1} m^{c_1}_{\mu,\mu''} m^{d}_{\mu'',\mu_1} V^1_{\lambda_1,\mu_1}$$

$m^a_{\lambda,\lambda''}$ are Gelfand-Tsetlin multiplicities if a is finite.
$m^{\infty}_{\lambda,\lambda''} = \lim_{a\to\infty} m^a_{\lambda,\lambda''}$.

Branching problem for $\mathrm{sl}(\infty)$

Theorem

If the embedding $\mathfrak{g}_2 \subset \mathfrak{g}_1$ satisfies the properties:

$$V_1 \cong V_2 \oplus \mathbb{C} \text{ and } V_{1*} \cong V_{2*} \oplus \mathbb{C}$$

Then for any $V^1_{\lambda_1,\mu_1} \subset V_1^{\otimes(p,q)}$

$$\overline{\mathrm{soc}}^{(r+1)} V^1_{\lambda_1,\mu_1} = \bigoplus_{\substack{|\lambda_2|=p-r \\ |\mu_2|=q-r}} m^1_{\lambda_1,\lambda_2} m^1_{\mu_1,\mu_2} V^2_{\lambda_2,\mu_2} \oplus$$

$$\bigoplus_{k=1}^{p-r} \bigoplus_{\substack{|\lambda_2|=p-r-k \\ |\mu_2|=q-r}} m^1_{\lambda_1,\lambda_2} m^1_{\mu_1,\mu_2} V^2_{\lambda_2,\mu_2} \oplus \bigoplus_{l=1}^{q-r} \bigoplus_{\substack{|\lambda_2|=p-r \\ |\mu_2|=q-r-l}} m^1_{\lambda_1,\lambda_2} m^1_{\mu_1,\mu_2} V^2_{\lambda_2,\mu_2}$$

Branching problem for $\mathrm{sl}(\infty)$

Theorem

If the embedding of $\mathfrak{g}' \subset \mathfrak{g}_1$ satisfies the property that $V_1 \cong kV' \oplus lV'_*$ and $V_{1*} \cong lV' \oplus kV'_*$, then for any simple $V^1_{\lambda_1,0} \subset V_1^{\otimes p}$ we have

$$\overline{\mathrm{soc}}^{(r+1)} V^1_{\lambda_1,0} \cong \bigoplus_{m=0}^{p} \bigoplus_{\substack{|\lambda'|=m-r \\ |\mu'|=p-m-r}} C^{\lambda_1}_{\lambda',\mu'} V'_{\lambda',\mu'}$$

where

$$C^{\lambda_1}_{\lambda',\mu'} = \bigoplus_{\substack{\alpha,\beta,\alpha',\beta',\gamma \\ \sigma_1,\dots,\sigma_k, \\ \tau_1,\dots\tau_l}} c^{\lambda_1}_{\alpha,\beta} c^{\alpha}_{\sigma_1,\dots,\sigma_k} c^{\beta}_{\tau_1,\dots,\tau_l} c^{\alpha'}_{\sigma_1,\dots,\sigma_k} c^{\beta'}_{\tau_1,\dots,\tau_l} c^{\alpha'}_{\gamma,\lambda'} c^{\beta'}_{\gamma,\mu'}$$

Branching problem for $\mathrm{sl}(\infty)$: modules $V_{\lambda,0}$

Theorem

For the general embedding $\mathfrak{g}' \subset \mathfrak{g}$ and for any $V_{\lambda,0} \subset V^{\otimes p}$ we get

$$\overline{\mathrm{soc}}^{(r+1)} V_{\lambda,0} \cong \bigoplus_{\lambda''} \bigoplus_{|\lambda_1|=|\lambda''|} \bigoplus_{m=0}^{p} \bigoplus_{\substack{|\lambda'|=m-r \\ |\mu'|=p-m-r}} m^{a}_{\lambda,\lambda''} C^{\lambda_1}_{\lambda',\mu'} V'_{\lambda',\mu'} \oplus$$

$$\bigoplus_{\lambda''} \bigoplus_{|\lambda_1|=|\lambda''|-1} \bigoplus_{m=0}^{p-1} \bigoplus_{\substack{|\lambda'|=m-r+1 \\ |\mu'|=p-m-r}} m^{a}_{\lambda,\lambda''} m^{b}_{\lambda'',\lambda_1} C^{\lambda_1}_{\lambda',\mu'} V'_{\lambda',\mu'} \oplus \cdots \oplus$$

$$\bigoplus_{\lambda''} \bigoplus_{|\lambda_1|=|\lambda''|-r} \bigoplus_{m=0}^{p-r} \bigoplus_{\substack{|\lambda'|=m \\ |\mu'|=p-r-m}} m^{a}_{\lambda,\lambda''} m^{b}_{\lambda'',\lambda_1} C^{\lambda_1}_{\lambda',\mu'} V'_{\lambda',\mu'}$$

The end

Thank you for your attention!