G-complete reducibility and semisimple modules

(on joint work with M. Bate, S. Herpel, and B. Martin)

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Setup

- G a connected reductive algebraic group over an algebraically closed field k of characteristic $p \geq 0$;
- $H \leq G$ is a closed subgroup of G; we say that H is reductive provided the identity component H° of H is reductive;
- V is a rational G-module and for simplicity assume that V is faithful, so that we may assume that $G \leq \mathrm{GL}(V)$;
- $\mathfrak{h} = \operatorname{Lie} H$ and $\mathfrak{g} = \operatorname{Lie} G$ are the corresponding Lie algebras.

In our next notion we replace a rational representation $H \to \mathrm{GL}(V)$ by a homomorphism $H \to G$ for an arbitrary reductive group G.

Serre's notion of G-complete reducibility

Definition (Serre 1998)

We say that $H \leq G$ is G-completely reducible (G-cr) provided whenever H is contained in a parabolic subgroup P of G there exists a Levi subgroup L of P so that $H \leq L$.

- If $G = \mathrm{GL}(V)$, then H is G-cr iff V is a semisimple H-module.
- Serre: If $G = \mathrm{SO}(V)$ or $\mathrm{Sp}(V)$ and $p \neq 2$, then H is G-cr iff V is a semisimple H-module.
- But this notion also makes sense for G of type E_8, say.
- Serre: if H is G-cr, then H is reductive.
- For $p = 0$, H is G-cr iff H is reductive.
- If $p > 0$, this is more subtle and depends on the embedding $H \hookrightarrow G$.

Examples

- If $H \leq G$ is linearly reductive $\quad \Rightarrow \quad H$ is G-cr;
- If $H \leq G$ is a Levi subgroup $\quad \Rightarrow \quad H$ is G-cr;
- If $H \leq G$ is maximal rank subgroup $\quad \Rightarrow \quad H$ is G-cr;
- Nagata: for $p > 0$ and semisimple H, there is a non-semisimple H-module $V \quad \Rightarrow \quad H$ is not $\mathrm{GL}(V)$-cr;
- Liebeck: let $m \geq 4$ even, $p = 2$,
  $H := \mathrm{Sp}_m(k) \hookrightarrow \mathrm{Sp}_m(k) \times \mathrm{Sp}_m(k) \hookrightarrow \mathrm{Sp}_{2m}(k) =: G$,
  then H is not G-cr
  (note also H is semisimple on k^{2m});
  if $p \neq 2$, then H is G-cr.

Richardson's Philosophy

G acts diagonally on $G^n = G \times \cdots \times G$ by simultaneous conjugation:

$$g \cdot (x_1, \ldots, x_n) = (gx_1g^{-1}, \ldots, gx_ng^{-1}),$$

for $g \in G$ and $\mathbf{x} = (x_1, \ldots, x_n) \in G^n$. Study subgroups $H \leq G$ indirectly by studying the G-orbit of a generating tuple $\mathbf{h} \in G^n$ of H.

Theorem (Richardson 1985, BMR 2005)

Let $H \leq G$ and let $\mathbf{h}$ in G^n be a generating tuple for H.
Then H is G-cr iff the G-orbit of $\mathbf{h}$ in G^n is closed.

This allows to use methods from Geometric Invariant Theory in the study of G-cr subgroups, e.g. optimal destabilizing parabolic subgroups (Kempf-Rousseau-Hesselink).
$\Rightarrow$ uniform and short proofs.

Criteria for G-complete reducibility

Let h be the Coxeter number of G.

Theorem (Serre 1998)

Suppose $p > 2h - 2$. Let $H \leq G$.

Then $\mathfrak{g}$ is a semisimple H-module iff H is G-cr.

Theorem (BMR 2005)

Let $H \leq G$ so that $\operatorname{Lie} C_G(H) = \mathfrak{c}_{\mathfrak{g}}(H)$.

If $\mathfrak{g}$ is a semisimple H-module, then H is G-cr.

Theorem (BMR-Tange 2010)

Suppose that p is very good for G.

If $\mathfrak{g}$ is a semisimple H-module, then H is G-cr.

Question: what about for an arbitrary G-module V?

Recall that $H \leq G \leq \mathrm{GL}(V)$.

Theorem (Serre 1998)

Suppose that $p > n(V)$.

Then H is G-completely reducible iff V is a semisimple H-module.

$n(V)$ is defined as follows:

let $T \leq G$ be a maximal torus of G and let λ be a T-weight of V.

Define $n(V) := \max_\lambda\{\sum_{\alpha>0}\langle\lambda, \alpha^\vee\rangle\}$.

E.g. $n(\mathfrak{g}) = 2h - 2$.

The proof of the theorem is elaborate and complicated; e.g. it depends on Serre's notion of saturation and on the full force of the next result.

G-complete reducibility and reductivity

Theorem (Serre 1998 (Jantzen, McNinch, Liebeck-Seitz))

Suppose that $p \geq a(G)$ and that $(H : H^\circ)$ is prime to p. Then H is reductive iff H is G-completely reducible.

$a(G)$ is defined as follows:
for G simple: $a(G) = \mathrm{rank}(G) + 1$;
for G reductive: $a(G) = \max(1, a(G_1), \ldots, a(G_r))$, where $G_1, \ldots, G_r$ are the simple components of G.
The theorem is a consequence of a number of deep theorems due to Jantzen (1997) and McNinch (1998) (G classical) and Liebeck-Seitz (1996) (G of exceptional type), where the latter involves complicated and long case-by-case analyses.

Goal: new concise and uniform proofs of the last two theorems, even at the cost of less favorable bounds on p.

Semisimplicity results due to Jantzen and Serre

Recall that V is a faithful rational G-module so that $G \leq \mathrm{GL}(V)$.

Theorem (Jantzen 1997)

If $p \geq \dim V$, then V is semisimple.

Theorem (Serre 1994)

Suppose $p > 2\dim V - 2$. If V is semisimple, then so is $V \otimes V^*$.

This is a very special case of Serre's Tensor Product Theorem.

In both instances, the bounds are sharp:
$p = 2,\ G = \mathrm{SL}_2(k),\ V = \mathfrak{g}; \quad p = 2,\ G = \mathrm{GL}_2(k),\ V = k^2.$

Separability

Definition (BMR 2005)

$H \leq G$ is said to be separable in G if its scheme-theoretic centralizer is smooth, i.e., if its global and infinitesimal centralizers have the same dimension: $\mathrm{Lie}\, C_G(H) = \mathfrak{c}_{\mathfrak{g}}(H)$.

If $G = \mathrm{GL}(V)$, then every subgroup H is separable.

If $p = 2$, $G = \mathrm{SL}_2$ is not separable in itself.

Theorem (BMR-Tange 2010)

Suppose that p is very good for G. Then H is separable in G.

Theorem (Herpel 2011)

Every closed subgroup scheme is separable in G iff p is pretty good for G.

Reductive pairs

Definition (Richardson 1967)

Let $H \leq G$ such that H is reductive. Then (G, H) is a reductive pair if $\mathfrak{h}$ is an H-module direct summand of $\mathfrak{g}$.

Lemma (BMR 2005)

Suppose that $(\mathrm{GL}(V), G)$ is a reductive pair. If V is a semisimple H-module, then H is G-completely reducible.

Lemma (BMRT 2010)

If $(\mathrm{GL}(V), G)$ is a reductive pair, then every $H \leq G$ is separable in G.

Reductive pairs (continued)

Lemma (Bate-Herpel-Martin-R 2011)

Suppose that $p > 2 \dim V - 2$. *Then* $(\mathrm{GL}(V), G)$ *is a reductive pair.*

Proof: $p > 2 \dim V - 2 \quad \Rightarrow \quad p \geq \dim V$.

Jantzen's Theorem $\Rightarrow \quad V$ is semisimple.

Serre's Theorem $\quad \Rightarrow \quad V \otimes V^* \cong \mathrm{Lie}(\mathrm{GL}(V))$ is also semisimple.

$\Rightarrow \quad \mathfrak{g}$ is a direct G-module summand of $\mathrm{Lie}(\mathrm{GL}(V))$.

Bound is sharp: $p = 2, G = \mathrm{SL}_2(k), V = k^2$; G not separable in itself.

Moral: "generically" V gives rise to a reductive pair $(\mathrm{GL}(V), G)$.

Herpel: p bad for G, then there always exists a non-separable $H \leq G$.

$\Rightarrow \quad (\mathrm{GL}(V), G)$ is not a reductive pair for any faithful G-module V!

Variations of Theorems of Serre

Recall that $H \leq G \leq \mathrm{GL}(V)$.

Theorem (BHMR 2011; analogue of "$n(V)$-Theorem")

(i) *Suppose that $p \geq \dim V$ and that $(H : H^\circ)$ is prime to p. If H is G-cr, then V is H-semisimple.*

(ii) *Suppose that $p > 2\dim V - 2$. If V is H-semisimple, then H is G-cr.*

Proof: (i): H is G-cr $\quad\Rightarrow\quad$ H° is reductive.
Jantzen's Theorem (applied to H°) $\quad\Rightarrow\quad$ V is H-semisimple.
(ii): $p > 2\dim V - 2 \quad\Rightarrow\quad (\mathrm{GL}(V), G)$ is a reductive pair.
V is H-semisimple $\quad\Rightarrow\quad$ H is G-cr, by lemma above.

Serre's bound $n(V)$ in general much better:
e.g. $G = E_6$, $V = L(\omega_1)$: $n(V) = 16$, while $\dim V = 27$.
However, Serre's bound $p > n(V)$ does not apply in case V admits a non-restricted composition factor.

Variations of Theorems of Serre (continued)

Theorem (BHMR 2011; analogue of "$a(G)$-Theorem")

Suppose that $p > 2\dim V - 2$ for a faithful G-module V and that $(H : H^\circ)$ is prime to p. Then H° is reductive iff H is G-cr.

Proof:

"$\Rightarrow$": $p > 2\dim V - 2 \quad \Rightarrow \quad p \geq \dim V$.

Jantzen's Theorem (applied to H°) $\quad \Rightarrow \quad$ V is H-semisimple.

Part (ii) of previous theorem $\quad \Rightarrow \quad$ H is G-cr.

"$\Leftarrow$": H is G-cr $\quad \Rightarrow \quad$ H is reductive.

Serre's bound $a(G)$ is much better again.

Generalizing Jantzen's Theorem

Theorem (BHMR 2011)

If $p \geq \dim V$ and H is G-cr, then V is a semisimple H-module.

Proof:

Jantzen's Theorem $\Rightarrow$ V is a semisimple G-module.

wlog: $V = L(\lambda)$ is simple $\lambda = \lambda_0 + p\lambda_1 + \cdots + p^r\lambda_r$, with λ_i restricted.

Set $L_i := L(p^i\lambda_i) \cong L(\lambda_i)^{[i]}$, the i-th Frobenius twist of $L(\lambda_i)$.

Then $V \cong L_0 \otimes L_1 \otimes \cdots \otimes L_r$, by Steinberg's Tensor Product Theorem.

$p \geq \dim V \;\Rightarrow\; p \geq \dim L_i = \dim L(\lambda_i) \;\Rightarrow\; p > n(L(\lambda_i))$ for each i.

Serre's Theorem $\Rightarrow$ $L(\lambda_i)$ is H-semisimple and so is L_i for each i.

$p \geq \dim V \quad\Rightarrow\quad p > \sum_i(\dim L_i - 1)$.

Serre's Tensor Product Theorem $\Rightarrow$ V is a semisimple H-module.

Recall $H \leq G \leq \mathrm{GL}(V)$.

Theorem (BHMR 2011)

Suppose H acts semisimply on $V \otimes V^*$.
Then H is G-completely reducible and H is separable in G.

Theorem is similar in spirit to Serre's Theorem, but strikingly it does not require any restriction on p.

Proof is based on a refinement of Richardson's tangent space argument.

Note: if $V \otimes V^*$ is H-semisimple, then so is V.

Thank You!