Deformations of tensor products and Schur positivity

Ghislain Fourier

Universität zu Köln

19.03.2012 / DFG 1388 - Schwerpunkttagung 2012

In this talk I want to relate two things:

- Fusion products by B. Feigin and S. Loktev - Deformed tensor products
- Schur positivity theorem by T. Lau et al

Furthermore, I want to show some implications to the theory of Weyl modules for truncated current algebras.

In this talk I want to relate two things:

- Fusion products by B. Feigin and S. Loktev - Deformed tensor products
- Schur positivity theorem by T. Lau et al

Furthermore, I want to show some implications to the theory of Weyl modules for truncated current algebras.

In this talk I want to relate two things:

- Fusion products by B. Feigin and S. Loktev - Deformed tensor products
- Schur positivity theorem by T. Lau et al

Furthermore, I want to show some implications to the theory of Weyl modules for truncated current algebras.

In this talk I want to relate two things:

- Fusion products by B. Feigin and S. Loktev - Deformed tensor products
- Schur positivity theorem by T. Lau et al

Furthermore, I want to show some implications to the theory of Weyl modules for truncated current algebras.

In this talk I want to relate two things:

- Fusion products by B. Feigin and S. Loktev - Deformed tensor products
- Schur positivity theorem by T. Lau et al

Furthermore, I want to show some implications to the theory of Weyl modules for truncated current algebras.

In this talk I want to relate two things:

- Fusion products by B. Feigin and S. Loktev - Deformed tensor products
- Schur positivity theorem by T. Lau et al

Furthermore, I want to show some implications to the theory of Weyl modules for truncated current algebras.

In this talk I want to relate two things:

- Fusion products by B. Feigin and S. Loktev - Deformed tensor products
- Schur positivity theorem by T. Lau et al

Furthermore, I want to show some implications to the theory of Weyl modules for truncated current algebras.

- Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$.
- Let $\lambda = n \geq 0$ a dominant integral weight.
- $L(n)$ the simple module of highest weight n.
- $L(n) \otimes L(m) \cong L(n+m) \oplus \ldots \oplus L(n+m-\min\{n,m\})$.

$\Rightarrow$ if $n_1 + n_2 = m_1 + m_2$ then

$$L(n_1) \otimes L(n_2) \twoheadrightarrow L(m_1) \otimes L(m_2) \Leftrightarrow \min\{n_1, n_2\} \geq \min\{m_1, m_2\}.$$

- Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$.
- Let $\lambda = n \geq 0$ a dominant integral weight.
- $L(n)$ the simple module of highest weight n.
- $L(n) \otimes L(m) \cong L(n+m) \oplus \ldots \oplus L(n+m-\min\{n,m\})$.

$\Rightarrow$ if $n_1 + n_2 = m_1 + m_2$ then

$$L(n_1) \otimes L(n_2) \twoheadrightarrow L(m_1) \otimes L(m_2) \Leftrightarrow \min\{n_1, n_2\} \geq \min\{m_1, m_2\}.$$

- Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$.
- Let $\lambda = n \geq 0$ a dominant integral weight.
- $L(n)$ the simple module of highest weight n.
- $L(n) \otimes L(m) \cong L(n+m) \oplus \ldots \oplus L(n+m-\min\{n,m\})$.

$\Rightarrow$ if $n_1 + n_2 = m_1 + m_2$ then

$$L(n_1) \otimes L(n_2) \twoheadrightarrow L(m_1) \otimes L(m_2) \Leftrightarrow \min\{n_1, n_2\} \geq \min\{m_1, m_2\}.$$

- Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$.
- Let $\lambda = n \geq 0$ a dominant integral weight.
- $L(n)$ the simple module of highest weight n.
- $L(n) \otimes L(m) \cong L(n+m) \oplus \ldots \oplus L(n+m-\min\{n,m\})$.

$\Rightarrow$ if $n_1 + n_2 = m_1 + m_2$ then

$$L(n_1) \otimes L(n_2) \twoheadrightarrow L(m_1) \otimes L(m_2) \Leftrightarrow \min\{n_1, n_2\} \geq \min\{m_1, m_2\}.$$

- Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$.
- Let $\lambda = n \geq 0$ a dominant integral weight.
- $L(n)$ the simple module of highest weight n.
- $L(n) \otimes L(m) \cong L(n+m) \oplus \ldots \oplus L(n+m-\min\{n,m\})$.

$\Rightarrow$ if $n_1 + n_2 = m_1 + m_2$ then

$$L(n_1) \otimes L(n_2) \twoheadrightarrow L(m_1) \otimes L(m_2) \Leftrightarrow \min\{n_1, n_2\} \geq \min\{m_1, m_2\}.$$

- Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$.
- Let $\lambda = n \geq 0$ a dominant integral weight.
- $L(n)$ the simple module of highest weight n.
- $L(n) \otimes L(m) \cong L(n+m) \oplus \ldots \oplus L(n+m-\min\{n,m\})$.

$\Rightarrow$ if $n_1 + n_2 = m_1 + m_2$ then

$$L(n_1) \otimes L(n_2) \twoheadrightarrow L(m_1) \otimes L(m_2) \Leftrightarrow \min\{n_1, n_2\} \geq \min\{m_1, m_2\}.$$

We define a partial order on $(P^+)^2$ by

$$(n_1, n_2) \preceq (m_1, m_2) \Leftrightarrow \min\{n_1, n_2\} \leq min\{m_1, m_2\}.$$

Generalize this:
Fix $K \in \mathbb{Z}_{\geq 0}$, for $1 \leq \ell \leq K$ we define $r_\ell : (P^+)^K \longrightarrow \mathbb{Z}$ by

$$r_\ell(n_1, \ldots, n_K) := \min_{i_1 < \ldots < i_\ell} \{n_{i_1} + \ldots + n_{i_\ell}\}.$$

We introduce a partial order on $(P^+)^K$

$$(n_1, \ldots, n_K) \preceq (m_1, \ldots, m_K)$$

$$:\Leftrightarrow r_\ell(n_1, \ldots, n_K) \leq r_\ell(m_1, \ldots, m_K) \text{ for all } 1 \leq \ell \leq K.$$

Lemma

If $(n_1, \ldots, n_K) \preceq (m_1, \ldots, m_K)$, then:

$$L(m_1) \otimes \ldots \otimes L(m_K) \to L(n_1) \otimes \ldots \otimes L(n_K).$$

We define a partial order on $(P^+)^2$ by

$$(n_1, n_2) \preceq (m_1, m_2) \Leftrightarrow \min\{n_1, n_2\} \leq min\{m_1, m_2\}.$$

Generalize this:

Fix $K \in \mathbb{Z}_{\geq 0}$, for $1 \leq \ell \leq K$ we define $r_\ell : (P^+)^K \longrightarrow \mathbb{Z}$ by

$$r_\ell(n_1, \ldots, n_K) := \min_{i_1 < \ldots < i_\ell} \{n_{i_1} + \ldots + n_{i_\ell}\}.$$

We introduce a partial order on $(P^+)^K$

$$(n_1, \ldots, n_K) \preceq (m_1, \ldots, m_K)$$

$$:\Leftrightarrow r_\ell(n_1, \ldots, n_K) \leq r_\ell(m_1, \ldots, m_K) \text{ for all } 1 \leq \ell \leq K.$$

Lemma

If $(n_1, \ldots, n_K) \preceq (m_1, \ldots, m_K)$, then:

$$L(m_1) \otimes \ldots \otimes L(m_K) \twoheadrightarrow L(n_1) \otimes \ldots \otimes L(n_K).$$

We define a partial order on $(P^+)^2$ by

$$(n_1, n_2) \preceq (m_1, m_2) \Leftrightarrow \min\{n_1, n_2\} \leq min\{m_1, m_2\}.$$

Generalize this:
Fix $K \in \mathbb{Z}_{\geq 0}$, for $1 \leq \ell \leq K$ we define $r_\ell : (P^+)^K \longrightarrow \mathbb{Z}$ by

$$r_\ell(n_1, \ldots, n_K) := \min_{i_1 < \ldots < i_\ell} \{n_{i_1} + \ldots + n_{i_\ell}\}.$$

We introduce a partial order on $(P^+)^K$

$$(n_1, \ldots, n_K) \preceq (m_1, \ldots, m_K)$$

$$:\Leftrightarrow r_\ell(n_1, \ldots, n_K) \leq r_\ell(m_1, \ldots, m_K) \text{ for all } 1 \leq \ell \leq K.$$

Lemma

If $(n_1, \ldots, n_K) \preceq (m_1, \ldots, m_K)$, then:

$$L(m_1) \otimes \ldots \otimes L(m_K) \to L(n_1) \otimes \ldots \otimes L(n_K).$$

We define a partial order on $(P^+)^2$ by

$$(n_1, n_2) \preceq (m_1, m_2) \Leftrightarrow \min\{n_1, n_2\} \leq min\{m_1, m_2\}.$$

Generalize this:
Fix $K \in \mathbb{Z}_{\geq 0}$, for $1 \leq \ell \leq K$ we define $r_\ell : (P^+)^K \longrightarrow \mathbb{Z}$ by

$$r_\ell(n_1, \ldots, n_K) := \min_{i_1 < \ldots < i_\ell} \{n_{i_1} + \ldots + n_{i_\ell}\}.$$

We introduce a partial order on $(P^+)^K$

$$(n_1, \ldots, n_K) \preceq (m_1, \ldots, m_K)$$

$$:\Leftrightarrow r_\ell(n_1, \ldots, n_K) \leq r_\ell(m_1, \ldots, m_K) \text{ for all } 1 \leq \ell \leq K.$$

Lemma

If $(n_1, \ldots, n_K) \preceq (m_1, \ldots, m_K)$, then:

$$L(m_1) \otimes \ldots \otimes L(m_K) \twoheadrightarrow L(n_1) \otimes \ldots \otimes L(n_K).$$

We could try to proof this lemma by using combinatorics, but here is a more conceptual idea:

The tensor product is not a cyclic module (at least not generated by the highest weight vectors), we want to change this.

- Denote $\mathfrak{g} \otimes \mathbb{C}[t]$, the Lie algebra of maps $\mathbb{C} \longrightarrow \mathfrak{g}$.
- Lie bracket given by $[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m}$.
- For $a \in \mathbb{C}$ denote the evaluation map

$$ev_a : \mathfrak{g} \otimes \mathbb{C}[t] \longrightarrow \mathfrak{g} \quad : \quad x \otimes p(t) \mapsto p(a)x.$$

- For a $\mathfrak{g}$-module V, denote $V_a := (ev_a)^* V$.

We could try to proof this lemma by using combinatorics, but here is a more conceptual idea:
The tensor product is not a cyclic module (at least not generated by the highest weight vectors), we want to change this.

- Denote $\mathfrak{g} \otimes \mathbb{C}[t]$, the Lie algebra of maps $\mathbb{C} \longrightarrow \mathfrak{g}$.
- Lie bracket given by $[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m}$.
- For $a \in \mathbb{C}$ denote the evaluation map

$$ev_a : \mathfrak{g} \otimes \mathbb{C}[t] \longrightarrow \mathfrak{g} \quad : \quad x \otimes p(t) \mapsto p(a)x.$$

- For a $\mathfrak{g}$-module V, denote $V_a := (ev_a)^* V$.

We could try to proof this lemma by using combinatorics, but here is a more conceptual idea:
The tensor product is not a cyclic module (at least not generated by the highest weight vectors), we want to change this.

- Denote $\mathfrak{g} \otimes \mathbb{C}[t]$, the Lie algebra of maps $\mathbb{C} \longrightarrow \mathfrak{g}$.
- Lie bracket given by $[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m}$.
- For $a \in \mathbb{C}$ denote the evaluation map

$$ev_a : \mathfrak{g} \otimes \mathbb{C}[t] \longrightarrow \mathfrak{g} \quad : \quad x \otimes p(t) \mapsto p(a)x.$$

- For a $\mathfrak{g}$-module V, denote $V_a := (ev_a)^* V$.

We could try to proof this lemma by using combinatorics, but here is a more conceptual idea:
The tensor product is not a cyclic module (at least not generated by the highest weight vectors), we want to change this.

- Denote $\mathfrak{g} \otimes \mathbb{C}[t]$, the Lie algebra of maps $\mathbb{C} \longrightarrow \mathfrak{g}$.
- Lie bracket given by $[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m}$.
- For $a \in \mathbb{C}$ denote the evaluation map

$$ev_a : \mathfrak{g} \otimes \mathbb{C}[t] \longrightarrow \mathfrak{g} \quad : \quad x \otimes p(t) \mapsto p(a)x.$$

- For a $\mathfrak{g}$-module V, denote $V_a := (ev_a)^* V$.

We could try to proof this lemma by using combinatorics, but here is a more conceptual idea:
The tensor product is not a cyclic module (at least not generated by the highest weight vectors), we want to change this.

- Denote $\mathfrak{g} \otimes \mathbb{C}[t]$, the Lie algebra of maps $\mathbb{C} \longrightarrow \mathfrak{g}$.
- Lie bracket given by $[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m}$.
- For $a \in \mathbb{C}$ denote the evaluation map

$$ev_a : \mathfrak{g} \otimes \mathbb{C}[t] \longrightarrow \mathfrak{g} \quad : \quad x \otimes p(t) \mapsto p(a)x.$$

- For a $\mathfrak{g}$-module V, denote $V_a := (ev_a)^* V$.

We could try to proof this lemma by using combinatorics, but here is a more conceptual idea:
The tensor product is not a cyclic module (at least not generated by the highest weight vectors), we want to change this.

- Denote $\mathfrak{g} \otimes \mathbb{C}[t]$, the Lie algebra of maps $\mathbb{C} \longrightarrow \mathfrak{g}$.
- Lie bracket given by $[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m}$.
- For $a \in \mathbb{C}$ denote the evaluation map

$$ev_a : \mathfrak{g} \otimes \mathbb{C}[t] \longrightarrow \mathfrak{g} \quad : \quad x \otimes p(t) \mapsto p(a)x.$$

- For a $\mathfrak{g}$-module V, denote $V_a := (ev_a)^* V$.

For $a \in \mathbb{C}$ and $n \in P^+$, $L(n)_a$ is a simple $\mathfrak{g} \otimes \mathbb{C}[t]$-module.

Moreover by construction:

$$\mathfrak{g} \otimes (t-a)\mathbb{C}[t].L(n)_a = 0.$$

More general for $a_1, \ldots, a_K \in \mathbb{C}$

$$\mathfrak{g} \otimes \prod (t-a_i)\mathbb{C}[t].L(n_1)_{a_1} \otimes \ldots \otimes L(n_K)_{a_K} = 0.$$

Hence, $L(n_1)_{a_1} \otimes \ldots \otimes L(n_K)_{a_K}$ is a simple $\mathfrak{g} \otimes \mathbb{C}[t]$-module.

Theorem (Chari-Pressley)

Any simple finite–dimensional $\mathfrak{g} \otimes \mathbb{C}[t]$-module is of this form.

For $a \in \mathbb{C}$ and $n \in P^+$, $L(n)_a$ is a simple $\mathfrak{g} \otimes \mathbb{C}[t]$-module. Moreover by construction:

$$\mathfrak{g} \otimes (t-a)\mathbb{C}[t].L(n)_a = 0.$$

More general for $a_1, \ldots, a_K \in \mathbb{C}$

$$\mathfrak{g} \otimes \prod (t-a_i)\mathbb{C}[t].L(n_1)_{a_1} \otimes \ldots \otimes L(n_K)_{a_K} = 0.$$

Hence, $L(n_1)_{a_1} \otimes \ldots \otimes L(n_K)_{a_K}$ is a simple $\mathfrak{g} \otimes \mathbb{C}[t]$-module.

Theorem (Chari-Pressley)

Any simple finite–dimensional $\mathfrak{g} \otimes \mathbb{C}[t]$-module is of this form.

For $a \in \mathbb{C}$ and $n \in P^+$, $L(n)_a$ is a simple $\mathfrak{g} \otimes \mathbb{C}[t]$-module. Moreover by construction:

$$\mathfrak{g} \otimes (t-a)\mathbb{C}[t].L(n)_a = 0.$$

More general for $a_1, \ldots, a_K \in \mathbb{C}$

$$\mathfrak{g} \otimes \prod (t-a_i)\mathbb{C}[t].L(n_1)_{a_1} \otimes \ldots \otimes L(n_K)_{a_K} = 0.$$

Hence, $L(n_1)_{a_1} \otimes \ldots \otimes L(n_K)_{a_K}$ is a simple $\mathfrak{g} \otimes \mathbb{C}[t]$-module.

Theorem (Chari-Pressley)

Any simple finite–dimensional $\mathfrak{g} \otimes \mathbb{C}[t]$-module is of this form.

For $a \in \mathbb{C}$ and $n \in P^+$, $L(n)_a$ is a simple $\mathfrak{g} \otimes \mathbb{C}[t]$-module. Moreover by construction:

$$\mathfrak{g} \otimes (t-a)\mathbb{C}[t].L(n)_a = 0.$$

More general for $a_1, \ldots, a_K \in \mathbb{C}$

$$\mathfrak{g} \otimes \prod (t-a_i)\mathbb{C}[t].L(n_1)_{a_1} \otimes \ldots \otimes L(n_K)_{a_K} = 0.$$

Hence, $L(n_1)_{a_1} \otimes \ldots \otimes L(n_K)_{a_K}$ is a simple $\mathfrak{g} \otimes \mathbb{C}[t]$-module.

Theorem (Chari-Pressley)

Any simple finite–dimensional $\mathfrak{g} \otimes \mathbb{C}[t]$-module is of this form.

We are in particular interested in cyclic modules:
$L(n_1)_{a_1} \otimes \ldots \otimes L(n_K)_{a_K}$ is cyclic generated by $v_1 \otimes \ldots \otimes v_K$.

- $\mathfrak{g} \otimes \mathbb{C}[t]$ is graded, $\mathbf{U}(\mathfrak{g} \otimes \mathbb{C}[t])$ is graded.
- Induced filtration on any cyclic module.
- Associated graded is again a $\mathfrak{g} \otimes \mathbb{C}[t]$-module.

B. Feigin-Loktev considered the associated graded module of $L(n_1)_{a_1} \otimes \ldots \otimes L(n_K)_{a_K}$. They called it *fusion product*, denoted

$$L(n_1)_{a_1} * \ldots * L(n_K)_{a_K}.$$

We are in particular interested in cyclic modules:
$L(n_1)_{a_1} \otimes \ldots \otimes L(n_K)_{a_K}$ is cyclic generated by $v_1 \otimes \ldots \otimes v_K$.

- $\mathfrak{g} \otimes \mathbb{C}[t]$ is graded, $\mathbf{U}(\mathfrak{g} \otimes \mathbb{C}[t])$ is graded.
- Induced filtration on any cyclic module.
- Associated graded is again a $\mathfrak{g} \otimes \mathbb{C}[t]$-module.

B. Feigin-Loktev considered the associated graded module of $L(n_1)_{a_1} \otimes \ldots \otimes L(n_K)_{a_K}$. They called it *fusion product*, denoted

$$L(n_1)_{a_1} * \ldots * L(n_K)_{a_K}.$$

We are in particular interested in cyclic modules:
$L(n_1)_{a_1} \otimes \ldots \otimes L(n_K)_{a_K}$ is cyclic generated by $v_1 \otimes \ldots \otimes v_K$.

- $\mathfrak{g} \otimes \mathbb{C}[t]$ is graded, $\mathbf{U}(\mathfrak{g} \otimes \mathbb{C}[t])$ is graded.
- Induced filtration on any cyclic module.
- Associated graded is again a $\mathfrak{g} \otimes \mathbb{C}[t]$-module.

B. Feigin-Loktev considered the associated graded module of $L(n_1)_{a_1} \otimes \ldots \otimes L(n_K)_{a_K}$. They called it *fusion product*, denoted

$$L(n_1)_{a_1} * \ldots * L(n_K)_{a_K}.$$

We are in particular interested in cyclic modules:
$L(n_1)_{a_1} \otimes \ldots \otimes L(n_K)_{a_K}$ is cyclic generated by $v_1 \otimes \ldots \otimes v_K$.

- $\mathfrak{g} \otimes \mathbb{C}[t]$ is graded, $\mathbf{U}(\mathfrak{g} \otimes \mathbb{C}[t])$ is graded.
- Induced filtration on any cyclic module.
- Associated graded is again a $\mathfrak{g} \otimes \mathbb{C}[t]$-module.

B. Feigin-Loktev considered the associated graded module of $L(n_1)_{a_1} \otimes \ldots \otimes L(n_K)_{a_K}$. They called it *fusion product*, denoted

$$L(n_1)_{a_1} * \ldots * L(n_K)_{a_K}.$$

We are in particular interested in cyclic modules:
$L(n_1)_{a_1} \otimes \ldots \otimes L(n_K)_{a_K}$ is cyclic generated by $v_1 \otimes \ldots \otimes v_K$.

- $\mathfrak{g} \otimes \mathbb{C}[t]$ is graded, $\mathbf{U}(\mathfrak{g} \otimes \mathbb{C}[t])$ is graded.
- Induced filtration on any cyclic module.
- Associated graded is again a $\mathfrak{g} \otimes \mathbb{C}[t]$-module.

B. Feigin-Loktev considered the associated graded module of $L(n_1)_{a_1} \otimes \ldots \otimes L(n_K)_{a_K}$. They called it *fusion product*, denoted

$$L(n_1)_{a_1} * \ldots * L(n_K)_{a_K}.$$

We are in particular interested in cyclic modules: $L(n_1)_{a_1} \otimes \ldots \otimes L(n_K)_{a_K}$ is cyclic generated by $v_1 \otimes \ldots \otimes v_K$.

- $\mathfrak{g} \otimes \mathbb{C}[t]$ is graded, $\mathbf{U}(\mathfrak{g} \otimes \mathbb{C}[t])$ is graded.
- Induced filtration on any cyclic module.
- Associated graded is again a $\mathfrak{g} \otimes \mathbb{C}[t]$-module.

B. Feigin-Loktev considered the associated graded module of $L(n_1)_{a_1} \otimes \ldots \otimes L(n_K)_{a_K}$. They called it *fusion product*, denoted

$$L(n_1)_{a_1} * \ldots * L(n_K)_{a_K}.$$

We are in particular interested in cyclic modules:
$L(n_1)_{a_1} \otimes \ldots \otimes L(n_K)_{a_K}$ is cyclic generated by $v_1 \otimes \ldots \otimes v_K$.

- $\mathfrak{g} \otimes \mathbb{C}[t]$ is graded, $\mathbf{U}(\mathfrak{g} \otimes \mathbb{C}[t])$ is graded.
- Induced filtration on any cyclic module.
- Associated graded is again a $\mathfrak{g} \otimes \mathbb{C}[t]$-module.

B. Feigin-Loktev considered the associated graded module of $L(n_1)_{a_1} \otimes \ldots \otimes L(n_K)_{a_K}$. They called it *fusion product*, denoted

$$L(n_1)_{a_1} * \ldots * L(n_K)_{a_K}.$$

$L(n_1)_{a_1} * \ldots * L(n_K)_{a_K}$ is a cyclic $\mathfrak{g} \otimes \mathbb{C}[t]$-module, hence there exists an ideal J, such that

$$L(n_1)_{a_1} * \ldots * L(n_K)_{a_K} \cong \mathbf{U}(\mathfrak{g} \otimes \mathbb{C}[t])/J.$$

Theorem (E. Feigin)

J is generated by

$$\mathfrak{n}^+ \otimes \mathbb{C}[t] \; ; \; h \otimes t^n - 0^n(n_1 + \ldots + n_K) \; ; \; (f \otimes 1)^{n_1 + \ldots + n_K + 1}$$

and

$$(f \otimes t^{K+1}) \; ; \; (f \otimes t^\ell)^{r_{K-\ell}(n_1,\ldots,n_k)} \text{ for } 1 \leq \ell \leq K$$

This implies the lemma.

$L(n_1)_{a_1} * \ldots * L(n_K)_{a_K}$ is a cyclic $\mathfrak{g} \otimes \mathbb{C}[t]$-module, hence there exists an ideal J, such that

$$L(n_1)_{a_1} * \ldots * L(n_K)_{a_K} \cong \mathbf{U}(\mathfrak{g} \otimes \mathbb{C}[t])/J.$$

Theorem (E. Feigin)

J is generated by

$$\mathfrak{n}^+ \otimes \mathbb{C}[t] \; ; \; h \otimes t^n - 0^n(n_1 + \ldots + n_K) \; ; \; (f \otimes 1)^{n_1 + \ldots + n_K + 1}$$

and

$$(f \otimes t^{K+1}) \; ; \; (f \otimes t^\ell)^{r_{K-\ell}(n_1, \ldots, n_k)} \text{ for } 1 \leq \ell \leq K$$

This implies the lemma.

$L(n_1)_{a_1} * \ldots * L(n_K)_{a_K}$ is a cyclic $\mathfrak{g} \otimes \mathbb{C}[t]$-module, hence there exists an ideal J, such that

$$L(n_1)_{a_1} * \ldots * L(n_K)_{a_K} \cong \mathbf{U}(\mathfrak{g} \otimes \mathbb{C}[t])/J.$$

Theorem (E. Feigin)

J is generated by

$$\mathfrak{n}^+ \otimes \mathbb{C}[t] \; ; \; h \otimes t^n - 0^n(n_1 + \ldots + n_K) \; ; \; (f \otimes 1)^{n_1 + \ldots + n_K + 1}$$

and

$$(f \otimes t^{K+1}) \; ; \; (f \otimes t^\ell)^{r_{K-\ell}(n_1, \ldots, n_k)} \text{ for } 1 \leq \ell \leq K$$

This implies the lemma.

$L(n_1)_{a_1} * \ldots * L(n_K)_{a_K}$ is a cyclic $\mathfrak{g} \otimes \mathbb{C}[t]$-module, hence there exists an ideal J, such that

$$L(n_1)_{a_1} * \ldots * L(n_K)_{a_K} \cong \mathbf{U}(\mathfrak{g} \otimes \mathbb{C}[t])/J.$$

Theorem (E. Feigin)

J is generated by

$$\mathfrak{n}^+ \otimes \mathbb{C}[t]\ ;\ h \otimes t^n - 0^n(n_1 + \ldots + n_K)\ ;\ (f \otimes 1)^{n_1 + \ldots + n_K + 1}$$

and

$$(f \otimes t^{K+1})\ ;\ (f \otimes t^\ell)^{r_{K-\ell}(n_1, \ldots, n_k)} \text{ for } 1 \leq \ell \leq K$$

This implies the lemma.

Try to do this more general:

- Let $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$.
- Denote P^+ the dominant integral weights.
- $L(\lambda)$ the simple finite–dimensional module.

We want to introduce a partial order on $(P^+)^K$, generalizing the $\mathfrak{sl}_2$ case.

Try to do this more general:

- Let $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$.
- Denote P^+ the dominant integral weights.
- $L(\lambda)$ the simple finite–dimensional module.

We want to introduce a partial order on $(P^+)^K$, generalizing the $\mathfrak{sl}_2$ case.

Try to do this more general:

- Let $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$.
- Denote P^+ the dominant integral weights.
- $L(\lambda)$ the simple finite–dimensional module.

We want to introduce a partial order on $(P^+)^K$, generalizing the $\mathfrak{sl}_2$ case.

Try to do this more general:

- Let $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$.
- Denote P^+ the dominant integral weights.
- $L(\lambda)$ the simple finite–dimensional module.

We want to introduce a partial order on $(P^+)^K$, generalizing the $\mathfrak{sl}_2$ case.

Try to do this more general:

- Let $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$.
- Denote P^+ the dominant integral weights.
- $L(\lambda)$ the simple finite–dimensional module.

We want to introduce a partial order on $(P^+)^K$, generalizing the $\mathfrak{sl}_2$ case.

Try to do this more general:

- Let $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$.
- Denote P^+ the dominant integral weights.
- $L(\lambda)$ the simple finite–dimensional module.

We want to introduce a partial order on $(P^+)^K$, generalizing the $\mathfrak{sl}_2$ case.

In fact, there is only one natural choice:

Set for all $\alpha > 0$, $1 \leq \ell \leq K$:

$$r_{\alpha,\ell}(\lambda_1, \ldots, \lambda_K) := \min_{i_1 < \ldots < i_\ell} \{(\lambda_{i_1} + \ldots + \lambda_{i_\ell})(h_\alpha)\}.$$

And define

$$(\lambda_1, \ldots, \lambda_K) \preceq (\mu_1, \ldots, \mu_K)$$

$$:\Leftrightarrow r_{\alpha,\ell}(\lambda_1, \ldots, \lambda_K) \leq r_{\alpha,\ell}(\mu_1, \ldots, \mu_K) \text{ for all } \alpha, \ell.$$

Conjecture

If $(\lambda_1, \ldots, \lambda_K) \preceq (\mu_1, \ldots, \mu_K)$, then

$$L(\mu_1) \otimes \ldots \otimes L(\mu_K) \to L(\lambda_1) \otimes \ldots \otimes L(\lambda_K).$$

In fact, there is only one natural choice:
Set for all $\alpha > 0$, $1 \leq \ell \leq K$:

$$r_{\alpha,\ell}(\lambda_1, \ldots, \lambda_K) := \min_{i_1 < \ldots < i_\ell} \{(\lambda_{i_1} + \ldots + \lambda_{i_\ell})(h_\alpha)\}.$$

And define

$$(\lambda_1, \ldots, \lambda_K) \preceq (\mu_1, \ldots, \mu_K)$$

$$:\Leftrightarrow r_{\alpha,\ell}(\lambda_1, \ldots, \lambda_K) \leq r_{\alpha,\ell}(\mu_1, \ldots, \mu_K) \text{ for all } \alpha, \ell.$$

Conjecture

If $(\lambda_1, \ldots, \lambda_K) \preceq (\mu_1, \ldots, \mu_K)$, then

$$L(\mu_1) \otimes \ldots \otimes L(\mu_K) \to L(\lambda_1) \otimes \ldots \otimes L(\lambda_K).$$

In fact, there is only one natural choice:
Set for all $\alpha > 0$, $1 \leq \ell \leq K$:

$$r_{\alpha,\ell}(\lambda_1, \ldots, \lambda_K) := \min_{i_1 < \ldots < i_\ell} \{(\lambda_{i_1} + \ldots + \lambda_{i_\ell})(h_\alpha)\}.$$

And define

$$(\lambda_1, \ldots, \lambda_K) \preceq (\mu_1, \ldots, \mu_K)$$

$$:\Leftrightarrow r_{\alpha,\ell}(\lambda_1, \ldots, \lambda_K) \leq r_{\alpha,\ell}(\mu_1, \ldots, \mu_K) \text{ for all } \alpha, \ell.$$

Conjecture

If $(\lambda_1, \ldots, \lambda_K) \preceq (\mu_1, \ldots, \mu_K)$, *then*

$$L(\mu_1) \otimes \ldots \otimes L(\mu_K) \twoheadrightarrow L(\lambda_1) \otimes \ldots \otimes L(\lambda_K).$$

As in the $\mathfrak{sl}_2(\mathbb{C})$ case, we have a relation to fusion products by considering the associated graded module of

$$L(\lambda_1)_{a_1} \otimes \ldots \otimes L(\lambda_K)_{a_K}$$

(which is by the way again a simple module due to Chari-Pressley).

Proposition

The fusion product is a quotient of $\mathbf{U}(\mathfrak{g} \otimes \mathbb{C}[t])/J$, where J is generated by

$$\mathfrak{n}^+ \otimes \mathbb{C}[t] \; ; \; h \otimes t^n - 0^n(\sum \lambda_i)(h) \; ; \; (x_{-\alpha} \otimes 1)^{\sum \lambda_i(h_\alpha)+1}$$

and

$$(x_{-\alpha} \otimes t^{K+1}) \; ; \; (x_{-\alpha} \otimes t^\ell)^{r_{\alpha,K-\ell}(n_1,\ldots,n_K)} \text{ for } 1 \leq \ell \leq K$$

As in the $\mathfrak{sl}_2(\mathbb{C})$ case, we have a relation to fusion products by considering the associated graded module of

$$L(\lambda_1)_{a_1} \otimes \ldots \otimes L(\lambda_K)_{a_K}$$

(which is by the way again a simple module due to Chari-Pressley).

Proposition

The fusion product is a quotient of $\mathbf{U}(\mathfrak{g} \otimes \mathbb{C}[t])/J$*, where* J *is generated by*

$$\mathfrak{n}^+ \otimes \mathbb{C}[t] \; ; \; h \otimes t^n - 0^n(\sum \lambda_i)(h) \; ; \; (x_{-\alpha} \otimes 1)^{\sum \lambda_i(h_\alpha)+1}$$

and

$$(x_{-\alpha} \otimes t^{K+1}) \; ; \; (x_{-\alpha} \otimes t^{\ell})^{r_{\alpha,K-\ell}(n_1,\ldots,n_k)} \text{ for } 1 \leq \ell \leq K$$

Proving that this is actually an isomorphism would imply the conjecture above, but this is done only in a few cases:

- If $\lambda_i = m_i\omega_{j_i}$ for all $1 \leq i \leq K$.
- For affine Demazure modules of a fixed level.
- For $K > |\sum \lambda_i|$.

The last point is in fact coming from the theory and results about Weyl modules.

Proving that this is actually an isomorphism would imply the conjecture above, but this is done only in a few cases:

- If $\lambda_i = m_i \omega_{j_i}$ for all $1 \leq i \leq K$.
- For affine Demazure modules of a fixed level.
- For $K > |\sum \lambda_i|$.

The last point is in fact coming from the theory and results about Weyl modules.

Proving that this is actually an isomorphism would imply the conjecture above, but this is done only in a few cases:

- If $\lambda_i = m_i\omega_{j_i}$ for all $1 \leq i \leq K$.
- For affine Demazure modules of a fixed level.
- For $K > |\sum \lambda_i|$.

The last point is in fact coming from the theory and results about Weyl modules.

Proving that this is actually an isomorphism would imply the conjecture above, but this is done only in a few cases:

- If $\lambda_i = m_i \omega_{j_i}$ for all $1 \leq i \leq K$.
- For affine Demazure modules of a fixed level.
- For $K > |\sum \lambda_i|$.

The last point is in fact coming from the theory and results about Weyl modules.

Proving that this is actually an isomorphism would imply the conjecture above, but this is done only in a few cases:

- If $\lambda_i = m_i \omega_{j_i}$ for all $1 \leq i \leq K$.
- For affine Demazure modules of a fixed level.
- For $K > |\sum \lambda_i|$.

The last point is in fact coming from the theory and results about Weyl modules.

- The category of finite–dimensional $\mathfrak{g} \otimes \mathbb{C}[t]$-modules is not–semisimple.
- To every simple module one can associate a "largest" module with this given simple quotient.
- Similar to modular representation theory: *Weyl module*.
- $L(\lambda)_0$ is simple (and graded), the corresponding Weyl module is denoted $W(\lambda)_0$.
- If $\sum \lambda_i = \lambda$, then

$$W(\lambda)_0 \twoheadrightarrow L(\lambda_1)_{a_1} * \ldots * L(\lambda_K)_{a_K}.$$

Theorem (Chari-Pressley, F-Littelmann)

If $\lambda = \sum m_i \omega_i$, then

$$W(\lambda_0) \cong L(\omega_1)_{a_{1,1}} * \ldots * L(\omega_1)_{a_{1,m_1}} * \ldots * L(\omega_n)_{a_{n,m_n}}.$$

- The category of finite–dimensional $\mathfrak{g} \otimes \mathbb{C}[t]$-modules is not–semisimple.
- To every simple module one can associate a ”largest” module with this given simple quotient.
- Similar to modular representation theory: *Weyl module*.
- $L(\lambda)_0$ is simple (and graded), the corresponding Weyl module is denoted $W(\lambda)_0$.
- If $\sum \lambda_i = \lambda$, then

$$W(\lambda)_0 \twoheadrightarrow L(\lambda_1)_{a_1} * \ldots * L(\lambda_K)_{a_K}.$$

Theorem (Chari-Pressley, F-Littelmann)

If $\lambda = \sum m_i \omega_i$, then

$$W(\lambda_0) \cong L(\omega_1)_{a_{1,1}} * \ldots * L(\omega_1)_{a_{1,m_1}} * \ldots * L(\omega_n)_{a_{n,m_n}}.$$

- The category of finite–dimensional $\mathfrak{g} \otimes \mathbb{C}[t]$-modules is not–semisimple.
- To every simple module one can associate a ”largest” module with this given simple quotient.
- Similar to modular representation theory: *Weyl module*.
- $L(\lambda)_0$ is simple (and graded), the corresponding Weyl module is denoted $W(\lambda)_0$.
- If $\sum \lambda_i = \lambda$, then

$$W(\lambda)_0 \twoheadrightarrow L(\lambda_1)_{a_1} * \ldots * L(\lambda_K)_{a_K}.$$

Theorem (Chari-Pressley, F-Littelmann)

If $\lambda = \sum m_i \omega_i$, then

$$W(\lambda_0) \cong L(\omega_1)_{a_{1,1}} * \ldots * L(\omega_1)_{a_{1,m_1}} * \ldots * L(\omega_n)_{a_{n,m_n}}.$$

- The category of finite–dimensional $\mathfrak{g} \otimes \mathbb{C}[t]$-modules is not–semisimple.
- To every simple module one can associate a ”largest” module with this given simple quotient.
- Similar to modular representation theory: *Weyl module*.
- $L(\lambda)_0$ is simple (and graded), the corresponding Weyl module is denoted $W(\lambda)_0$.
- If $\sum \lambda_i = \lambda$, then

$$W(\lambda)_0 \twoheadrightarrow L(\lambda_1)_{a_1} * \ldots * L(\lambda_K)_{a_K}.$$

Theorem (Chari-Pressley, F-Littelmann)

If $\lambda = \sum m_i \omega_i$, then

$$W(\lambda_0) \cong L(\omega_1)_{a_{1,1}} * \ldots * L(\omega_1)_{a_{1,m_1}} * \ldots * L(\omega_n)_{a_{n,m_n}}.$$

- The category of finite–dimensional $\mathfrak{g} \otimes \mathbb{C}[t]$-modules is not–semisimple.
- To every simple module one can associate a "largest" module with this given simple quotient.
- Similar to modular representation theory: *Weyl module*.
- $L(\lambda)_0$ is simple (and graded), the corresponding Weyl module is denoted $W(\lambda)_0$.
- If $\sum \lambda_i = \lambda$, then

$$W(\lambda)_0 \twoheadrightarrow L(\lambda_1)_{a_1} * \ldots * L(\lambda_K)_{a_K}.$$

Theorem (Chari-Pressley, F-Littelmann)

If $\lambda = \sum m_i \omega_i$, then

$$W(\lambda_0) \cong L(\omega_1)_{a_{1,1}} * \ldots * L(\omega_1)_{a_{1,m_1}} * \ldots * L(\omega_n)_{a_{n,m_n}}.$$

- The category of finite–dimensional $\mathfrak{g} \otimes \mathbb{C}[t]$-modules is not–semisimple.
- To every simple module one can associate a ”largest” module with this given simple quotient.
- Similar to modular representation theory: *Weyl module*.
- $L(\lambda)_0$ is simple (and graded), the corresponding Weyl module is denoted $W(\lambda)_0$.
- If $\sum \lambda_i = \lambda$, then

$$W(\lambda)_0 \twoheadrightarrow L(\lambda_1)_{a_1} * \ldots * L(\lambda_K)_{a_K}.$$

Theorem (Chari-Pressley, F-Littelmann)

If $\lambda = \sum m_i \omega_i$, then

$$W(\lambda_0) \cong L(\omega_1)_{a_{1,1}} * \ldots * L(\omega_1)_{a_{1,m_1}} * \ldots * L(\omega_n)_{a_{n,m_n}}.$$

- The category of finite–dimensional $\mathfrak{g} \otimes \mathbb{C}[t]$-modules is not–semisimple.
- To every simple module one can associate a ”largest” module with this given simple quotient.
- Similar to modular representation theory: *Weyl module*.
- $L(\lambda)_0$ is simple (and graded), the corresponding Weyl module is denoted $W(\lambda)_0$.
- If $\sum \lambda_i = \lambda$, then

$$W(\lambda)_0 \twoheadrightarrow L(\lambda_1)_{a_1} * \ldots * L(\lambda_K)_{a_K}.$$

Theorem (Chari-Pressley, F-Littelmann)

If $\lambda = \sum m_i \omega_i$, *then*

$$W(\lambda_0) \cong L(\omega_1)_{a_{1,1}} * \ldots * L(\omega_1)_{a_{1,m_1}} * \ldots * L(\omega_n)_{a_{n,m_n}}.$$

- More general, we could replace $\mathbb{C}[t]$ by any commutative algebra A.
- The simple modules are again tensor products of evaluation modules (Chari-F-Khandai).
- Define Weyl modules by homological properties.
- Try to compute them.
- The decomposition as $\mathfrak{g}$-modules is known in regular points (of some symmetric group action).
- In singular points hard to compute, even in the $\mathfrak{sl}_2$-case.
- Focus on finite–dimensional algebras...

- More general, we could replace $\mathbb{C}[t]$ by any commutative algebra A.
- The simple modules are again tensor products of evaluation modules (Chari-F-Khandai).
- Define Weyl modules by homological properties.
- Try to compute them.
- The decomposition as $\mathfrak{g}$-modules is known in regular points (of some symmetric group action).
- In singular points hard to compute, even in the $\mathfrak{sl}_2$-case.
- Focus on finite–dimensional algebras...

- More general, we could replace $\mathbb{C}[t]$ by any commutative algebra A.
- The simple modules are again tensor products of evaluation modules (Chari-F-Khandai).
- Define Weyl modules by homological properties.
- Try to compute them.
- The decomposition as $\mathfrak{g}$-modules is known in regular points (of some symmetric group action).
- In singular points hard to compute, even in the $\mathfrak{sl}_2$-case.
- Focus on finite–dimensional algebras...

- More general, we could replace $\mathbb{C}[t]$ by any commutative algebra A.
- The simple modules are again tensor products of evaluation modules (Chari-F-Khandai).
- Define Weyl modules by homological properties.
- Try to compute them.
- The decomposition as $\mathfrak{g}$-modules is known in regular points (of some symmetric group action).
- In singular points hard to compute, even in the $\mathfrak{sl}_2$-case.
- Focus on finite–dimensional algebras...

- More general, we could replace $\mathbb{C}[t]$ by any commutative algebra *A*.
- The simple modules are again tensor products of evaluation modules (Chari-F-Khandai).
- Define Weyl modules by homological properties.
- Try to compute them.
- The decomposition as $\mathfrak{g}$-modules is known in regular points (of some symmetric group action).
- In singular points hard to compute, even in the $\mathfrak{sl}_2$-case.
- Focus on finite–dimensional algebras...

- More general, we could replace $\mathbb{C}[t]$ by any commutative algebra A.
- The simple modules are again tensor products of evaluation modules (Chari-F-Khandai).
- Define Weyl modules by homological properties.
- Try to compute them.
- The decomposition as $\mathfrak{g}$-modules is known in regular points (of some symmetric group action).
- In singular points hard to compute, even in the $\mathfrak{sl}_2$-case.
- Focus on finite–dimensional algebras...

- More general, we could replace $\mathbb{C}[t]$ by any commutative algebra A.
- The simple modules are again tensor products of evaluation modules (Chari-F-Khandai).
- Define Weyl modules by homological properties.
- Try to compute them.
- The decomposition as $\mathfrak{g}$-modules is known in regular points (of some symmetric group action).
- In singular points hard to compute, even in the $\mathfrak{sl}_2$-case.
- Focus on finite–dimensional algebras...

- If $A \cong \mathbb{C}[t]/\prod(t-a_i)\mathbb{C}[t]$ for some $a_i \neq a_j$
- $\Rightarrow \mathfrak{g} \otimes A$ is semi–simple.
- $\Rightarrow$ Weyl modules are simple modules.
- If $A = \mathbb{C}[t]/t^K$
- $\Rightarrow$ Unique simple module of highest weight λ: $L(\lambda)_0$.
- What about the corresponding Weyl module $W(\lambda, K)$?

Conjecture

$$W(\lambda, K) \cong L(\lambda^{\max 1})_{a_1} * \ldots * L(\lambda^{\max K})_{a_K}$$

for certain $\lambda^{\max j}$.

- If $A \cong \mathbb{C}[t]/\prod(t-a_i)\mathbb{C}[t]$ for some $a_i \neq a_j$
- $\Rightarrow \mathfrak{g} \otimes A$ is semi–simple.
- $\Rightarrow$ Weyl modules are simple modules.
- If $A = \mathbb{C}[t]/t^K$
- $\Rightarrow$ Unique simple module of highest weight λ: $L(\lambda)_0$.
- What about the corresponding Weyl module $W(\lambda, K)$?

Conjecture

$$W(\lambda, K) \cong L(\lambda^{\max 1})_{a_1} * \ldots * L(\lambda^{\max K})_{a_K}$$

for certain $\lambda^{\max j}$.

- If $A \cong \mathbb{C}[t]/\prod(t-a_i)\mathbb{C}[t]$ for some $a_i \neq a_j$
- $\Rightarrow \mathfrak{g} \otimes A$ is semi–simple.
- $\Rightarrow$ Weyl modules are simple modules.
- If $A = \mathbb{C}[t]/t^K$
- $\Rightarrow$ Unique simple module of highest weight λ: $L(\lambda)_0$.
- What about the corresponding Weyl module $W(\lambda, K)$?

Conjecture

$$W(\lambda, K) \cong L(\lambda^{\max 1})_{a_1} * \ldots * L(\lambda^{\max K})_{a_K}$$

for certain $\lambda^{\max j}$.

- If $A \cong \mathbb{C}[t]/\prod(t-a_i)\mathbb{C}[t]$ for some $a_i \neq a_j$
- $\Rightarrow \mathfrak{g} \otimes A$ is semi–simple.
- $\Rightarrow$ Weyl modules are simple modules.
- If $A = \mathbb{C}[t]/t^K$
- $\Rightarrow$ Unique simple module of highest weight λ: $L(\lambda)_0$.
- What about the corresponding Weyl module $W(\lambda, K)$?

Conjecture

$$W(\lambda, K) \cong L(\lambda^{\max 1})_{a_1} * \ldots * L(\lambda^{\max K})_{a_K}$$

for certain $\lambda^{\max j}$.

- If $A \cong \mathbb{C}[t]/\prod(t-a_i)\mathbb{C}[t]$ for some $a_i \neq a_j$
- $\Rightarrow \mathfrak{g} \otimes A$ is semi–simple.
- $\Rightarrow$ Weyl modules are simple modules.
- If $A = \mathbb{C}[t]/t^K$
- $\Rightarrow$ Unique simple module of highest weight λ: $L(\lambda)_0$.
- What about the corresponding Weyl module $W(\lambda, K)$?

Conjecture

$$W(\lambda, K) \cong L(\lambda^{\max 1})_{a_1} * \ldots * L(\lambda^{\max K})_{a_K}$$

for certain $\lambda^{\max j}$.

- If $A \cong \mathbb{C}[t]/\prod(t-a_i)\mathbb{C}[t]$ for some $a_i \neq a_j$
- $\Rightarrow \mathfrak{g} \otimes A$ is semi–simple.
- $\Rightarrow$ Weyl modules are simple modules.
- If $A = \mathbb{C}[t]/t^K$
- $\Rightarrow$ Unique simple module of highest weight λ: $L(\lambda)_0$.
- What about the corresponding Weyl module $W(\lambda, K)$?

Conjecture

$$W(\lambda, K) \cong L(\lambda^{\max 1})_{a_1} * \ldots * L(\lambda^{\max K})_{a_K}$$

for certain $\lambda^{\max j}$.

- If $A \cong \mathbb{C}[t]/\prod(t-a_i)\mathbb{C}[t]$ for some $a_i \neq a_j$
- $\Rightarrow \mathfrak{g} \otimes A$ is semi–simple.
- $\Rightarrow$ Weyl modules are simple modules.
- If $A = \mathbb{C}[t]/t^K$
- $\Rightarrow$ Unique simple module of highest weight λ: $L(\lambda)_0$.
- What about the corresponding Weyl module $W(\lambda, K)$?

Conjecture

$$W(\lambda, K) \cong L(\lambda^{\max 1})_{a_1} * \ldots * L(\lambda^{\max K})_{a_K}$$

for certain $\lambda^{\max j}$.

What are the $\lambda^{\max j}$?

Proposition

For $\lambda \in P^+$ denote

$$(P^+)^K_\lambda = \{(\lambda_1, \ldots, \lambda_K) \mid \sum \lambda_i = \lambda\}.$$

There is a maximal element $(\lambda^{\max 1}, \ldots, \lambda^{\max K})$ in the poset $(P^+)^K_\lambda$ satisfying

$$\dim L(\lambda^{\max 1}) \otimes \ldots \otimes L(\lambda^{\max K}) \geq \dim L(\mu_1) \otimes \ldots \otimes L(\mu_K)$$

for any $(\mu_1, \ldots, \mu_K) \in (P^+)^K_\lambda$.

There is an explicit formula for this tuple.

What are the $\lambda^{\max j}$?

Proposition

For $\lambda \in P^+$ denote

$$(P^+)^K_\lambda = \{(\lambda_1, \ldots, \lambda_K) \mid \sum \lambda_i = \lambda\}.$$

There is a maximal element $(\lambda^{\max 1}, \ldots, \lambda^{\max K})$ in the poset $(P^+)^K_\lambda$ satisfying

$$\dim L(\lambda^{\max 1}) \otimes \ldots \otimes L(\lambda^{\max K}) \geq \dim L(\mu_1) \otimes \ldots \otimes L(\mu_k)$$

for any $(\mu_1, \ldots, \mu_K) \in (P^+)^K_\lambda$.

There is an explicit formula for this tuple.

What are the $\lambda^{\max j}$?

Proposition

For $\lambda \in P^+$ denote

$$(P^+)^K_\lambda = \{(\lambda_1, \dots, \lambda_K) \mid \sum \lambda_i = \lambda\}.$$

There is a maximal element $(\lambda^{\max 1}, \dots, \lambda^{\max K})$ in the poset $(P^+)^K_\lambda$ satisfying

$$\dim L(\lambda^{\max 1}) \otimes \dots \otimes L(\lambda^{\max K}) \geq \dim L(\mu_1) \otimes \dots \otimes L(\mu_k)$$

for any $(\mu_1, \dots, \mu_K) \in (P^+)^K_\lambda$.

There is an explicit formula for this tuple.

What are the $\lambda^{\max j}$?

Proposition

For $\lambda \in P^+$ denote

$$(P^+)^K_\lambda = \{(\lambda_1, \ldots, \lambda_K) \mid \sum \lambda_i = \lambda\}.$$

There is a maximal element $(\lambda^{\max 1}, \ldots, \lambda^{\max K})$ in the poset $(P^+)^K_\lambda$ satisfying

$$\dim L(\lambda^{\max 1}) \otimes \ldots \otimes L(\lambda^{\max K}) \geq \dim L(\mu_1) \otimes \ldots \otimes L(\mu_k)$$

for any $(\mu_1, \ldots, \mu_K) \in (P^+)^K_\lambda$.

There is an explicit formula for this tuple.

S. Fomin et al, Okounkov, T. Lascoux et al have also obtained this tuple by the transpose of a row shuffle of a tuple of partitions.
They conjectured that there exists a surjective map

$$L(\lambda^{\max 1}) \otimes \ldots \otimes L(\lambda^{\max K}) \twoheadrightarrow L(\mu_1) \otimes \ldots \otimes L(\mu_k)$$

for any tuple $(\mu_1, \ldots, \mu_K) \in (P^+)^K_\lambda$.
This is now a theorem by T. Lam et al.

- This gives some evidence about the fusion product decomposition of Weyl modules.
- This also gives evidence to the Schur positivity conjecture (order on $(P^+)^K_\lambda$ implies surjective maps).

S. Fomin et al, Okounkov, T. Lascoux et al have also obtained this tuple by the transpose of a row shuffle of a tuple of partitions.
They conjectured that there exists a surjective map

$$L(\lambda^{\max 1}) \otimes \ldots \otimes L(\lambda^{\max K}) \twoheadrightarrow L(\mu_1) \otimes \ldots \otimes L(\mu_k)$$

for any tuple $(\mu_1, \ldots, \mu_K) \in (P^+)^K_\lambda$.
This is now a theorem by T. Lam et al.

- This gives some evidence about the fusion product decomposition of Weyl modules.
- This also gives evidence to the Schur positivity conjecture (order on $(P^+)^K_\lambda$ implies surjective maps).

S. Fomin et al, Okounkov, T. Lascoux et al have also obtained this tuple by the transpose of a row shuffle of a tuple of partitions.
They conjectured that there exists a surjective map

$$L(\lambda^{\max 1}) \otimes \ldots \otimes L(\lambda^{\max K}) \twoheadrightarrow L(\mu_1) \otimes \ldots \otimes L(\mu_k)$$

for any tuple $(\mu_1, \ldots, \mu_K) \in (P^+)^K_\lambda$.
This is now a theorem by T. Lam et al.

- This gives some evidence about the fusion product decomposition of Weyl modules.
- This also gives evidence to the Schur positivity conjecture (order on $(P^+)^K_\lambda$ implies surjective maps).

S. Fomin et al, Okounkov, T. Lascoux et al have also obtained this tuple by the transpose of a row shuffle of a tuple of partitions.
They conjectured that there exists a surjective map

$$L(\lambda^{\max 1}) \otimes \ldots \otimes L(\lambda^{\max K}) \twoheadrightarrow L(\mu_1) \otimes \ldots \otimes L(\mu_k)$$

for any tuple $(\mu_1, \ldots, \mu_K) \in (P^+)^K_\lambda$.
This is now a theorem by T. Lam et al.

- This gives some evidence about the fusion product decomposition of Weyl modules.
- This also gives evidence to the Schur positivity conjecture (order on $(P^+)^K_\lambda$ implies surjective maps).

The proof of the conjectures is still work in progress...

- It is natural to consider $K = 2$.
- Proof for $\mathfrak{sl}_3(\mathbb{C})$, by given a bijection on highest weight vectors (Young tableaux).
- Most of the presentation works also for other classical types: the poset, the dimension formula.
- To get a general proof, a more intelligent approach might be useful.

Thank you!

The proof of the conjectures is still work in progress...

- It is natural to consider $K = 2$.
- Proof for $\mathfrak{sl}_3(\mathbb{C})$, by given a bijection on highest weight vectors (Young tableaux).
- Most of the presentation works also for other classical types: the poset, the dimension formula.
- To get a general proof, a more intelligent approach might be useful.

Thank you!

The proof of the conjectures is still work in progress...

- It is natural to consider $K = 2$.
- Proof for $\mathfrak{sl}_3(\mathbb{C})$, by given a bijection on highest weight vectors (Young tableaux).
- Most of the presentation works also for other classical types: the poset, the dimension formula.
- To get a general proof, a more intelligent approach might be useful.

Thank you!

The proof of the conjectures is still work in progress...

- It is natural to consider $K = 2$.
- Proof for $\mathfrak{sl}_3(\mathbb{C})$, by given a bijection on highest weight vectors (Young tableaux).
- Most of the presentation works also for other classical types: the poset, the dimension formula.
- To get a general proof, a more intelligent approach might be useful.

Thank you!

The proof of the conjectures is still work in progress...

- It is natural to consider $K = 2$.
- Proof for $\mathfrak{sl}_3(\mathbb{C})$, by given a bijection on highest weight vectors (Young tableaux).
- Most of the presentation works also for other classical types: the poset, the dimension formula.
- To get a general proof, a more intelligent approach might be useful.

Thank you!

The proof of the conjectures is still work in progress...

- It is natural to consider $K = 2$.
- Proof for $\mathfrak{sl}_3(\mathbb{C})$, by given a bijection on highest weight vectors (Young tableaux).
- Most of the presentation works also for other classical types: the poset, the dimension formula.
- To get a general proof, a more intelligent approach might be useful.

Thank you!

The proof of the conjectures is still work in progress...

- It is natural to consider $K = 2$.
- Proof for $\mathfrak{sl}_3(\mathbb{C})$, by given a bijection on highest weight vectors (Young tableaux).
- Most of the presentation works also for other classical types: the poset, the dimension formula.
- To get a general proof, a more intelligent approach might be useful.

Thank you!